23

Summary, conclusions and outlook

23.1 General

Relativistic quantum field theory has been very successful in describing strong, electromagnetic and weak interactions, in the region of small couplings by perturbation theory, within the framework of the standard model.

However, the region of strong coupling, like the hadronic spectrum and various scattering phenomena of hadrons within QCD, is still largely unsolved.

A large variety of methods have been used to address this question, including lattice gauge simulations, light-cone quantization, low energy effective Lagrangians like the Skyrme model and chiral Lagrangians, large N approximation, techniques of conformal invariance, the integrable model approach, supersymmetric models, string theory approach, QCD sum rules, etc. In spite of this major effort the gap between the phenomenology and the basic theory has only been partially bridged, and the problem is still open.

The goals of this book are to provide a detailed description of the tool box of non-perturbative techniques, to apply them on simplified systems, mainly of gauge dynamics in two dimensions, and to examine the lessons one can learn from those systems about four-dimensional QCD and hadron physics.

The study of two-dimensional problems to improve the understanding of four-dimensional physical systems was found to be fruitful. This follows two directions, one is the utilization of non-perturbative methods on simpler setups and the second is extracting the physical behavior of hadrons in one space dimension.

Obviously, physics in two dimensions is simpler than that of the real world since the underlying manifold is simpler and since the number of degrees of freedom of each field is smaller. There are some additional simplifying features in two-dimensional physics. In one space dimension there is no rotation symmetry and no angular momentum. The light-cone is disconnected and is composed of left moving and right moving branches. Therefore, massless particles are either on one branch or the other. These two properties are the basic building blocks of the idea of transmutation between systems of different statistics. Also, the ultra-violet behavior is more convergent in two dimensions, making for instance QCD_2 a superconvergent theory.

In this summary chapter we go over several notions, concepts and methods with emphasis on the comparison between the two- and four-dimensional worlds and what one can deduce about the latter from the former. In particular we deal

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with conformal invariance, integrability, bosonization, solitons and topological charges, confinement versus screening and finally the hadronic spectrum and scattering.

23.2 Conformal invariance

From the outset there is a very dramatic difference between conformal invariance in two and four dimensions. The former is characterized by an infinite-dimensional algebra, the Virasoro algebra, whereas the latter is associated with the finite-dimensional algebra of $SO(4,2)$. This basic difference stems from the fact that whereas the conformal transformations in four dimensions are global, in two dimensions the parameters of conformal transformations are holomorphic functions (or anti-holomorphic), see Section 17.5 versus 2.1. Nevertheless there are several features of conformal invariance which are common to the two cases. We will now compare various aspects of conformal invariance in two and four dimensions:

- The notion of a primary field and correspondingly a highest weight state is used both in two-dimensional conformal field theories as well as for the four-dimensional collinear algebra. It is expressed in the former as (17.38),

$$L_0[\phi(0)|0>] = h[\phi(0)|0>] \qquad L_n[\phi(0)|0>] = 0, \qquad n > 0 \tag{23.1}$$

and for the latter,

$$L_0[\Phi(0)|0>] = j[\Phi(0)|0>] \qquad L_-[\Phi(0)|0>] = 0. \tag{23.2}$$

The difference is of course the infinite set of annihilation operators L_n versus the single annihilation operator L_- in four dimensions.

- The COPE, the conformal operator product expansion has a compact form in two dimensional CFT (Section 2.12)

$$\mathcal{O}_i(z,\bar{z})\mathcal{O}_j(w,\bar{w}) \sim \sum_k C_{ijk}(z-w)^{h_k-h_i-h_j}(\bar{z}-\bar{w})^{\bar{h}_k-\bar{h}_i-\bar{h}_j}\mathcal{O}_k(w,\bar{w}), \tag{23.3}$$

where C_{ijk} are the *product coefficients*, while in four dimensions it reads,

$$A(x)B(0) = \sum_{n=0}^{\infty} C_n \left(\frac{1}{x^2}\right)^{1/2(t_A+t_B-t_n)} \frac{x_-^{n+s_1+s_2-s_A-s_B}}{B(j_A-j_B+j_n, j_B-j_A+j_n)} \times \int_0^1 du u^{(j_A-j_B+j_n-1)}(1-u)^{(j_B-j_A+j_n-1)}\mathcal{O}_n^{j_1,j_2}(ux_-), \tag{23.4}$$

where the definitions of the various quantities are in Chapter 17. Again there is a striking difference between the simple formula in two dimensions and the complicated one in four dimensions.

