A MAGNUS THEOREM FOR FREE PRODUCTS OF LOCALLY INDICABLE GROUPS

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(Received 19 August, 1988)

1. Introduction. A one-relator product G of groups A and B is defined to be the quotient of their free product $A * B$ by the normal closure, $\langle W \rangle^{A*B}$, of a single element W, which is assumed to be cyclically reduced and of length at least 2. For convenience, the group G will be denoted by $(A * B)/W$.

Since any infinite one-relator group can be regarded as a one-relator product of two free groups, it is natural therefore to try to extend known results from the theory of one-relator groups to other one-relator products. For further discussion and references in connection with this see [**5**] and [**6**].

A group is said to be locally indicable if each of its non-trivial, finitely generated subgroups admits an epimorphism onto the infinite cyclic group. The class of locally indicable groups is closed under subgroups and the formation of free products. We shall use these facts in the sequel without explicit reference. In this note we prove the following.

THEOREM. *Let A, B be locally indicable groups, and let $R, S \in A * B$ be cyclically reduced words each of length at least 2. If $\langle R \rangle^{A*B} = \langle S \rangle^{A*B}$ then R is a congugate in $A * B$ of $S^{\pm 1}$.*

This result generalises a theorem of Magnus [**8**] which says that if R and S are two cyclically reduced words in a free group whose normal closures coincide, then R is a cyclic permutation of $S^{\pm 1}$.

Since the identity map on $A * B$ induces an isomorphism from $(A * B)/R$ to $(A * B)/S$ if and only if $\langle R \rangle^{A*B} = \langle S \rangle^{A*B}$, we can regard the theorem as a result on one-relator products and hence as part of the programme described above.

Our approach is purely group-theoretical, being, in fact, a combination of the methods used in [**4**] and in [**1**]. Let $G = (A * B)/U$ be a one-relator product of locally indicable groups. With this same approach we have obtained straightforward proofs of the following results: if U is not a proper power then g is torsion-free [**3**, Theorem 4.2]; if $U = V^k$ $(k > 1)$, where V itself is not a proper power, then all elements of G of finite order are conjugates of powers of V [**4**, Corollary 8]. (Note that an easy consequence of the theorem is that V has order precisely k in G.)

Finally we mention the following corollary to the above results which doesn't seem to have appeared in any of the literature.

COROLLARY. *Let A, B, R and S be as in the theorem. If $(A * B)/R^n$ is isomorphic to $(A * B)/S^n$ $(n \geqslant 1)$ then $(A * B)/R$ is isomorphic to $(A * B)/S$.*

Glasgow Math. J. **31** (1989) 383–387.

This generalises Corollary 4.13.1 in [**9**], and can be obtained in exactly the same way.

2. Proof of Theorem. If A_0, B_0 denote the subgroups of A, B (respectively) generated by those elements of A and B that appear in R or S (or both) then by the Freiheitssatz for locally indicable groups [**2**] the natural maps $(A_0 * B_0)/R \to (A * B)/R$ and $(A_0 * B_0)/S \to (A * B)/S$ are monomorphisms. It follows from the comment made in the introduction that it is sufficient to deal with A_0 and B_0.

The proof is by induction on $l = \rho + \sigma$, where ρ and σ denote the lengths of R and S respectively, in $A * B$. There are four cases.

Case 0. A_0 and B_0 are free. This was done by Magnus in [**8**].

Case 1. R and S have length 2 in $A * B$ (i.e. the initial case $l = 4$ of the induction). Let $R = a_1 b_1$ and $S = a_2 b_2$ $(a_1, a_2 \in A,\ b_1, b_2 \in B)$. Then since A and B are torsion-free, $(A * B)/R$ is a free product with amalgamation. It follows from the theory of amalgamated products that $\langle a_1, a_2 \rangle$, $\langle b_1, b_2 \rangle$ are cyclic subgroups of A, B (respectively), whence A_0 and B_0 are both free and we are in case 0.

