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The Dirichlet Problem for the Slab with Entire Data and a Difference Equation for Harmonic Functions

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Abstract. It is shown that the Dirichlet problem for the slab $(a, b) \times \mathbb{R}^d$ with entire boundary data has an entire solution. The proof is based on a generalized Schwarz reflection principle. Moreover, it is shown that for a given entire harmonic function g, the inhomogeneous difference equation $h(t+1, y) - h(t, y) = g(t, y)$ has an entire harmonic solution h.

1 Introduction

It is well known that the Dirichlet problem for unbounded domains differs in many respects from the case of bounded domains due to the non-uniqueness of solutions. An excellent discussion of the Dirichlet problem for general unbounded domains can be found in [10].

Maybe the simplest example of this kind is the Dirichlet problem for the strip $(a, b) \times \mathbb{R}$ that was considered by Widder in [24]; see also [6]. A discussion of the Dirichlet problem for half-spaces can be found in [9, 22], and for a cylinder in [20].

In this paper we are concerned with the harmonic extendibility of the solution of the Dirichlet problem for entire data on the slab (see [5])

$$S_{a,b} := (a, b) \times \mathbb{R}^d.$$

We say that a function $f: \mathbb{R}^d \to \mathbb{C}$ is entire if there exists an analytic function $F: \mathbb{C}^d \to \mathbb{C}$ such that $F(x) = f(x)$ for all $x \in \mathbb{R}^d$. Thus, an entire function $f: \mathbb{R}^d \to \mathbb{C}$ is real analytic, and it possesses an everywhere convergent power series expansion. It is well known that every harmonic function $h: \mathbb{R}^d \to \mathbb{C}$ is entire.

Our first main result in this paper is the following theorem.

Theorem 1.1 *Let h be a solution of the Dirichlet problem for the slab $S_{a,b}$ for entire data f_0, $f_1: \mathbb{R}^d \to \mathbb{C}$;* i.e., *$h$ is harmonic on $S_{a,b}$ and $\lim_{t \to a} h(t, y) = f_0(y)$ and $\lim_{t \to b} h(t, y) = f_1(y)$. Then h extends to all of $\mathbb{R}^{d+1}$ as a harmonic function.*

A similar result holds for the Dirichlet problem for the ellipsoid: H. S. Shapiro and the first author established in [18] that for each entire data function there exists a solution of the Dirichlet problem that extends to a harmonic function defined on $\mathbb{R}^d$; see also [2] for further extensions. For the case of a cylinder with ellipsoidal base it

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is not yet known whether for any entire data function there exists an entire harmonic solution; see [11,19] for partial results. We refer the reader to a discussion in [7,14,17,21] regarding the question of which domains Ω allow entire extensions for entire data.

From Theorem 1.1 we will derive our second main result. (The related problem of studying entire extensions of solutions to the Cauchy problem is discussed in [16, 23].

Theorem 1.2 *If $g: \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$ is harmonic, then the difference equation*

$$h(t+1, y) - h(t, y) = g(t, y)$$

has a harmonic solution $h: \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$.

Let us recall some notations and definitions. A function $f: \Omega \to \mathbb{C}$ defined on a domain Ω in the Euclidean space $\mathbb{R}^d$ is called *harmonic* if f is twice continuously differentiable and $\Delta f(x) = 0$ for all $x \in \Omega$, where

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_d^2}$$

is the Laplace operator. We also write Δ_x instead of Δ to indicate the variables for differentiation. We say that a function $g: \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$ is even (resp. odd) at t_0 if

$$g(t_0 + t, y) = g(t_0 - t, y),$$

and $g(t_0 + t, y) = -g(t_0 - t, y)$, respectively, for all $t \in \mathbb{R}$ and $y \in \mathbb{R}^d$.

