SIMPLICIAL AND HOMOTOPICAL COHOMOLOGY OF POLYHEDRA

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1. Introduction. It is well known that, on the category of finite polyhedra, any two cohomology theories, satisfying the Eilenberg–Steenrod axioms, are isomorphic. Examples of such theories are simplicial cohomology and homotopical cohomology (the latter is defined by means of homotopy classes of maps into Eilenberg–MacLane spaces). In the case of polyhedra, using triple sequences and spectral sequences, one obtains a deep insight into the relationship between general cohomology theories (without the dimension axiom) and ordinary simplicial cohomology (**1**, p. 66). As a corollary the above-mentioned uniqueness of cohomology theories satisfying the dimension axiom is obtained.

Section 2 contains an entirely elementary proof of the isomorphism between simplicial and homotopic cohomology in the case of an arbitrary, possibly infinite, polyhedron. We need, in fact, only the exact cohomology sequence of cofibrations and the homotopy sequence of a fibration. The theory given here may also serve as an introduction to the author's work on copolyhedra (**6**, pp. 215–254).

In §3, we discuss incidence numbers in the framework of homotopy theory. We give this notion a new, purely homotopical interpretation, making use of homotopical transgression.

In §4, the theory dual to the preceding one is developed. Duality is meant in the sense of Eckmann and Hilton (**1**, pp. 59–73). We consider here copolyhedra. These spaces have been investigated by the author (**6**, pp. 215–254). A polyhedron is a space Y for which a sequence of cofibrations

$$Y_0 \xrightarrow{i_1} Y_1 \xrightarrow{i_2} \ldots \to Y$$

exists (the inclusions of the m-skeletons), such that all the cofibres are built up of Moore spaces (wedge products of spheres). Dual to polyhedra are copolyhedra. Roughly speaking, a copolyhedron is a space X, for which a sequence of fibre maps

$$X \xrightarrow{p_1} X_1 \xrightarrow{p_2} X_2 \xrightarrow{p_3} \ldots$$

exists, such that all the fibres are Eilenberg–MacLane spaces. Some additional

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properties are required; cf. §4. If the fibres are topological products of Eilenberg–MacLane spaces $K(Z, n)$, Z = infinite cyclic group, we call the copolyhedron a special copolyhedron. We develop a theory of incidence numbers for such special copolyhedra. The ordinary homotopy groups of those special copolyhedra whose fibres are the products of a finite number of spaces $K(Z, n)$ are then determined by incidence matrices as are the cohomology groups of a finite polyhedron.

2. Simplicial and homotopical cohomology of polyhedra. Consider a polyhedron X, denote by X_m its m-skeleton, by $i_m : X_{m-1} \rightarrow X_m$, $j_m : X_{m-1} \rightarrow X$ the natural inclusions and by Coker i_m and Coker j_m the spaces X_m/X_{m-1} and X/X_{m-1} respectively. Given an abelian group G, a simplicial cochain f^m with coefficients in G is a function on the oriented m-simplexes $\{\sigma_m\}$ with values $g = f^m(\sigma_m)$ in G. We can also represent this element g by an element of $\pi_m(K(G, m))$, where $K(G, m)$ is the Eilenberg–MacLane space corresponding to G and m. In other words, g is just a homotopy class of maps

$$(\sigma_m, |\partial\sigma_m|) \longrightarrow (K(G, m), *),$$

which map the boundary sphere $|\partial\sigma_m|$ of σ_m into the base point $*$ of $K(G, m)$. Since the quotient space X_m/X_{m-1} is just a wedge product of m-spheres, each m-sphere corresponding to exactly one m-simplex of X_m, we see that a simplicial cochain $f^m \in C^m(X, G)$ determines a homotopy class

$$F(f^m) \in \Pi(\text{Coker } i_m, K(G, m)).$$

It is clear that we obtain an isomorphism between the two groups $C^m(X, G)$ and $\Pi(\text{Coker } i_m, K(G, m))$ in this way. We have thus proved:

PROPOSITION 2.1. *The simplicial cochain group $C^m(X, G)$ of a polyhedron X is isomorphic to the homotopical cochain group*

$$C_h^m(X, G) = \Pi(\text{Coker } i_m, K(G, m)).$$

We shall now build up a part of the homotopical cohomology theory of polyhedrons. The main task is to define a coboundary operator δ_h and study its properties. Let us denote the group $\Pi(\text{Coker } j_m, K(G, m))$ by $Z_h^m(X, G)$. It will turn out that this is the group of m-cocycles.