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- As an example let us compare the OPE of two currents. Recall from the discussion of Chapter 3 the expression in two dimensions reads,

$$J^a(z)J^b(w) = \frac{k\delta^{ab}}{(z-w)^2} + i\frac{f_c^{ab}J^c(w)}{(z-w)} + \text{finite terms}, \tag{23.5}$$

for any non-abelian group, and in particular for the abelian case the second term on the right-hand side is missing. For comparison the OPE of the transverse components of the electromagnetic currents given in Chapter 17 takes the form,

$$J^T(x)J^T(0) \sim$$
$$\sum_{n=0}^{\infty} C_n \left(\tfrac{1}{x^2}\right)^{(6-t_n)/2} (-ix_-)^{n+1} \tfrac{\Gamma(2j_n)}{\Gamma(j_n)\Gamma(j_n)} \int_0^1 \mathrm{d}u[u(1-u)]^{j_n-1} \mathcal{Q}_n^{1,1}(ux_-). \tag{23.6}$$

- The conformal Ward identity associated with the dilatation operator in four dimensions (17.60),

$$\sum_i^N (l_\phi + \gamma(g^*) + x_i\partial_i) <T\phi(x_1)...\phi(x_N)>= 0, \tag{23.7}$$

where l_ϕ is the canonical dimension and $\gamma(g^*)$ is the anomalous dimension, seems very similar to the one in two dimensions,

$$\sum_i (z_i\partial_i + h_i) <0|\phi_1(z_1,\bar{z}_1)...\phi_n(z_n,\bar{z}_n)|0>= 0. \tag{23.8}$$

In both cases one has to determine the full quantum conformal dimensions of the various operators. However, as was shown in Section 2.7, in certain CFT models, like the unitary minimal models, there are powerful tools based on unitarity which enable us to determine exactly the dimensions h_i of all the primary operators and hence all the operators of the model. On the other hand, it is a non-trivial task to determine the anomalous dimensions in other models in two dimensions, and of course four-dimensional operators. In certain supersymmetric theories there are operators whose dimension is protected, but generically one has to use perturbative calculations to determine the anomalous dimensions of gauge theories to a given order in the coupling constant.

Using the Ward identity one can extract the form of the two-point function of operators of spin s in four dimensions. It is given by,

$$<\phi(x_1)\phi(x_2)>= N_2(g^*)(\mu^*)^{-2\gamma(g^*)} \left[\frac{1}{(x_1-x_2)^2}\right]^{l_\phi+\gamma(g^*)} \left(\frac{(x_1-x_2)_+}{(x_1-x_2)_-}\right)^s. \tag{23.9}$$

The corresponding two-point function in two dimensions, which depends only on the conformal dimension of the operator h, reads,

$$G_2(z_1,\bar{z}_1,z_2,\bar{z}_2) \equiv <0|\phi_1(z_1,\bar{z}_1)\phi_1(z_2,\bar{z}_2)|0> = \frac{c_2}{(z_1-z_2)^{2h_1}(\bar{z}_1-\bar{z}_2)^{2\bar{h}_1}}. \tag{23.10}$$

- As for higher point functions, we have seen in Section 2.9 that one can use the local Ward identities together with Virasoro null vectors to write down partial differential equations. The result for a four-point function (2.63) was later used to determine the four-point function of the Ising model (2.94). Two dimensional conformal field theories are further invariant under affine Lie algebra transformations, and as we have shown in Section 3.6 those can be combined with null vectors to derive the so-called Knizhnik–Zamolodchikov equations (3.69), which were later used to solve for the four-point function of the $SU(N)$ WZW model in Section 4.4. These types of differential equations that fully determine correlation functions are obviously absent in four-dimensional interacting conformal field theories.

23.3 Integrability

Integrability was discussed in Chapter 5 in the context of two-dimensional models and in Chapter 18 in four-dimensional gauge theories. For systems with a finite number of degrees of freedom, like spin chain models, there is a finite number of conserved charges, equal of course to the number of degrees of freedom. For integrable field theories there is an infinite countable number of conserved charges. Furthermore, the scattering processes of those models always involve a conservation of the number of particles.

In two dimensions we have encountered continuous integrable models like the sine-Gordon model as well as discretized ones like the XXX spin chain model. The integrable sectors of gauge dynamical systems discussed in Chapter 18 are based on identifying an exact map between certain properties of the systems and a spin chain structure. In two dimensions the spin chain models follow from a discretization of the space coordinate, by placing a spin variable on each site that can take several values and imposing periodicity. In the four-dimensional $\mathcal{N} = 4$ super YM theory discussed in Section 18.1 the spin chain corresponds to a trace of field operators and in the process of high-energy scattering of Section 18.2 it is a "chain" of reggeized gluons exchanged in the t-channel of a scattering process. A summary of the comparison among the basic two-dimensional spin chain, the "spin chains" associated with the planar $\mathcal{N} = 4$ SYM, and the high-energy scattering in QCD, is given in Table 23.1. A powerful method to solve all these spin chain models is the use of the algebraic Bethe ansatz. This was discussed in detail for the the $XXX_{1/2}$ model in Section 5.14. The solutions of the energy eigenvalues needed for the high-energy scattering process was based on generalizing this method to the case of spin s Heisenberg model (see Section 18.2) and for the $\mathcal{N} = 4$ to the case of an $SO(6)$ invariance.