2 The Dirichlet Problem on the Slab with Entire Data

Suppose $h: [a, b] \times \mathbb{R}^d \to \mathbb{C}$ is continuous and harmonic in the open slab $(a, b) \times \mathbb{R}^d$ such that $h(a, y) = h(b, y) = 0$ for all $y \in \mathbb{R}^d$. Then it is a well known consequence of the Schwarz reflection principle that h extends to a harmonic function on $\mathbb{R}^{d+1}$ that is periodic in the variable t with period $2(b-a)$, *i.e.,*

$$h\big(t + 2(b-a), y\big) = h(t, y).$$

In order to obtain a similar result with arbitrary entire boundary data, we need the following extension of the Schwarz reflection principle.

Theorem 2.1 *Suppose that Ω is a domain in $\mathbb{R}^{d+1}$ such that for each $x = (x_1, \ldots, x_{d+1}) \in \Omega$ the vector $\widetilde{x} = (-x_1, x_2, \ldots, x_{d+1}) \in \Omega$, and let $\Omega_+, \Omega_0, \Omega_-$ denote the sets of points $x \in \Omega$ for which x_1 is positive, zero, and negative (respectively). Suppose that $y \mapsto F(y)$ for $y = (x_2, \ldots, x_{d+1}) \in \mathbb{R}^d$ is an entire function, and assume that h is harmonic on Ω_- such that for all $y \in \Omega_0$ we have $h(x) \to F(y)$ as $x \to y \in \Omega_0$. Then h has a harmonic extension to Ω.*

Proof First, we recall that the Cauchy problem for Laplace's equation with entire data posed on a hyperplane has a unique entire solution. As stated in [15, p. 80, Example 11.2], this fact can be proved using the Bony–Schapira theorem or the Cauchy–Kovalevskaya theorem with estimates [15, Thm. 2.1] (*cf.* [12, Ch. IX]). We now provide some details regarding the latter approach. Using the notation in [15, Thm. 2.1], the coefficients of the differential operator and the data are all analytic in a polydisk $\overline{D(0,R)}$ of radius R for every R. Given $0 < t < 1$, [15, Thm. 2.1] states that the solution is holomorphic in a polydisk of radius tR in each coordinate except the last, which has radius δtR, and $\delta > 0$ depends on the size of the modulus of each coefficient in the polydisk $\overline{D(0,R)}$ but not on the size of the data. Since each coefficient is constant, the supremum of the modulus of each coefficient over the polydisk $\overline{D(0,R)}$ is a constant independent of R, and this implies that δ is independent of R. Hence, letting R tend to infinity we obtain the desired result.

Applying this fact, we have in particular that there is a unique entire harmonic function H such that $H(0,y) = F(y)$ and $\frac{\partial}{\partial x} H(0,y) = 0$ for all $y \in \mathbb{R}^d$. Moreover, from the uniqueness part of the Cauchy–Kovalevskaya Theorem, it follows that H is even at $t_0 = 0$, since $H(-t,y)$ solves the same Cauchy problem as $H(t,y)$. Consider the function

$$f(t,y) := h(t,y) - H(t,y)$$

for $(t,y) \in \Omega_-$. Then for each $y \in \mathbb{R}^d$, we have $f(t,y) \to 0$ as $t \to 0$, and by the Schwarz reflection principle (see [3, p. 8]) f extends to a harmonic function $\widetilde{f}$ on Ω by the formula $\widetilde{f}(t,y) = -f(-t,y)$ for all $(t,y) \in \Omega_+$. Then

$$\widetilde{h}(t,y) := \widetilde{f}(t,y) + H(t,y)$$

is a harmonic extension of h from Ω_- to Ω, and for $t > 0$ we have

$$\widetilde{h}(t,y) = \widetilde{f}(t,y) + H(t,y) = -f(-t,y) + H(t,y) = -h(-t,y) + 2H(t,y). \tag{2.1}$$ ∎

Our first main result stated in the introduction is a consequence of the next theorem.

Theorem 2.2 *Assume that $h \in C([a,b]\times\mathbb{R}^d)$ is harmonic in the slab $(a,b)\times\mathbb{R}^d$ such that $y \mapsto h(a,y)$ and $y \mapsto h(b,y)$ are entire. Then there exists a harmonic extension $\widetilde{h}: \mathbb{R}^{d+1} \to \mathbb{C}$.*

Proof The following provides an inductive step.