Since j_{m+1} is a cofibration, the diagram

$$\begin{array}{ccc} X_m & \longrightarrow & \text{Coker } i_m \\ \Big\downarrow{\scriptstyle j_{m+1}} & & \\ X & & \end{array}$$

may be completed by the induced cofibration ϕ_m:

$$\begin{array}{ccc} X_m & \longrightarrow & \operatorname{Coker} i_m \\ \Big\downarrow j_{m+1} & & \Big\downarrow \phi_m \\ X & \longrightarrow & X/X_{m-1} \end{array}$$

It is known that the cofibre of ϕ_m is the same as that of j_{m+1}; i.e. equal to X/X_m.

Consider now the exact sequence of homotopical cohomology of the cofibration ϕ_m:

$$(2.1)\quad \ldots \to H_h^m(\operatorname{Coker} j_{m+1}, G) \to H_h^m(\operatorname{Coker} j_m, G) \to H_h^m(\operatorname{Coker} i_m, G) \xrightarrow{\delta'} H_h^{m+1}(\operatorname{Coker} j_{m+1}, G) \to H_h^{m+1}(\operatorname{Coker} j_m, G) \to H_h^{m+1}(\operatorname{Coker} i_m, G) \to \ldots .$$

All spaces Coker j_m, Coker i_m are CW-complexes in a natural way. Coker $j_{m+1} = X/X_m$ has no m-cells (if $m > 0$). It follows that $H_h^m(\operatorname{Coker} j_{m+1}, G) = 0$. Similarly $H_h^{m+1}(\operatorname{Coker} i_m, G) = 0$. Remembering the definitions of $C_h^m(X, G)$ and $Z_h^m(X, G)$, we see that the sequence (2.1) becomes

$$(2.2)\quad 0 \to Z_h^m(X, G) \to C_h^m(X, G) \xrightarrow{\delta'} Z_h^{m+1}(X, G) \to H_h^{m+1}(\operatorname{Coker} j_m, G) \to 0.$$

The group $H_h^{m+1}(\operatorname{Coker} j_m, G)$ can be interpreted in a different manner. For this purpose, we look at the homotopical cohomology sequence of the cofibration j_m:

$$\ldots \to H_h^m(X_{m-1}, G) \to H_h^{m+1}(\operatorname{Coker} j_m, G) \to H_h^{m+1}(X, G) \to H_h^{m+1}(X_{m-1}, G) \to \ldots .$$

The $(m-1)$-skeleton X_{m-1} of X has no simplexes of dimension $> m-1$; hence the groups $H_h^m(X_{m-1}, G)$ and $H_h^{m+1}(X_{m-1}, G)$ consist of the zero element alone. Therefore the groups $H_h^{m+1}(\operatorname{Coker} j_m, G)$ and $H_h^{m+1}(X, G)$ are isomorphic. If we identify them under this isomorphism, the sequence (2.2) becomes

$$(2.3)\quad 0 \to Z_h^m(X, G) \to C_h^m(X, G) \xrightarrow{\delta'} Z_h^{m+1}(X, G) \to H_h^{m+1}(X, G) \to 0.$$

If we connect two such sequences of dimension m and dimension $m + 1$, we get:

$$\begin{array}{ccccccccc} 0 \to Z_h^m(X, G) \to C_h^m(X, G) & \xrightarrow{\delta'} & Z_h^{m+1}(X, G) & \to & H_h^{m+1}(X, G) \to 0 \\ & & \Big\downarrow Id & & \\ & 0 \to & Z_h^{m+1}(X, G) & \xrightarrow{i} & C_h^{m+1}(X, G) \to \ldots . \end{array}$$

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We then define the homotopical coboundary operator by $\delta_h = i \circ \delta'$. Because of the exactness of the sequences it is a trivial matter to verify

PROPOSITION 2.2. (a) $\delta_h \circ \delta_h = 0$,
(b) $\operatorname{Ker} \delta_h \cong Z_h^m(X, G)$,
(c) $H_h^{m+1}(X, G) \cong Z_h^{m+1}(X, G)/\operatorname{Im} \delta_h$.