There is one conceptual difference between the spin chains of the two-dimensional models and those associated with the $\mathcal{N} = 4$ SYM in four dimensions. In the former the models are non conformal, involving a scale, and hence

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Table 23.1. *Spin chain structure of the two-dimensional model and the four-dimensional gauge systems of $\mathcal{N} = 4$ SYM and of high-energy behavior of scattering amplitudes in QCD.*

Spin chain	Planar $\mathcal{N} = 4$ SYM	High energy scattering in QCD
Cyclic spin chain	Single trace operator	Reggeized guons in t-channel
Spin at a site	Field operator	$SL(2)$ spin
Number of sites	Number of operators	Number of gluons
Hamiltonian	Anomalous dilatation operator	H_{BFKL}
Energy eigenvalue	Anomalous dimension $g^{-2}\delta\mathcal{D}$	$\sim \frac{1}{\lambda}\frac{\log \mathcal{A}}{\log s}$
Evolution time	Global time	The total rapidity $\log s$
Zero momentum $U = 1$	Cyclicity constraint	

also with particles and an S-matrix. The integrable sectors of four-dimensional gauge theories, however, are conformal invariant.

The study of integrable models in two dimensions is quite mature, whereas the application of integrability to four-dimensional systems is at an infant stage. The concepts of multi-local charges described in Section 5.11 and of quantum groups discussed in Section 5.13 have been applied only slightly to gauge dynamical systems in four dimensions.

23.4 Bosonization

Bosonization is the formulation of fermionic systems in terms of bosonic variables and fermionization is just the opposite process. The study of bosonized physical systems offers several advantages:

(1) It is usually easier to deal with commuting fields rather than anti-commuting ones.
(2) In certain examples, like the Thirring model, the fermionic strong coupling regime turns into the weak coupling one in its bosonic version, the sine-Gordon model (see Section 6.2).
(3) The non-abelian bosonization, especially in the product scheme (see Section 6.3.4), offers a separation between colored and flavored degrees of freedom, which is very convenient for analyzing low lying spectrum.
(4) Baryons composed of N_{C} quarks are a many-body problem in the fermion language, while simple solitons are in the boson language.
(5) One loop fermionic computations involving the currents turn into tree level consideration in the bosonized version. The best-known example of the latter are the chiral (or axial) anomalies (see Section 9.1).

In four dimensions, spin is obviously non-trivial and one cannot constitute generically a bosonization equivalence. However, in certain circumstances a four-dimensional system can be described approximately by fields that depend only on the time and on one space direction. In those cases one can apply the bosonization technique. Examples of such scenarios are monopole induced proton decay, and fractional charges induced on monopoles by light fermions. In these cases the relevant degrees of freedom are in an s-wave and hence taken to depend only on the time and the radial direction. This enables one to use the corresponding bosonized field. There is a slight difference with two dimensions, as the radial coordinate goes from zero to infinity, so "half" a line. Appropriate boundary conditions enable us to use a reflection, so as to extend to a full line.

23.5 Topological field configurations

- The topological charges in any dimensions are conserved regardless of the equations of motion of the corresponding systems. In two dimensions it is very easy to write down a current which is conserved without the use of the equations of motion. This is referred to as a topological conservation. Consider a scalar field ϕ or its non-abelian analog ϕ^a that transforms in the adjoint representation of a group, then the following currents are abelian and non-abelian conserved currents,

$$J_\mu = \epsilon_{\mu\nu}\partial^\nu \phi \qquad J^a_\mu = \epsilon_{\mu\nu}\partial^\nu \phi^a . \tag{23.11}$$

Recall that for a system that admits, for instance an abelian case, also a current $J_\mu = \partial_\mu \phi$ that is conserved upon the use of the equations of motion, one can then replace the two currents with left and right conserved currents $J_\pm = \partial_\pm \phi$ or $J = \partial\phi$ and $\bar{J} = \bar{\partial}\phi$, as was discussed in Chapter 1. The charge associated with the topological conserved current is given by,

$$Q_{\text{top}} = \int \mathrm{d}x \phi' = [\phi(t, +\infty) - \phi(t, -\infty)] \equiv \phi_+ - \phi_-, \tag{23.12}$$

where the space dimension is taken to be $\mathcal{R}$. For a compactified space dimension, namely an S^1 this charge vanishes, except for cases where the field is actually an angle variable, in which case the charge is 2π. The latter appears in the case of $U(1)$ gauge theory in two dimensions, where there is a winding number.