Claim. There is an extension $\widetilde{h} \in C([a, 2b-a] \times \mathbb{R}^d)$ of h which is harmonic in $(a, 2b-a) \times \mathbb{R}^d$ such that $y \mapsto \widetilde{h}(2b-a, y)$ is entire.

Using the claim and induction, one obtains a harmonic extension on

$$\big(a, b + n(b-a)\big) \times \mathbb{R}^d$$

for each natural number n such that $y \mapsto \widetilde{h}(a + n(b-a), y)$ is entire. Similarly, there is a harmonic extension on $(a - n(b-a), b) \times \mathbb{R}^d$ for each natural number n, and the proof is complete.

In order to establish the claim, we assume that $a < b = 0$. Theorem 2.1 provides an extension $\widetilde{h}(t,y)$ of $h(t,y)$ to $[a,-a] \times \mathbb{R}^d$. Moreover, from the proof of Theorem 2.1, $\widetilde{h}(t,y)$ is given by equation (2.1)

$$\widetilde{h}(t,y) = -h(-t,y) + 2H(t,y),$$

where H is an entire harmonic function. This implies that the restriction $\widetilde{h}(-a,y)$ is entire, since $h(a,y)$ is assumed entire, and the result then follows. ∎

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Corollary 2.3 *Given entire functions $f_0, f_1: \mathbb{R}^d \to \mathbb{C}$, there exists a harmonic function $h: \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$ such that $h(a, y) = f_0(y)$ and $h(b, y) = f_1(y)$ for all $y \in \mathbb{R}^d$.*

Proof It is known that the Dirichlet problem for the slab with continuous data has a solution h; see *e.g.,* [10]. By Theorem 2.2 the function h has an entire extension. ∎

3 The Difference Equation for Harmonic Functions

We recall from complex analysis [4, p. 407] that the inhomogeneous difference equation

$$f(z+1) - f(z) = G(z) \tag{3.1}$$

for a given entire function $G(z)$ has an entire solution $f(z)$. This is a classical fact, and the solution given in [4, p. 407] uses Bernoulli polynomials, an idea that goes back to the work of Guichard, Appel, and Hurwitz more than a century ago (see [1, 13]).

Taking the real part of both sides and recalling that any harmonic function $g(t, y)$ in the plane is the real part $\Re e\,\{G(t + iy)\}$ of some entire function $G(z)$, it follows that the difference equation

$$h(t+1, y) - h(t, y) = g(t, y) \tag{3.2}$$

for a given harmonic function g on $\mathbb{R}^2$ has a harmonic solution h defined on $\mathbb{R}^2$.

In this section, we will generalize this result to all dimensions of the variable $y \in \mathbb{R}^d$. Our approach is based on solving the Dirichlet problem for the slab $[0, 1/2] \times \mathbb{R}^d$, and thus we do not need special functions as in the above-mentioned classical studies.

It is a remarkable fact that equation (3.1) can be solved for meromorphic functions as well. It would be interesting to extend our results to include the difference equation (3.2) for g with singularities (say of a controlled type).

As an intermediate step toward solving the difference equation (3.2), we provide a solution in the case when $g(t, y)$ is even.

Theorem 3.1 *Let $g: \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$ be harmonic and even. Then there exists an entire harmonic solution of the difference equation*

$$h(t+1, y) - h(t, y) = g(t, y).$$

Namely, any solution $h(t, y)$ of the Dirichlet problem for the slab $[0, 1/2] \times \mathbb{R}^d$ with data

$$h(0, y) = -\frac{1}{2} g(0, y) \quad \text{and} \quad h\Big(\frac{1}{2}, y\Big) = 0$$

for all $y \in \mathbb{R}^d$ induces such an entire harmonic solution of the difference equation.