Once we have proved that the homotopically defined coboundary operator δ_h coincides with the ordinary simplicial coboundary δ, if we identify $C_h^m(X, G)$ with $C^m(X, G)$ (see Proposition 2.1), it follows that

$$Z_h^m(X, G) = \operatorname{Ker} \delta_h = \operatorname{Ker} \delta = Z^m(X, G)$$

and also that $\operatorname{Im} \delta_h = \operatorname{Im} \delta$. Proposition 2.2 then gives the result

$$H_h^{m+1}(X, G) = Z_h^{m+1}(X, G)/\operatorname{Im} \delta_h = Z^{m+1}(X, G)/\operatorname{Im} \delta = H^{m+1}(X, G),$$

i.e. the isomorphism of the homotopical and the simplicial cohomology.

A cochain $[f] \in C_h^m(X, G)$ is a homotopy class of a map

$$f: X_m/X_{m-1} \to K(G, m).$$

We identify the space $K(G, m)$ with the loop space $\Omega K(G, m+1)$. Instead of considering f, we may also look at the diagram

$$\begin{array}{ccccc}
X_m/X_{m-1} & \xrightarrow{f} & \Omega K(G, m+1) & \to & EK(G, m+1) \\
\downarrow & & \downarrow & & \downarrow \rho \\
* & \longrightarrow & * & \longrightarrow & K(G, m+1)
\end{array}$$

Here ρ is the fibre map associated with the paths on $K(G, m+1)$, starting at the base point $*$; $\delta' f$ (see diagram 2.3) is obtained by augmentation of the above diagram so that we obtain the diagram

$$\begin{array}{ccccccc}
X_m/X_{m-1} & \xrightarrow{Id} & X_m/X_{m-1} & \xrightarrow{f} & \Omega K(G, m+1) & \xrightarrow{i} & EK(G, m+1) \\
\downarrow \phi_m & & \downarrow & & \downarrow & & \downarrow \rho \\
X/X_{m-1} & \longrightarrow & * & \longrightarrow & * & \longrightarrow & K(G, m+1)
\end{array}$$

followed by a deformation of the map $i \circ f$ into the trivial map. Since ϕ_m is a cofibration, the deformation of $\rho \circ i \circ f$ may be extended to X/X_{m-1}. There results a map $f' : X/X_{m-1} \to K(G, m+1)$, which transforms X_m/X_{m-1} into the base point. The corresponding element in $\Pi(X/X_m, K(G, m+1))$ is precisely $\delta' f$. $\delta_h f$ is obtained by restriction of $\delta' f$ to $X_{m+1}/X_m \subset X/X_m$.

After this description of $\delta_h f$ we are going to prove that $\delta_h f$ attaches to an $(m+1)$-simplex σ_{m+1} the value of f on the boundary $\partial\sigma_{m+1}$, in the same way as in the simplicial theory. To achieve this, we consider the restriction of $\delta_h f$ to the $(m+1)$-simplex σ_{m+1}. Here $\delta_h f$ is considered as a map

$$\sigma_{m+1}/|\partial\sigma_{m+1}| \to X_{m+1}/X_m \to K(G, m+1).$$

Let $\partial : \pi_{m+1}(K(G, m + 1)) \rightarrow \pi_m(\Omega K(G, m + 1))$ be the isomorphism in the homotopy sequence of the fibration ρ. We identify these two groups under the isomorphism ∂. Then $\delta_h f(\sigma_{m+1})$ and $\partial(\delta_h f(\sigma_{m+1}))$ are the same element of G. But looking at the definition of ∂, one sees easily that $\partial(\delta_h f(\sigma_{m+1}))$ is nothing else but the homotopy class of the restriction of $i \circ f: X_m/X_{m-1} \rightarrow EK(G, m)$ to the m-skeleton of σ_{m+1}, i.e. to the boundary $\partial\sigma_{m+1}$ of σ_{m+1}. Applying the homotopy addition theorem (**3**, pp. 108–122), one sees that because X_{m-1} is mapped onto the base point, this element of

$$\Pi(\partial\sigma_{m+1}/\text{its } (m-1)\text{-skeleton}, K(G, m))$$

equals

$$\sum_{j=0}^{m+1} [\sigma_{m+1} : \sigma_m{}^j][i \circ f|\sigma_m{}^j],$$

where $[\sigma_{m+1} : \sigma_m{}^j]$ are the ordinary incidence numbers. So we have

$$(\delta_h f)(\sigma_{m+1}) = \sum_{j=0}^{m+1} [\sigma_{m+1} : \sigma_m{}^j] f(\sigma_m{}^j).$$

In the same way the simplicial coboundary δ is defined, so that $\delta_h = \delta$.