- Obviously one cannot have such topologically conserved currents and charges in four dimensions. However, for theories that are invariant under a non-abelian group, one can construct also in four dimensions a topological current and charge, as for the cases of Skyrmions, magnetic monopoles and instantons. For the Skyrmions the topological current is given by,

$$J^\mu_{\text{skyre}} = \frac{i\epsilon^{\mu\nu\rho\sigma}}{24\pi^2}\, \mathrm{Tr}\,[L_\nu L_\rho L_\sigma]. \tag{23.13}$$

Table 23.2. *Topological classical field configurations in two and four dimensions*

classical field	dim.	map	topological current
soliton	two	[1]	$\epsilon^{\mu\nu}\partial_\nu\phi$
baryon	two	$S^1 \to S^1$	$\epsilon^{\mu\nu}Tr[g^{-1}\partial_\nu g]$
Skyrmion	four	$S^3 \to S^3$	$\frac{i\epsilon^{\mu\nu\rho\sigma}}{24\pi^2}Tr[L_\nu L_\rho L_\sigma]$
monopole	four	$S^2_{space} \to S^2_{G/H}$	$\frac{1}{8\pi}\epsilon_{\mu\nu\rho\sigma}\epsilon^{abc}\partial^\nu\hat{\Phi}^a\partial^\rho\hat{\Phi}^b\partial^\sigma\hat{\Phi}^c$
instanton	four	$S^3_s \to S^3_g$	$\frac{i\epsilon^{\mu\nu\rho\sigma}}{16\pi^2}Tr[A_\nu\partial_\rho A_\sigma + \frac{2}{3}A_\nu A_\rho A_\sigma]$

- The topological charges, for compact spaces, are the winding numbers of the corresponding topological configurations. For a compact one space dimension, we have the map of $S^1 \to S^1$ related to the homotopy group $\pi_1(S^1)$. In two space dimensions, the windings are associated with the map $S_2 \to S_2^{G/H}$, as for the magnetic monopoles. For three space dimensions, it is $S^3 \to S^3$ for the Skyrmions at $N_f = 2$, and the non-abelian instantons for the gauge group $SU(2)$. The topological data of the various models is summarized in Table 23.2.
- According to Derrick's theorem (see Section 5.3), for a theory of a scalar field with an ordinary kinetic term with two derivatives, and any local potential at $D \geq 2$, the only non singular time-independent solutions of finite energy are the vacua. However, as we have seen in Chapters 20, 21 and 22, there are solitons in the form of Skyrmions and monopoles and instantons. Those configurations bypass Derrick's theorem by introducing higher derivative terms or including non-abelian gauge fields.
- As was emphasised in Chapter 20, the extraction of the baryonic properties in the Skyrme model is very similar to the one for the baryons in the bosonized theory in two dimensions. Unlike the latter which is exact in the strong coupling limit, one cannot derive the former starting from the underlying theory. Another major difference between the two models is of course the existence of angular momentum only in the four-dimensional case.
- A non-trivial task associated with topological configurations is the construction of configurations that carry multipole topological charge, for instance a multi-baryon state both of the bosonized QCD_2 as well as of the Skyrme model, a multi-monopole solution and a multi-instanton solution. For the two-dimensional baryons (as discussed in Section 13.6) the construction is a straightforward generalization of the configuration of baryon number one. For the multi-monopole solutions we presented Nahm's construction, and for the multi-instantons the ADHM construction. These constructions, which are in fact related, are much more complicated than that for the two-dimensional muti-baryons.

[1] Depends on the type of the soliton. See Section 5.3.

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- A very important phenomenon that occurs in both two and four dimensions is the strong-weak duality, and the duality between a soliton and an elementary field. In two dimensions we have encountered this duality in the relation between the Thirring model and the sine-Gordon model, where the coupling of the latter β is related to that of the former g as (6.27),

$$\frac{\beta^2}{4\pi} = \frac{1}{1+\frac{g}{\pi}}. \tag{23.14}$$

This also relates the elementary fermion field of the Thirring model with the soliton of the sine-Gordon model. In particular for $g = 0$ corresponding to $\beta^2 = 4\pi$, the Thirring model describes a free Dirac fermion, while the soliton of the corresponding sine-Gordon theory is the same fermion in its bosonization disguise. An analog in four dimensions is the Olive–Montonen duality discussed in Section 21.8, which relates the electric charge e with the magnetic one $e_{\mathrm{M}} = \frac{4\pi}{e}$, where the former is carried by the elementary states $W^{\pm}$ and the latter by the magnetic monopoles.