Since A, B are locally indicable groups, each of A_0, B_0 admits an epimorphism onto the infinite cycle group $\langle z \rangle$. A suitable combination of these two epimorphisms yields an epimorphism $\phi: A_0 * B_0 \to \langle z \rangle$ such that $R \in \ker \phi$. Now $S \in \langle R \rangle^{A_0 * B_0}$, and so $S \in \ker \phi$ also.

Case 2. $\phi(B_0) = 1$. Then $\phi(A_0) = \langle z \rangle$ so there exists $a \in A_0$ with $\phi(a) = z$. Define $K = A_0 \cap \ker \phi$, $C_i = a^i B_0 a^{-i}$ $(i \in \mathbb{Z})$. Then by the Kurosh subgroup theorem

$$\ker \phi = K * \Big(\mathop{\ast}_i C_i \Big).$$

Rewriting R and S in terms of this free product decomposition of $\ker \phi$ we find that at least 2 of the groups C_i are involved for at least one of R, S. Suppose otherwise. Then, for some fixed i, j we have that $R, S \in K * C_i$, $K * C_j$ (respectively), whence $a^{-i} R a^i$, $a^{-j} S a^j \in K * B_0$. If $i \neq 0$, then, since R is cyclically reduced of length at least 2, one of the letters a^{-i}, a^i must appear in the reduced form of $a^{-i} R a^i$; this forces $a^i \in K$, a contradiction. Hence $i = 0$ and similarly $j = 0$ whence, $R, S \in K * B_0$ contrary to the assumption that A_0 is generated by letters occurring in R and S. We can assume therefore that at least two of the C_i are involved in the rewrite of R.

Consider the following groups

$$P = \Big\langle K * \Big(\mathop{\ast}_i C_i \Big) \,\Big|\, R_i (i \in \mathbb{Z}) \Big\rangle$$

and

$$Q = \Big\langle K * \Big(\mathop{\ast}_i C_i \Big) \,\Big|\, S_i (i \in \mathbb{Z}) \Big\rangle,$$

where R_i, S_i is the rewrite of $a^i R a^{-i}$, $a^i S a^{-i}$ (respectively) $(i \in \mathbb{Z})$.

https://doi.org/10.1017/S001708950000793X Published online by Cambridge University Press

Let λ denote the least, and L the greatest subscript i such that C_i is involved in R_0, and let the corresponding values for S_0 be μ, M. Then λ is the least subscript i such that C_i is involved in $S_{\lambda-\mu}$. We show that L is the largest subscript of the C_i involved in $S_{\lambda-\mu}$, that is, we show that $L-\lambda=M-\mu$.

Suppose $L-\lambda>M-\mu$. Let P_i ($i \in \mathbb{Z}$) denote the group

$$P_i=\langle K*C_{\lambda+i}*\ldots*C_{L+i} \mid R_i\rangle.$$

Then

$$P_{\mu-\lambda}=\langle K*C_\mu*\ldots*C_{\mu+L-\lambda} \mid R_{\mu-\lambda}\rangle.$$

Since $L-\lambda>M-\mu$ we have $\mu+L-\lambda>M$, and so by the Freiheitssatz for locally indicable groups $K*C_\mu*\ldots*C_{\mu+L-\lambda-1}$ embeds naturally into $P_{\mu-\lambda}$. Likewise this group embeds into $P_{\mu-\lambda-1}$. Hence

$$P_{\mu-\lambda-1,\mu-\lambda}=\langle K*C_{\mu-1}*\ldots*C_{\mu+L-\lambda} \mid R_{\mu-\lambda-1}, R_{\mu-\lambda}\rangle$$

is the free product of $P_{\mu-\lambda-1}$ and $P_{\mu-\lambda}$ amalgamating $K*C_\mu*\ldots*C_{\mu+L-\lambda-1}$. Observe that $K*C_{\mu+1}*\ldots*C_{\mu+L-\lambda}$ embeds naturally into both $P_{\mu-\lambda-1,\mu-\lambda}$ and $P_{\mu-\lambda+1}$, so we can form the amalgamated free product

$$P_{\mu-\lambda-1,\mu-\lambda+1}=\langle K*C_{\mu-1}*\ldots*C_{\mu+L-\lambda+1} \mid R_{\mu-\lambda-1}, R_{\mu-\lambda}, R_{\mu-\lambda+1}\rangle.$$