We have proved

PROPOSITION 2.3. *The homotopical coboundary operator δ_h coincides with the simplicial coboundary operator δ. The value $(\delta_h f^m)(\sigma_{m+1})$ is obtained as the homotopy class of the restriction of f^m to the boundary of σ_{m+1}.*

COROLLARY. *The homotopical cohomology groups*

$$H_h{}^m(X, G) = \Pi(X, K(G, m))$$

are isomorphic to the ordinary simplicial cohomology groups $H^m(X, G)$.

3. Incidence numbers in the case of polyhedra. Let X be a polyhedron, σ_m an m-simplex of X. The boundary $\partial\sigma_m$ is a linear combination of $(m-1)$-simplexes:

$$\partial\sigma_m = \sum_{j=0}^{m} [\sigma_m : \sigma_{m-1}{}^j]\sigma_{m-1}{}^j.$$

This defines the incidence numbers $[\sigma_m : \sigma_{m-1}{}^j]$ in the ordinary way. Instead of using the boundary operator for chains, we may consider integral-valued simplicial cochains $f^m \in C^m(X, Z)$. Suppose the simplexes are indexed by some variable k. If we denote by $f_k{}^m$ the cochain taking the value $+1$ on $\sigma_m{}^k$ and the value 0 on all $\sigma_m{}^{k'}$, $k' \neq k$, then $f_k{}^m(\sigma_m{}^{k'}) = \delta_{kk'}$. For the coboundary follows:

$$(\delta f_k{}^m)(\sigma_{m+1}) = f_k{}^m\Big(\sum_{k'} [\sigma_{m+1} : \sigma_m{}^{k'}]\sigma_m{}^{k'}\Big)$$
$$= \sum_{k'} [\sigma_{m+1} : \sigma_m{}^{k'}]\cdot f_k{}^m(\sigma_m{}^{k'}) = \sum_{k'} [\sigma_{m+1} : \sigma_m{}^{k'}]\delta_{kk'} = [\sigma_{m+1} : \sigma_m{}^k].$$

This proves

PROPOSITION 3.1. *The incidence number* $[\sigma_{m+1} : \sigma_m{}^k]$ *belonging to an* $(m + 1)$*-simplex* σ_{m+1} *and an* m*-simplex* $\sigma_m{}^k$ *is equal to the value of the coboundary of the characteristic cochain belonging to* $\sigma_m{}^k$ *on the simplex* σ_{m+1}.

We now interpret this result by passing to the homotopical treatment of simplicial cohomology given in §2. Let X_m denote the m-skeleton of X. A general integral-valued m-cochain is represented by a homotopy class of a map $f : X_m/X_{m-1} \to K(Z, m)$. The characteristic cochain $f_k{}^m$ is zero on each m-simplex different from $\sigma_m{}^k$. This means that it may be represented by the homotopy class of a map $f_k{}^m$, which transforms everything of X_m/X_{m-1}, with the exception of $\sigma_m{}^k$ into the base point of $K(Z, m)$. The restriction of the map to $\bar{\sigma}_m{}^k$ represents $1 \in Z$. Proposition 2.3 gives now the result:

PROPOSITION 3.2. $[\sigma_{m+1} : \sigma_m{}^k]$ *is the homotopy class of the restriction of* $f_k{}^m$ *to the boundary of* σ_{m+1}.

Since X_m/X_{m-1} is a wedge product of spheres,

$$X_m/X_{m-1} = \oplus_k K'_k(m, Z),$$

we may introduce for a fixed k_0 the projection $\tilde{p}_{k_0}$ of

$$\oplus_k K'_k(m, Z)$$

onto the factor $K'_{k0}(m, Z)$. (This is the map that equals the identity on $K'_{k0}(m, Z)$ and maps the rest onto the base point.)