23.6 Confinement versus screening

Naive intuition tells us that dynamical quarks in the fundamental representation can screen external sources in the fundamental representation, dynamical adjoint quarks can screen adjoint sources, but that dynamical adjoint cannot screen fundamentals. The picture that emerged from our two dimensional calculations (Chapter 14) showed that this was not the case. We found that massless adjoint quarks could screen an external source in the fundamental representation. Moreover we have seen that any massless dynamical field will necessarily be in the screening phase. The argument for that was that in all cases we have considered we have found that the string tension is proportional to the mass of the dynamical quarks,

$$\sigma \sim mg, \tag{23.15}$$

where m is the mass of the quark and g is the gauge coupling, and hence for the massless case it vanishes. This was shown in Chapter 14 based on performing a chiral rotation that enabled us to eliminate the external sources and computing the string tension as the difference between the Hamiltonian of the system with the external sources and the one without them namely (14.12),

$$\sigma = <H> - <H_0>\ . \tag{23.16}$$

It seems as though the situation in two dimensions is very different from that in four dimensions. From the onset there is a dramatic difference between two and four dimensions relating to the concept of confining theory. In two dimensions both the Coulomb abelian potential and the non-abelian one are linear with the separation distance L, whereas obviously in four dimensions the Coulomb potential between two particles behaves as $1/L$. The confining potential is linear with L in both two and four dimensions. However, that does not explain the difference between two and four dimensions, it merely means that in two dimensions the coulomb and confining potentials behave in the same manner.[2] The determination of the string tension in two dimensions cannot be repeated in four dimensions. The reason is that in the latter case the anomaly is not linear in the gauge field and thus one cannot use the chiral rotation to eliminate the external quark anti-quark pair. That does not imply that the situation in four dimensions differs from the two-dimensional one, it just means that one has to use different methods to compute the string tension in four dimensions.

What are the four-dimensional systems that might resemble the two-dimensional case of dynamical adjoint matter and external fundamental quarks? A system with external quarks in the fundamental representation in the context of pure YM theory seems a possible analog since the dynamical fields, the gluons, are in the adjoint representation, though they are vector fields and not fermions. An alternative is the $\mathcal{N} = 1$ SYM where in addition to the gluons there are also gluinos which are Majorana fermions in the adjoint representation. Both these cases should correspond to the massless adjoint case in two dimensions. The latter admits a screening behavior where as the four-dimensional models seem to be in the confining phase. This statement is supported by several different types of calculations in particular for the non supersymmetric case this behavior is found in lattice simulations.

At this point we cannot provide a satisfactory intuitive explanation why the behavior in two and four dimensions is so different. There is also no simple picture of how the massless adjoint dynamical quarks in two dimensions are able to screen external charges in the fundamental representation.

It is worth mentioning that there is ample evidence that four-dimensional hadronic physics is well described by a string theory. This is based for instance on realizing that mesons and baryons in nature admit Regge trajectory behavior which is an indication of a stringy nature. Any string theory is by definition a two-dimensional theory and hence a very basic relation between four-dimensional hadron physics and two-dimensional physics.

In addition to the ordinary string tension which relates to the potential between a quark and anti-quark in the fundamental representation, one defines the k string that connects a set of k quarks with a set of k anti-quarks. This object has been examined in four-dimensional YM as well as four-dimensional $\mathcal{N} = 1$ SYM. These two cases seem to be the analog of the two-dimensional QCD theory with adjoint quarks and with external quarks in a representation that is characterized by k boxes in the Young tableau description. In Chapter 14 we have derived an expression for the string tension as a function of the representation of the external and dynamical quarks and in particular for dynamical adjoint

[2] Note that the linear potential in two dimensions is already there at lowest order, while obviously in four dimensions it is a highly non-perturbative effect.

fermions and external quarks in the k representation. If there is any correspondence between the four-dimensional adjoint matter field and the two-dimensional adjoint quarks it must be with massive adjoint quarks since for the massless case, as was mentioned above, the two-dimensional string tension vanishes whereas the four-dimensional one does not. Thus one may consider a correspondence for a softly broken $\mathcal{N}=1$ case where the gluinos are massive.

In two dimensions for the pure YM case we found that the string tension behaves like $\sigma \sim g^2 k_{\rm ext}^2$ whereas a Wilson line calculation yields $\sigma \sim g^2 C_2(R)$ where $C_2(R)$ is the second Casimir operator in the R representation of the external quarks. For the QCD$_2$ case of general k external charges and adjoint dynamical quarks, one can derive from (14.49) that,

$$\sigma_k^{2d} \sim \sin^2\left(\frac{\pi k}{N_{\rm c}}\right), \tag{23.17}$$

whereas in four dimensions it is believed that for general k, the string tension either follows a Casimir law or a sinusoidal rule as follows,

$$\sigma_k^{\rm cas} \sim \frac{k(N_{\rm c}-k)}{N_{\rm c}} \qquad \sigma_k^{\rm sin} \sim \sin\left(\frac{\pi k}{N_{\rm c}}\right). \tag{23.18}$$

It is an open problem which of these holds.

As expected all these expressions are invariant under $k \to N-k$ which corresponds for antisymmetric representations to replacing a quark with an antiquark.