Proof By Corollary 2.3 there exists an entire harmonic function $h(t, y)$ such that $h(0, y) = -\frac{1}{2} g(0, y)$ and $h(\frac{1}{2}, y) = 0$. The second of these two equations implies, by the Schwarz reflection principle, that

$$h\Big(\frac{1}{2} + t, y\Big) = -h\Big(\frac{1}{2} - t, y\Big). \tag{3.3}$$

Inserting $t = \frac{1}{2}$ in equation (3.3) gives $h(1, y) = -h(0, y) = \frac{1}{2}g(0, y)$. Now we consider the harmonic function

$$F(t, y) = h(t, y) - \frac{1}{2}g(t-1, y).$$

Then $F(1, y) = 0$, and by Schwarz's reflection principle, $F(1+t, y) = -F(1-t, y)$ for $y \in \mathbb{R}^d$. Then

$$\begin{aligned} h(1+t, y) &= F(1+t, y) + \frac{1}{2}g(t, y) = -F(1-t, y) + \frac{1}{2}g(t, y) \\ &= -h(1-t, y) + \frac{1}{2}g(-t, y) + \frac{1}{2}g(t, y) \end{aligned}$$

Since g is even, we have $\frac{1}{2}g(-t, y) + \frac{1}{2}g(t, y) = g(t, y)$ and

$$h(1-t, y) = h\Big(\frac{1}{2} + \frac{1}{2} - t, y\Big) = -h\Big(\frac{1}{2} - \Big(\frac{1}{2} - t\Big), y\Big) = -h(t, y).$$

It follows that $h(1+t, y) = h(t, y) + g(t, y)$. ■

The next result is surely a part of mathematical folklore; we include an elementary proof for the reader's convenience.

Lemma 3.2 *Let $g(t, y)$ be an entire harmonic function. Then there exists an entire harmonic function $u(t, y)$ such that*

$$\frac{\partial}{\partial t}u(t, y) = g(t, y).$$

If $g(t, y)$ is odd, then $u(t, y)$ can be chosen to be even.

Proof Define $h(t, y) := \int_0^t g(\tau, y)d\tau$. Then $\frac{\partial}{\partial t}h(t, y) = g(t, y)$ and

$$\Delta_y \frac{\partial}{\partial t}h(t, y) = \Delta_y g(t, y) = -\frac{\partial^2}{\partial t^2}g(t, y).$$

We conclude that $\frac{\partial}{\partial t}(\Delta_y h(t, y) + \frac{\partial}{\partial t}g(t, y)) = 0$, and it follows that

$$f(y) := \Delta_y h(t, y) + \frac{\partial}{\partial t}g(t, y)$$

only depends on y and not on t. Obviously, for any entire function $f(y)$, there exists an entire function $G(y)$ such that $\Delta_y G(y) = f(y)$. Then

$$u(t, y) := h(t, y) - G(y)$$

is a solution of the equation $\frac{\partial}{\partial t}u(t, y) = g(t, y)$, and $u(t, y)$ is harmonic, since we have

$$\begin{aligned} \Delta_{t,y}u(t, y) &= \frac{\partial^2}{\partial t^2}h(t, y) + \Delta_y h(t, y) - \Delta_y G(y) \\ &= \frac{\partial^2}{\partial t^2}h(t, y) + \Delta_y h(t, y) - \Delta_y h(t, y) - \frac{\partial}{\partial t}g(t, y) = 0. \end{aligned}$$

If $g(t, y)$ is odd, then $h(t, y)$ and hence $u(t, y)$ are both even. ■

Now we are able to prove our second main result.

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Theorem 3.3 *If $g: \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$ is harmonic, then the difference equation $h(t+1, y) - h(t, y) = g(t, y)$ has a harmonic solution $h: \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$.*

Proof Write the harmonic function g as a sum $g_0 + g_e$, where g_0 is odd and g_e is even. For the right-hand side g_e, there exists a solution $h_e(t, y)$ by Theorem 3.1, and so it suffices to solve the difference equation with right-hand side g_0. By Lemma 3.2 there exists an even harmonic function $u(t, y)$ such that

$$\frac{\partial}{\partial t} u(t, y) = g_0(t, y).$$

By Theorem 3.1, there exists a harmonic entire function $H(t, y)$ such that

$$H(t+1, y) - H(t, y) = u(t, y).$$

Differentiating $H(t, y)$ with respect to t, we thus find the solution of the difference equation with right-hand side $g_0(t, y)$. ∎

Finally, although the solution to the difference equation is far from unique, we are able to prove the following result.