We can then construct the free product with amalgamation

$$P_{\mu-\lambda-2,\mu-\lambda+1}=\langle K*C_{\mu-2}*\ldots*C_{\mu+L-\lambda+1} \mid R_{\mu-\lambda-2}, R_{\mu-\lambda-1}, R_{\mu-\lambda}, R_{\mu-\lambda+1}\rangle$$

where this time the amalgamated subgroup is $K*C_{\mu-1}*\ldots*C_{\mu+L-\lambda-2}$ of $P_{\mu-\lambda-1,\mu-\lambda+1}$ and $P_{\mu-\lambda-2}$.

Continuing this way we obtain a chain of groups whose union is P. Thus $P_{\mu-\lambda}$ is a subgroup of P. Since $\mu+L-\lambda>M$ we have that $K*C_\mu*\ldots*C_M$ embeds naturally in $P_{\mu-\lambda}$ and so in P. But the natural map between $(A_0*B_0)/R$ and $(A_0*B_0)/S$ induces an isomorphism $a^ixa^{-i}\mapsto a^ixa^{-i}$, $y\mapsto y$ ($x\in B_0$, $y\in K$) between P and Q. Whence $K*C_\mu*\ldots*C_M$ embeds naturally in Q, a contradiction. So we have that $L-\lambda \not> M-\mu$. If we suppose that $M-\mu>L-\lambda$ then at least two of the C_i are involved in S_0 and we can argue as above with the roles of R and S reversed.

Now we have the two isomorphic groups

$$\langle K*C_\lambda*\ldots*C_L \mid R_0\rangle$$

and

$$\langle K*C_\lambda*\ldots*C_L \mid S_{\lambda-\mu}\rangle.$$

Then $R_0, S_{\lambda-\mu}$ when regarded as words in $F*C_L$, where F denotes the group $K*C_\lambda*\ldots*C_{L-1}$, have cyclically reduced conjugates of length $\rho_0<\rho$, $\sigma_0<\sigma$, respectively, for otherwise $\lambda=L$, a contradiction. Moreover, if $\rho_0\leqslant 1$ then R_0 is conjugate in $F*C_L$ to an element of F or of C_L. The first contradicts the choice of L; the second contradicts the fact that $\rho\geqslant 2$. Similarly we have that $\sigma_0\geqslant 2$. By the induction hypothesis R_0 is a conjugate of $S_{\lambda-\mu}$. Thus the rewrite of R is equal to the rewrite of $Ua^{\lambda-\mu}Sa^{-(\lambda-\mu)}U^{-1}$ for some word U in A_0*B_0. By the properties of rewriting this means that R is conjugate to S, as required.

Case 3. $\phi A_0 \neq 1 \neq \phi B_0$. By case 0 we can assume that A_0 is not free. Choose letters $\alpha \in A_0$, $\beta \in B_0$ with α appearing in R or S and with $\phi(\alpha) = z^{n_1}$, $\phi(\beta) = z^{n_2}$ $(n_1, n_2 \neq 0)$. (The existence of such letters is ensured by the hypotheses.) Let us assume α appears in R. After replacing R by a cyclic permutation, if necessary, and possibly z by z^{-1}, we have $R = \alpha\beta_1 \ldots \alpha_k\beta_k$ $(k \geqslant 2)$ and $n_1 > 0$. Let $A_1 = \langle A_0, a \mid a^{n_1} = \alpha \rangle$. Then A_1 is locally indicable by [7, Theorem 9], and ϕ extends to A_1 by setting $\phi(a) = z$. Similarly we can extend ϕ to the locally indicable group $B_1 = \langle B_0, b \mid b^{n_2} = \beta \rangle$.