23.7 Hadronic phenomenology of two dimensions versus four dimensions

QCD_2 was addressed first in the fermionic formulation. In his seminal work 't Hooft deduced the mesonic spectrum in the large $N_{\rm C}$ limit as is described in Chapter 10. We further presented three additional approaches to the hadronic spectra in two dimensions, the currentization method for massless quarks for the entire plane of $N_{\rm C}$ and $N_{\rm f}$, the DLCQ approach to extract the mesonic spectrum for the case of fundamental as well as adjoint quarks and finally the bosonized formulation in the strong coupling limit to determine the baryonic spectrum. As for the four-dimensional hadronic spectrum we described the use of the large $N_{\rm C}$ planar limit and the analysis of the baryonic world using the Skyrme model. It is worth mentioning again that whereas in the four-dimensional case the Skyrme approach is only an approximated model derived by an "educational guess", in two dimensions the action in the strong coupling regime is exact.

23.7.1 Mesons

As was just mentioned the two-dimensional mesonic spectrum was extracted using the large $N_{\rm C}$ approximation in the fermionic formulation for $N_{\rm f}=1$

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('t Hooft model), the currentization for massless quarks and the DLCQ approach for both cases of quarks in the fundamental and the adjoint representation. For the particular region of $N_{\mathrm{C}} >> N_{\mathrm{f}}$ and $m = 0$ the fermionic large N_{c} and the currentization treatments yielded identical results. In fact this result is achieved also using the DLCQ method for adjoint fermions upon a truncation to a single parton and replacing g^2 with $2g^2$ (see (12.42)). For massive fundamental quarks the DLCQ results match very nicely those of lattice simulations and the large N_c calculations as can be seen from Figs (12.1) and (12.2).

In all these methods the corresponding equations do not admit exact analytic solutions for the whole range of parameters and thus one has to resort to numerical solutions. However, in certain domains one can determine the analytic behavior of the wavefunctions and masses.

The spectrum of mesons in two dimensions is characterized by the dependence of the meson masses M_{mes} on the gauge coupling g, the number of colors N_{c}, the number of flavors N_{f}, the quark mass m_{q} and the excitation number n. In four-dimensional QCD the meson spectra depend on the same parameters apart from the fact that Λ_{QCD}, the QCD scale, is replacing the two-dimensional gauge coupling and of course some additional quantum numbers. The following lines summarize the properties of the spectrum

- The highly excited states $n \gg 1$, are characterized by,
$$M_{\mathrm{mes}}^2 \sim \pi g^2 N_{\mathrm{c}} n. \tag{23.19}$$
This seems to fit the behaviors of mesons in nature. This behavior is referred to as a Regge trajectory and it follows easily from a bosonic string model of the mesons. Following this analogy, the role of the string tension in a two-dimensional model is played by $g^2 N_{\mathrm{c}}$. This seems to be in contradiction with the statement that the string tension is proportional to $m_q g$, as seen in the discussion of screening versus confinement.

  It is very difficult to derive the Regge trajectory behavior from direct calculations in four-dimensional QCD.
- The opposite limit of low-lying states and in particular the ground state can be deduced in the limit of large quark masses $m_q \gg g$ and small quark masses $g \gg m_q$. For the ground state in the former limit we find,
$$M_{\mathrm{mes}}^0 \cong m_{q_1} + m_{q_2}, \tag{23.20}$$
where m_{q_i} are the masses of the quark and anti-quark. In the opposite limit of $m_q \ll g$
$$(M_{\mathrm{mes}}^0)^2 \cong \frac{\pi}{3}\sqrt{\frac{g^2 N_{\mathrm{c}}}{\pi}}(m_1 + m_2). \tag{23.21}$$
For the special case of massless quarks we find a massless meson. This is very reminiscent of the four-dimensional picture for the massless pions. For small

masses this is similar to the pseudo-Goldstone boson relation where,

$$m_\pi^2 \sim \frac{<\bar{\psi}\psi>}{f_\pi^2}(m_1 + m_2). \tag{23.22}$$

Note that in two dimensions the massless mesons decouple.

- The 't Hooft model cannot be used to explore the dependence on N_f the number of flavors. This can be done from the 't Hooft-like equations derived in Chapter 11. It was found that for the first massive state there is a linear dependence of the meson mass squared on N_f

$$M_{\mathrm{mes}}^2 \sim N_\mathrm{f}. \tag{23.23}$$

We are not aware of a similar behavior of the mesons in four dimensions.