Theorem 3.4 *Let $g: \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$ be a harmonic function and let $h_j: \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$ for $j = 1, 2$ be harmonic solutions of the difference equation*

$$h_j(t+1, y) - h_j(t, y) = g(t, y). \tag{3.4}$$

Assume that we have the estimate

$$|h_1(t, y) - h_2(t, y)| = o(|y|^{(1-d)/2} e^{2\pi|y|}), \tag{3.5}$$

as $|y| \to \infty$ (uniformly in t). Then there exists a harmonic function $r: \mathbb{R}^d \to \mathbb{C}$ such that

$$h_1(t, y) = h_2(t, y) + r(y)$$

for all $y \in \mathbb{R}^d$ and $t \in \mathbb{R}$.

Proof Define $h(t, y) := h_1(t, y) - h_2(t, y)$. Then h is periodic in t with period 1, since h_1 and h_2 each solve the difference equation (3.4). Let us also define $H(t, y) := H_y(t) := h(t/(2\pi), y/(2\pi))$ for $y \in \mathbb{R}^d$, $t \in \mathbb{R}$. Then H_y is a 2π-periodic function in t, so it has a Fourier series $\sum_{k=-\infty}^{\infty} a_k(y) e^{ikt}$. Applying the Laplace operator to the Fourier coefficients

$$a_k(y) = \frac{1}{2\pi} \int_0^{2\pi} H_y(t) e^{ikt} dt, \tag{3.6}$$

we have

$$\Delta_y a_k(y) = \frac{1}{2\pi} \int_0^{2\pi} \Delta_y H(t, y) e^{ikt} dt.$$

Since $H(t, y)$ is harmonic, we know that

$$\Delta_y H(t, y) = -\frac{\partial^2}{\partial t^2} H(t, y).$$

Integration by parts then shows that

$$\Delta_y a_k(y) = -\frac{1}{2\pi} \int_0^{2\pi} \frac{\partial^2 H}{\partial t^2}(t, y) \cdot e^{ikt} dt = k^2 a_k(y).$$

Hence, $a_k(y)$ is a solution of the Helmholtz equation $\Delta_y a_k = k^2 a_k$. In view of (3.6), the estimate (3.5) carries over to $a_k(y)$. Since $k^2 > 0$, a classical result [8, p. 228], going back to work of I. Vekua and F. Rellich in the 1940's, yields that $a_k = 0$ for all $k \neq 0$. ■