Let G be the locally indicable group $A_1 * B_1 * \langle c \rangle$, where $\langle c \rangle$ denotes the infinite cyclic group, and extend ϕ to G by setting $\phi(c) = z$. Then $R, S \in \ker \phi = K_A * K_B * \langle a_i, b_i (i \in \mathbb{Z}) \rangle$ where $K_A = \ker \phi \cap A_0$, $K_B = \ker \phi \cap B_0$, $a_i = c^i a c^{-1-i}$ and $b_i = c^i b c^{-1-i}$.

Consider the groups

$$\tilde{P} = \langle K_A * K_B * \langle a_1, b_i (i \in \mathbb{Z}) \rangle \mid \tilde{R}_i (i \in \mathbb{Z}) \rangle$$

and

$$\tilde{Q} = \langle K_A * K_B * \langle a_i, b_i (i \in \mathbb{Z}) \rangle \mid \tilde{S}_i (i \in \mathbb{Z}) \rangle$$

where $\tilde{R}_i$, $\tilde{S}_i$ denote the rewrites, in terms of the free product decomposition of $\ker \phi$, of $c^i R c^{-i}$, $c^i S c^{-i}$ respectively. In fact

$$\tilde{R}_0 = a_0 \ldots a_{n_1 - 1} Y_1 \bar{\beta}_1 Z_1 \ldots W_k \bar{\alpha}_k X_k Y_k \bar{\beta}_k Z_k$$

for some elements $\bar{\alpha}_j \in K_A$, $\bar{\beta}_j \in K_B$ and words W_i, X_i in the a_j and Y_i, Z_i in the b_j.

Let λ, L denote the least and greatest indices i such that a_i appears in $\tilde{R}_0$. Then $\lambda \leqslant 0 \leqslant L$. Let μ, M be the corresponding values for $\tilde{S}_0$. Clearly by the Freiheitssatz and the fact that the natural map again induces an isomorphism $\tilde{P} \to \tilde{Q}$, we see that $L = \lambda$ if and only if $M = \mu$. Suppose then that $L \neq \lambda$. As in case 1 we can show, using an argument similar to the one used there, that $L - \lambda = M - \mu$. Thus the natural map induces an isomorphism between the groups

$$\langle K_A * K_B * \{a_\lambda, \ldots, a_L, b_i (i \in \mathbb{Z})\} \rangle \mid \tilde{R}_0 \rangle$$

and

$$\langle K_A * K_B * \{a_\lambda, \ldots, a_L, b_i\ (i \in \mathbb{Z})\} \rangle \mid \tilde{S}_{\lambda-\mu} \rangle.$$

Let $\tilde{F} = K_B * \langle a_\lambda, \ldots, a_L, b_i (i \in \mathbb{Z}) \rangle$, and let $\tilde{\tilde{R}}_0$, $\tilde{\tilde{S}}_{\lambda-\mu}$ denote the rewrites of $\tilde{R}_0$, $\tilde{S}_{\lambda-\mu}$, respectively, in terms of $K_A * \tilde{F}$. Let the length of $\tilde{\tilde{R}}_0$, $\tilde{\tilde{S}}_{\lambda-\mu}$ be ρ_1, σ_1, respectively. Then $\rho_1 \leqslant 1$ if and only if $\sigma_1 \leqslant 1$ only if $\tilde{\tilde{R}}_0$, $\tilde{\tilde{S}}_{\lambda-\mu} \in \tilde{F}$ only if $A_0 \leqslant gp(a)$, contradicting the assumption that A_0 is not free. Furthermore, $\rho_1 < \rho$ and the result now follows by the inductive hypothesis.

Acknowledgement. This paper is a response to a question of Jim Howie whom I wish to thank for correcting my mistakes in an earlier draft.

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https://doi.org/10.1017/S001708950000793X Published online by Cambridge University Press