- The 't Hooft model (Chapter 10) provides the solution of the meson spectrum in the planar limit in two dimensions. The planar, namely large N_c limit, in four dimensions is too complicated to be similarly solved. As we have seen in Chapter 19 one can extract the scaling in N_c dependence of certain hadronic properties like the mass the size and scattering amplitude but the full determination of the hadronic spectrum and scattering is still an unresolved mystery.
- Tremendous progress has been made in the understanding of the supersymmetric theory of $\mathcal{N} = 4$ partly by demonstrating that certain sectors of it can be described by integrable spin chain models (Section 18.1).
- As was demonstrated in Chapter 12 the DLCQ method has been found very effective to address the spectrum of mesons of two-dimensional QCD. This raises the question of whether one can use the DLCQ method to handle the spectrum of four-dimensional QCD. This task is clearly much more difficult. On route to the extraction of the hadronic spectrum of QCD_4 an easier system has been analyzed. It is that of the collinear QCD (see Chapter 17) where in the Hamiltonian of the system one drops off all interaction terms that depend on the transverse momenta. In this effective two-dimensional setup the transverse degrees of freedom of the gluon are retained in the form of two scalar fields. This system which was not described in the book has been solved in [14] where a complete bound and continuum spectrum was extracted as well as the Fock space wavefunctions.

23.7.2 Baryons

In Chapter 13 we have described the spectrum of baryons in multiflavor two-dimensional QCD in the strong coupling limit $\frac{m_q}{e_c} \to 0$. The four-dimensional baryonic spectrum was discussed in the large N_c limit in Chapter 19 and using the Skyrme model approach in Chapter 20. We would like now to compare these spectra and to investigate the possibility of predicting four-dimensional baryonic properties from the simpler two-dimensional model. In the former case the mass is a function of the QCD scale Λ_{QCD}, the number of colors N_c and the number

Table 23.3. *Scaling of baryon masses with N_c in two and four dimensions*

	two dimensions	four dimensions
Classical baryon mass	N_c	N_c
Quantum correction	N_c^0	N_c^{-1}

of flavors N_f and in the latter it is a function of e_c, N_c and N_f. Thus it seems that the dimensionful gauge coupling in two dimensions is the analog of Λ_{QCD} in four dimensions.

- In two dimensions the mass of the baryon was found to be,

$$E = 4m\sqrt{\frac{2N_c}{\pi}} + m\sqrt{2}\sqrt{\left(\frac{\pi}{N_c}\right)^3}\left[C_2 - N_c^2\frac{(N_f-1)}{2N_f}\right], \tag{23.24}$$

where the classical mass m is given by

$$m = \left[N_c c m_q \left(\frac{e_c\sqrt{N_f}}{\sqrt{2\pi}}\right)^{\Delta_C}\right]^{\frac{1}{1+\Delta_C}}, \tag{23.25}$$

with $\Delta_c = \frac{N_c^2-1}{N_c(N_c+N_f)}$. Due to the fact that in two dimensions there is no spin, the structure of the spectrum with respect to the flavor group is obviously different in two and four dimensions. For instance the lowest allowed state for $N_c = N_f = 3$ is in two dimensions the totally symmetric representation **10**, whereas it is the mixed representation **8** in four dimensions.
- Let us discuss now the scaling with N_c in the large N_c limit. The classical term behaves like N_c, while the quantum correction like 1. This classical result is in accordance with four dimensions, derived when the large N expansion is applied to the baryonic system (see Chapter 19), that the baryon mass is linear in N_c and with the Skyrmion result (see Chapter 20). However, whereas in two dimensions the quantum correction behaves like N_c^0, namely suppressed by a factor of $\frac{1}{N_c}$ compared to the classical term, in four dimensions it behaves like $\frac{1}{N_c}$ namely a suppression of $\frac{1}{N_c^2}$. This is summarized in Table 23.3.
- In terms of the dependence on the number of flavors, it is interesting to note that both in two dimensions and in four dimensions, the contribution to the mass due to the quantum fluctuations has a term proportional to the second Casimir operator associated with the representation of the baryonic state under the $SU(N_f)$ flavor group (compare (23.24) with (20.68)).
- Another property of the baryonic spectrum that can be compared between the two- and four-dimensional cases is the flavor content of the various states. In Chapter 13 we have computed the $\bar{u}u, \bar{d}d$ and $\bar{s}s$ content for the Δ^+ and Δ^{++} states. Recall that in the two-dimensional model for $N_c = N_f = 3$ we do not have a state in the **8** representation but only in the **10** so strictly speaking there

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Table 23.4. *Flavor content of two-dimensional and four-dimensional baryons*

	two dimensions state	value	four dimensions state	value
$\langle \bar{u}u \rangle$	Δ^+	$\frac{1}{2}$	p	$\frac{2}{5}$
$\langle \bar{d}d \rangle$	Δ^+	$\frac{1}{3}$	p	$\frac{11}{30}$
$\langle \bar{s}s \rangle$	Δ^+	$\frac{1}{6}$	p	$\frac{7}{30}$
$\langle \bar{s}s \rangle$	Δ^{++}	$\frac{1}{6}$	Δ	$\frac{7}{24}$
$\langle \bar{s}s \rangle$			Ω^-	$\frac{5}{24}$

is no exact analog of the proton. Instead we take the charge $= +1\Delta^+$ as the two-dimensional analog of the proton. In the Skyrme model one can compute in a similar manner the flavor content of the four-dimensional baryons. The two- and four-dimensional states compare as is summarized in Table 23.4.