We may also consider the inclusion

$$\tilde{i}_{k0} : K'_{k0}(m, Z) \to \oplus_k K'_k(m, Z).$$

Let $\bar{p}_m : X_m \to X_m/X_{m-1}$ be the canonical projection. Then we have the commutative diagram:

$$\begin{array}{ccccc} X_{m-1} & \longrightarrow & * & & \\ \Big\downarrow i_m & & \Big\downarrow & & \\ X_m & \xrightarrow[\bar{p}_m]{} & \oplus_k K'(m, Z) & \xrightarrow[\tilde{p}_{k0}]{} & K'_{k0}(m, Z). \end{array}$$

The characteristic cochain of $\sigma_m{}^{k_0}$ can then be represented by a composite map

$$X_m/X_{m-1} = \oplus_k K'(m, Z) \xrightarrow{\tilde{p}_{k0}} K'_{k0}(m, Z) \xrightarrow{h_{k0}} K(Z, m).$$

The homotopy class of h_{k_0} is determined by the equation $h_{k0*}(1) = 1 \in Z$. Let S^m be the m-sphere. Since $\pi_m(S^m) = Z = \pi_m(K(Z, m))$, to the map

$$h_{k_0} \circ \tilde{p}_{k_0} : X_m/X_{m-1} \to K(Z, m)$$

considered above, corresponds a unique class of maps

$$X_m/X_{m-1} = \oplus_k K'_k(m, Z) \xrightarrow{\tilde{p}_{k0}} K'_{k0}(m, Z) \xrightarrow{r_{k0}} S^m.$$

It is then clear that the incidence number $[\sigma_{m+1} : \sigma_m{}^{k_0}]$ is obtained by restricting the map $r_{k_0} \circ \tilde{p}_{k_0}$ to $\partial\sigma_{m+1}$. We shall obtain a new interpretation of the incidence

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numbers by applying a theorem about homotopical transgression (**2**, p. 622, Proposition 4) to the following situation:

$$\text{(I)}\qquad \begin{array}{ccccccc} X_m & \xrightarrow{\bar{p}_m} & \oplus_k K'_k(m, Z) & \xrightarrow{\tilde{p}_{k0}} & K'_{k0}(m, Z) & \xrightarrow{r_{k0}} & S^m \\ \downarrow{\scriptstyle i_{m+1}} & & & & & & \\ X_{m+1} & \xrightarrow{\bar{p}_{m+1}} & \oplus_l K'_l(m+1, Z) & \xleftarrow{\tilde{i}_{l0}} & K'_{l0}(m+1, Z) & \xleftarrow{r_{l0}^{-1}} & S^{m+1} \end{array}$$

$\tilde{i}_{l0}$ is the natural inclusion map. If we suppose the polyhedron X_m to be simply connected, then we may apply (**2**, Proposition 4) to the cofibration i_{m+1} since Z is free. Since $\pi_1(X_m) \cong \pi_1(X)$ for $m \geqslant 2$ (**3**, p. 239, 6.3.3 corollary), we shall assume henceforth that $m \geqslant 2$ and $\pi_1(X) = 0$. The map h mentioned in (**2**, p. 622) is in our case

$$\begin{array}{ccc} X_m & \longrightarrow & * \\ {\scriptstyle i_{m+1}}\downarrow & \xrightarrow{h} & \downarrow \\ X_{m+1} & \longrightarrow & \oplus_l K'_l(m+1, Z) \end{array}$$

and the proposition states that

$$h_* : \pi_{m+1}(i_{m+1}) \to \pi_{m+1}(\oplus_l K'_l(m+1, Z))$$

is an isomorphism. Noting that ∂ leads from $\pi_{m+1}(i_{m+1})$ to $\pi_m(X_m)$, we thus obtain the diagram

$$\text{(II)}\qquad \begin{array}{ccccccc} \pi_m(X_m) & \xrightarrow{\bar{p}_{m*}} & \pi_m(\oplus_k K'_k(m, Z)) & \xrightarrow{\tilde{p}_{k0*}} & \pi_m(K'_{k0}(m, Z)) & \xrightarrow{r_{k0*}} & \pi_m(S^m) \\ \uparrow{\scriptstyle \sigma = \partial \circ h_*^{-1}} & & & & & & \\ \pi_{m+1}(\oplus_l K'_l(m+1, Z)) & \xleftarrow[\tilde{i}_{l0*}]{} & \pi_{m+1}(K'_{l0}(m+1, Z)) & \xleftarrow[r_{l0*}^{-1}]{} & \pi_{m+1}(S^{m+1}). & & \end{array}$$