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References

[1] P. Appell, *Sur les fonctions périodiques de deux variables*. J. Math. Pures et Appl. **7**(1891), 157–219.
[2] D. H. Armitage, *The Dirichlet problem when the boundary function is entire.* J. Math. Anal. Appl. **291**(2004), no. 2, 565–577. http://dx.doi.org/10.1016/j.jmaa.2003.11.016
[3] D. H. Armitage and S. J. Gardiner, *Classical potential theory*. Springer Monographs in Mathematics, Springer-Verlag, London, 2001. http://dx.doi.org/10.1007/978-1-4471-0233-5
[4] C. A. Berenstein and R. Gay, *Complex analysis and special topics in harmonic analysis.* Springer-Verlag, New York, 1995. http://dx.doi.org/10.1007/978-1-4613-8445-8
[5] F. T. Brawn, *The Poisson integral and harmonic majorization in* $R^n \times]0,1[$. J. London Math. Soc. **3**(1971), 747–760. http://dx.doi.org/10.1112/jlms/s2-3.4.747
[6] W. Durand, *On some boundary value problems on a strip in the complex plane.* Rep. Math. Phys. **52**(2003), no. 1, 1–23. http://dx.doi.org/10.1016/S0034-4877(03)90001-2
[7] P. Ebenfelt, D. Khavinson, and H. S. Shapiro, *Algebraic aspects of the Dirichlet problem.* In: Quadrature domains and their applications, Oper. Theory Adv. Appl., 156, Birkhäuser, Basel, 2005, pp. 151–172. http://dx.doi.org/10.1007/3-7643-7316-4_7
[8] A. Friedman, *On n-metaharmonic functions and functions of infinite order.* Proc. Amer. Math. Soc. **8**(1957), 223–229.
[9] S. J. Gardiner, *The Dirichlet and Neumann problems for harmonic functions in half-spaces.* J. London Math. Soc. **24**(1981), no. 3, 502–512. http://dx.doi.org/10.1112/jlms/s2-24.3.502
[10] ———, *The Dirichlet problem with non-compact boundary.* Math. Z. **213**(1993), no. 1, 163–170. http://dx.doi.org/10.1007/BF03025715
[11] S. J. Gardiner and H. Render, *Harmonic functions which vanish on a cylindrical surface.* J. Math. Anal. Appl. **433**(2016), no. 2, 1870–1882. http://dx.doi.org/10.1016/j.jmaa.2015.08.077
[12] L. Hörmander, *The analysis of linear partial differential operators.* Vol. I, Springer-Verlag, Berlin-Heidelberg, New York, 1983.
[13] A. Hurwitz, *Sur l'intégrale finie d'une fonction entière*. Acta Math. **20**(1897), no. 1, 285–312. http://dx.doi.org/10.1007/BF02418035
[14] D. Khavinson, *Singularities of harmonic functions in* C^n. In: Several complex variables and complex geometry, Part 3 (Santa Cruz, CA, 1989), Proc. Sympos. Pure Math., 52, Amer. Math. Soc., Providence, RI, 1991, pp. 207–217. http://dx.doi.org/10.1090/pspum/052.3/1128595
[15] ———, *Holomorphic partial differential equations and classical potential theory.* Universidad de La Laguna, Departamento de Análisis Matemático, La Laguna 1996.
[16] ———, *Cauchy's problem for harmonic functions with entire data on a sphere.* Canad. Math. Bull. **40**(1997), no. 1, 60–66. http://dx.doi.org/10.4153/CMB-1997-007-3
[17] D. Khavinson and E. Lundberg, *The search for singularities of solutions to the Dirichlet problem: recent developments.* In: Hilbert spaces of analytic functions, CRM Proc. Lecture Notes, 51, American Mathematical Society, Providence, RI, 2010, pp. 121–132.
[18] D. Khavinson and H. S. Shapiro, *Dirichlet's problem when the data is an entire function*. Bull. London Math. Soc. **24**(1992), no. 5, 456–468. http://dx.doi.org/10.1112/blms/24.5.456
[19] D. Khavinson, E. Lundberg, and H. Render, *Dirichlet's problem with entire data posed on an ellipsoidal cylinder*. arxiv:1602.02837
[20] I. Miyamoto, *A type of uniqueness of solutions for the Dirichlet problem on a cylinder.* Tôhoku Math. J. **48**(1996), no. 2, 267–292. http://dx.doi.org/10.2748/tmj/1178225381
[21] H. Render, *Real Bargmann spaces, Fischer decompositions and sets of uniqueness for polyharmonic functions.* Duke Math. J. **142**(2008), no. 2, 313–351. http://dx.doi.org/10.1215/00127094-2008-008
[22] D. Siegel and E. O. Talvila, *Uniqueness for the n-dimensional half space Dirichlet problem.* Pacific J. Math. **175**(1996), no. 2, 571–587. http://dx.doi.org/10.2140/pjm.1996.175.571

[23] H. S. Shapiro, *An algebraic theorem of E. Fischer and the holomorphic Goursat problem*. Bull. London Math. Soc. **21**(1989), no. 6, 513–537. http://dx.doi.org/10.1112/blms/21.6.513
[24] D. V. Widder, *Functions harmonic in a strip*. Proc. Amer. Math. Soc. **12**(1961), 67–72. http://dx.doi.org/10.1090/S0002-9939-1961-0132838-8

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