23.8 Outlook

We can imagine future developments associated with the topics covered in the book in three different directions: Further progress in the application of the methods discussed in the book to unravel the mysteries of gauge dynamics in nature; applications of the methods in other domains of physics not related to four-dimensional gauge theories; and improving our understanding of the strong interaction and hadron physics due to other non-perturbative techniques that are not discussed in the book. Let us now briefly fantasize on hypothetical developments in those three avenues.

23.8.1 Further progress in the application of the methods discussed in the book

- A lesson that follows from the book is that the exploration of physical systems on one space dimension is both simpler to handle and sheds light on the real world so there are plenty of other unresolved questions that could be explored first in two dimensions. This includes exploration of the full standard model and the physics beyond the standard model including supersymmetry and its dynamical breaking, large extra dimensions, compositeness etc.
- There has been tremendous development in recent years in applying methods of integrable models and in particular of spin chains, like the thermal Bethe ansatz, to $\mathcal{N} = 4$ SYM theory, namely, in the context of supersymmetric conformal gauge theory. We have no doubt that there will be further development in computing the anomalous dimensions of gauge invariant operators and correlators.

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- Moreover, one can identify in a similar manner to $\mathcal{N} = 4$ SYM theory spin chain structures in gauge theories which are confining and with less or even no supersymmetries. In that case the spin chain Hamiltonian would not correspond to the dilatation operator but rather be associated with the excitation energies of hadrons.
- It is plausible that the full role of magnetic monopoles and of instantons has not yet been revealed. They have already had several reincarnations and there may be more. For instance there was recently a proposal to describe baryons as instantons which are solitons of a five-dimensional flavor gauge theory in curved five dimensions.

23.8.2 Applications to other domains

- A very important application of two-dimensional conformal symmetry has been to superstring theories. A great part of the developments in superstring theories is attributed to the infinite-dimensional conformal symmetry algebra. In fact it went in both directions and certain progress in understanding the structure of conformal invariance has emerged from the research of string theories. A similar symbiotic evolution took place with regard to the affine Lie algebras.
- String theories and in particular the string theory on $AdS_5 \times S^5$ have recently been analyzed using the tools of integrable models like mapping to spin chains, using the Bethe ansatz equations, identifying a set of infinitely many conserved charges and using structure of Yangian symmetry.
- Spin chain models have been suggested to describe systems of "real" spins in condensed matter physics. As was discussed in this book the application of the corresponding tools to field theory systems has been quite fruitful. The opposite direction will presumably also take place and the use of properties of integrability that were understood in field theories will shed new light on certain condensed matter systems.
- The application of conformal invariance to condensed matter systems at criticality has a long history. There has been recently an intensive effort to further develop the understanding of systems like various superconductors, the fractional Hall effect and other systems using modern conformal symmetry techniques.

23.8.3 Developments in gauge dynamics due to other methods

- An extremely important framework for analyzing gauge theories has been supersymmetry. Regardless if it is realized in nature or not, it is evident that there are more tools to handle supersymmetric gauge theories and hence they are much better understood than non supersymmetric ones. One can gain novel insight about non supersymmetric theories by introducing supersymmetry breaking terms to well understood supersymmetric models. For instance one can start with the Seiberg Witten solution of $\mathcal{N} = 2$ [192] where the structure of vacua is known and extract confinement behavior in $\mathcal{N} = 1$ and non supersymmetric theories.

- A breakthrough in the understanding of gauge theories in the strong coupling regime took place with the discovery by Maldacena of the AdS/CFT holographic duality [158]. The strongly coupled $\mathcal{N} = 4$ in the large N and large 't Hooft parameter λ is mapped into a weakly curved supergravity background. Thousands of research papers that followed develop this map in many different directions and in particular also in relation to the pure YM theory and QCD in four dimensions. There is very little doubt that further exploration of the duality will shed new light on QCD and on hadron physics.
- String theory has been born as a possible theory of hadron physics. It then underwent a phase transition into a candidate for the theory of quantum gravity and even a unifying theory for everything. In recent years, mainly due to the AdS/CFT duality there is a renaissance of the idea that hadrons at low energies should be described as strings. This presumably combined with the duality seems to be a useful tool that will improve our understanding of gauge dynamics.
- The computations of scattering amplitudes in gauge theories has been boosted in recent years due to various developments including the use of techniques based on twistors, on a novel T-duality in the context of the Ads/CFT duality and on a conjectured duality between Wilson lines and scattering amplitudes. One does not need a wild imagination to foresee further progress in the industry of computing scattering amplitudes.

To summarize, non-perturbative methods have always been very important tools in exploring the physical world. We have no doubt that they will continue to be a very essential ingredient in future developments of science in general and physics in particular.