Consider the element $1 \in Z = \pi_{m+1}(S^{m+1})$. Its image in $\pi_{m+1}(\oplus_l K'_l(m+1, Z))$ (see diagram II) may be represented by

$$\begin{array}{ccc} S^m & \longrightarrow & * \\ \downarrow & & \downarrow \\ CS^m & \longrightarrow & \oplus_l K'_l(m+1, Z) \end{array}$$

and hence the corresponding element $h_*^{-1}(i_{l0*} \circ r_{l0*}^{-1}(1))$ by the inclusion map

$$\begin{array}{ccccc} S^m & \longrightarrow & \partial\sigma_{m+1}^{l0} & \xrightarrow{f_1} & X_m \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle i_{m+1}} \\ CS^m & \longrightarrow & \sigma_{m+1}^{l0} & \xrightarrow{f_2} & X_{m+1} \end{array}$$

f_1 and f_2 are the natural inclusions. The transgression being $\sigma = \partial \circ h_*^{-1}$, we see that $\sigma(i_{l_0*} \circ r_{l_0*}^{-1}(1))$ is represented by $S^m \to \partial\sigma_{m+1}{}^{l_0} \to X_m$. If we compose this map with $r_{k_0} \circ \tilde{p}_{k_0} \circ \bar{p}_m : X_m \to S^m$, corresponding to the characteristic co-chain of $\sigma_m{}^{k_0}$, we obtain:

$$S^m \longrightarrow \partial\sigma_{m+1}{}^{l_0} \xrightarrow{f_1} X_m \xrightarrow{\bar{p}_m} \oplus_k K'_k(m, Z) \xrightarrow{\tilde{p}_{k_0}} K'_{k_0}(m, Z) \xrightarrow{r_{k_0}} S^m.$$

The degree of this map is the incidence number $[\sigma_{m+1} : \sigma_m{}^{k_0}]$. This proves:

PROPOSITION 3.3. *In a simply connected polyhedron X the incidence number $[\sigma_{m+1}{}^{l_0} : \sigma_m{}^{k_0}]$ is for $m \geqslant 2$ the image of $1 \in \pi_{m+1}(S^{m+1})$ in $\pi_m(S^m)$ under the map indicated by diagram (II).*

4. Incidence numbers in the theory of copolyhedra. We first recall the definition of a copolyhedron (**6**, p. 222). A copolyhedron is a space X, fibred by a sequence of fibre maps

$$X \xrightarrow{p_1} X_1 \xrightarrow{p_2} X_2 \xrightarrow{p_3} \ldots$$

satisfying the following conditions:

(α) Each fibre Ker p_m is an Eilenberg–MacLane space $K(Q_m, m)$.

(β) $\pi_{m-1}(X_m) = 0$ for each $m \geqslant 1$.

(γ) For all abelian groups G, we have $\text{Ext}(G, Q_m) = 0$, i.e. the groups Q_m are divisible (**5**, pp. 92–93).

We also recall that the homotopy groups $\pi_m(G; X)$ of a copolyhedron X with coefficients in an abelian group G may be calculated in the following way: Put

$$\mathbf{C}^m(G; X) = \pi_m(G; \text{Ker } p_m).$$

Then there exists a differential operator

$$d_m : \mathbf{C}^m(G, X) \to \mathbf{C}^{m-1}(G, X)$$

and we have

$$\pi_m(G; X) = \text{Ker } d_m/\text{Im } d_{m+1};$$

i.e. the homotopy groups are calculated as the homology groups of the chain complex $(\mathbf{C}(G, X), d)$, see (**6**, p. 222, Satz 1.4′).

We shall be interested in spaces X for which only the conditions (α), (β) of a copolyhedron are fulfilled. The groups Q_m will be direct products of infinite cyclic groups. We call such spaces *special copolyhedra*. As for the coefficient group G, we shall consider only the case $G = Z$. The remark following (**6**, p. 222, Satz 1.4′) shows that the homotopy groups $\pi_n(X) = \pi_n(Z; X)$ can still be computed as the homology groups of the above-mentioned chain complex.

In the previous section we assigned incidence numbers $[\sigma_{m+1}{}^{l_0} : \sigma_m{}^{k_0}]$ to pairs of simplexes $\sigma_{m+1}{}^{l_0}$ and $\sigma_m{}^{k_0}$, or equivalently to their images $K'_{l_0}(m + 1, Z)$

and $K'_{k0}(m, Z)$ in the cofibres X_{m+1}/X_m and X_m/X_{m-1}. Since the notion of a copolyhedron is dual to that of a polyhedron, to a cofibre

$$X_m/X_{m-1} = \oplus_k \; K'_k(m, Z)$$

of the polyhedron X corresponds dually a fibre

$$\text{Ker } p_m = \prod_k K_k(Z, m)$$

of the copolyhedron; Z is the additive group of integers. Each factor $K_{k0}(Z, m)$ of $\prod_k K_k(Z, m)$ is said to correspond to a cosimplex of X.

We now develop the theory dual to the one expounded in §3, i.e. we shall define incidence numbers between a factor

$$\sigma^{m+1}{}_{l0} = K_{l0}(Z, m+1) \subset \text{Ker } p_{m+1}$$

and a factor

$$\sigma^m{}_{k0} = K_{k0}(Z, m) \subset \text{Ker } p_m$$

and then prove some theorems corresponding to theorems of §3. First we prove that any element $[f] \in \mathbf{C}^m(G, X)$ defines a function on the cosimplexes $K_k(Z, m)$ with values in $\text{Hom}(G, Z)$.

f is a map

$$f : K'(m, G) \to \text{Ker } P_m = \prod_k K_k(Z, m).$$

Let $\tilde{p}_{k0}$ denote the projection

$$\prod_k K_k(Z, m) \to K_{k0}(Z, m).$$

Then the value $\phi(K_{k0}(Z, m))$ is defined to be the homotopy class of the map

$$K'(m, G) \xrightarrow{f} \text{Ker } p_m = \prod_k K_k(Z, m) \xrightarrow{\tilde{p}_{k0}} K_{k0}(Z, m).$$

By the universal coefficient theorem for homotopy groups, this is an element of $\pi_m(G; K_{k0}(Z, m)) = \text{Hom}(G, Z)$. Conversely, suppose we are given for each cosimplex $\sigma^m{}_k = K_k(Z, m)$ a value

$$g_k^* \in \text{Hom}(G, Z) = \pi_m(G, K_{k0}(Z, m)).$$

Each g_k^* determines a homotopy class

$$(g_k^*) \in \Pi(K'(m, G), K_k(Z, m))$$

and the product yields a homotopy class

$$\prod_k (g_k^*) \in \Pi(K'(m, G), \prod_k K_k(Z, m)) = \mathbf{C}^m(G, X).$$

This readily implies

PROPOSITION 4.1. *The group* $\mathbf{C}^m(G, X)$ *defined homotopically as*

$$\Pi(K'(m, G), \text{Ker } p_m)$$

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is isomorphic to the group of infinite chains of cosimplexes with values in Hom (G, Z).

From now on we restrict our attention to the case of special copolyhedra ($G = Z$). Since Hom $(Z, Z) = Z$, there exists a map

$$f : K'(m+1, Z) \to \prod_l K_l(Z, m+1)$$

whose components equal the constant map for $l \neq l_0$ and whose l_0th component is in the homotopy class $1 \in \pi_{m+1}(K(Z, m+1))$. The homotopy class

$$[f] \in \mathbf{C}^{m+1}(Z, X)$$

is the characteristic chain of the cosimplex $K_{l_0}(Z, m+1)$. The map f may be represented as

$$S^{m+1} = K'(m+1, Z) \xrightarrow{h_{l_0}} K_{l_0}(Z, m+1) \xrightarrow{\tilde{\imath}_{l_0}} \prod {}_l K_l(Z, m+1),$$

i.e. $$f = h_{l_0} \circ \tilde{\imath}_{l_0}, \qquad [h_{l_0}] = 1 \in \pi_{m+1}(K(Z, m+1)).$$

Definition. Let σ_l^{m+1}, σ_k^m be cosimplexes of the special copolyhedron X. Let

$$[f] \in \mathbf{C}^{m+1}(Z, X)$$

be the characteristic chain of σ_l^{m+1}. The incidence number $[\sigma^{m+1}{}_l : \sigma^m{}_k]$ is defined to be the value of $d_{m+1}[f]$ on the cosimplex $\sigma^m{}_k$.

Remark. Since the value of $d_{m+1}[f]$ on $\sigma^m{}_k$ is the homotopy class of a map $K'(m, Z) \to K(Z, m)$, it is an element of $\pi_m(K(Z, m))$, i.e. an integer.

We now give an explicit description of $[\sigma^{m+1}{}_l : \sigma^m{}_k]$. Consider the diagram

$$\text{(I)} \qquad \begin{array}{ccccccc} S^m & \xleftarrow{h_{k_0}^{-1}} & K_{k_0}(Z, m) & \xleftarrow{\tilde{p}_{k_0}} & \operatorname{Ker} p_m & \longrightarrow & p_m^{-1}(\operatorname{Ker} p_{m+1}) \\ & & & & & & \downarrow \tau_m \\ S^{m+1} & \xrightarrow{h_{l_0}} & K_{l_0}(Z, m+1) & & \xrightarrow{\quad i_{l_0} \quad} & & \operatorname{Ker} p_{m+1} \end{array}$$

τ_m is the restriction of the fibre map p_m to the subspace $p_m^{-1}(\operatorname{Ker} p_{m+1}) \subset X_{m-1}$, $[h_{k_0}] = 1 \in \pi_m(K_{k_0}(Z, m))$ and $\tilde{p}_{k_0}$ the canonical projection. Since τ_m is a fibre map with fibre Ker τ_m = Ker p_m, we obtain the homomorphism of homotopy groups described in diagram II:

$$\text{(II)} \qquad \begin{array}{ccccc} Z = \pi_m(S^m) & \xleftarrow{h_{k_0*}^{-1}} & \pi_m(K_{k_0}(Z, m)) & \xleftarrow{\tilde{p}_{k_0*}} & \pi_m(\operatorname{Ker} p_m) \\ & & & & \uparrow \partial \\ Z = \pi_{m+1}(S^{m+1}) & \xrightarrow[h_{l_0*}]{} & \pi_{m+1}(K_{l_0}(Z, m+1)) & \xrightarrow[i_{l_0*}]{} & \pi_{m+1}(\operatorname{Ker} p_{m+1}) \end{array}$$

PROPOSITION 4.2. *The incidence number* $[\sigma^{m+1}{}_{l_0} : \sigma^m{}_{k_0}]$ *is the image of* $1 \in \pi_{m+1}(S^{m+1})$ *under the map indicated in diagram* (II).

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Proof. The proposition follows immediately from the definition of $[f]$, $d_{m+1}[f]$, and the incidence number.

Now, because of Hurewicz's theorem, it is also clear that diagram (II) may be interpreted as a sequence of homomorphisms of homology groups:

$$\text{(II}'\text{)}\quad \begin{array}{ccccc} Z = H_m(S^m) & \xleftarrow{h_{k0*}^{-1}} & H_m(K_{k_0}(Z, m)) & \xleftarrow{\tilde{p}_{k0*}} & H_m(\text{Ker } p_m) \\ & & & & \uparrow \partial \\ Z = H_{m+1}(S^{m+1}) & \xrightarrow[h_{l0*}]{} & H_{m+1}(K_{l_0}(Z, m+1)) & \xrightarrow[\tilde{i}_{l0*}]{} & H_{m+1}(\text{Ker } p_{m+1}). \end{array}$$

Clearly, Proposition 4.2 remains valid; but now $1 \in H_{m+1}(S^{m+1})$. This suggests that cohomology should give us the incidence number $[\sigma^{m+1}{}_{l_0} : \sigma^m{}_{k0}]$ as homology and homotopy do. This is indeed true. Since the fibre Ker τ_m is equal to Ker p_m and is therefore $(m-1)$-connected, and because the base Ker p_{m+1} is certainly simply connected, we may apply (**2**, p. 622, 5.c). Thus diagram (I) leads to diagram (III):

$$\text{(III)}\quad \begin{array}{ccccc} H^m(S^m) & \xrightarrow{(h_{k_0}{}^{-1})^*} & H^m(K_{k_0}(Z, m)) & \xrightarrow{\tilde{p}_{k0}{}^*} & H^m(\text{Ker } p_m) \\ & & & & \downarrow \tau = h^{*-1} \circ J \\ H^{m+1}(S^{m+1}) & \xleftarrow[h_{l0*}]{} & H^{m+1}(K(Z, m+1)) & \xleftarrow[\tilde{\tilde{i}}_{l0*}]{} & H^{m+1}(\text{Ker } p_{m+1}) \end{array}$$

τ is the cohomological transgression. The following proposition is dual to Proposition 3.2.

PROPOSITION 4.3. *The incidence number* $[\sigma^{m+1}{}_{l_0} : \sigma^m{}_{k0}]$ *is the image of* $1 \in H^m(S^m)$ *under the homomorphism indicated in diagram* III.

Proof. The proposition is a consequence of the invariance of the Kronecker index and the explicit homotopical representation of the homomorphisms occurring in diagram III; see (**2**, p. 622, 5.c).

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