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ISOMETRIC PREDUALS OF JAMES SPACES

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A Banach space B is called an *isometric predual*, or simply a *predual*, of a Banach space X if the dual B^* of B is isometrically isomorphic to X. A Banach space X is said to have a *unique* (*isometric*) *predual* if X has a predual and all preduals are mutually isometrically isomorphic. In general a Banach space does not have a unique predual even if it has a predual. A simple example of this is the space l^1, because c_0 and c are isometric preduals of l^1 but not isometrically isomorphic. A. Grothendieck [**3**] first noticed that L^∞-spaces have unique preduals, and then S. Sakai generalized this to von Neumann algebras (see p. 30 of [**9**]). Recently one of the authors [**4**] has shown that every quotient space of a von Neumann algebra by a σ-weakly closed subspace, as a Banach space with quotient norm, has a unique predual. Also T. Ando [**1**] has shown that the space H^∞ has a unique predual and P. Wojtasczcyk has also proved this result independently. Evidently, these are the only known non-reflexive Banach spaces with unique preduals. See the Addendum.

In this paper we prove uniqueness of preduals of James quasi-reflexive spaces. In particular, we are interested in James spaces having norms presented in [**5**] and [**6**]. Note that quasi-reflexive spaces have a different character from L^∞-spaces and the spaces mentioned previously.

We use the following standard notation. We shall always regard a Banach space X as a subspace of its second dual X^{**} in the canonical way. A subspace means a closed linear subspace. For a subset A of a Banach space X, $A^\perp$ denotes the annihilator of A in the dual X^*. If A is a subset of a dual Banach space X^*, then $A_\perp$ denotes the set of all elements in X annihilated by A. For a subset A of a Banach space X, $[A]$ denotes the closed linear span of A in X, and $X = A \oplus B$ means that X is the direct sum of subspaces A and B.

The proof of our results is based on the following idea: If X is a Banach space, then $X^{***} = X^\perp \oplus X^*$ where X^* is norm 1 complemented in X^{***}. That is, the projection from X^{***} onto X^* associated with this decomposition has norm 1. Thus a sufficient condition for X^* to have a unique predual is that $X^\perp$ is the only norm 1 complement of X^* in X^{***}. In order to show this, it is sufficient to show that if $\varphi \in X^{***}$ and

$$(1) \qquad ||\varphi + x^*|| \geqq ||x^*|| \quad \text{for all } x^* \in X^*$$

then $\varphi \in X^\perp$.

As an illustration of this method, we present a proof (different from the usual

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proof) that l^∞ has a unique predual. If $X = l^1$, then we have $X^* = l^\infty$ and $X^{***} = X^\perp \oplus X^* = X^\perp \oplus l^\infty$. A straightforward argument, using the representation $X^* = l^\infty = C(\beta N)$ and

$$X^{**} = C(\beta N)^* = M(\beta N) = M(\beta N \backslash N) \oplus M(N) = M(\beta N \backslash N) \oplus X$$

where βN is the Stone Cech compactification of the set N of positive integers, yields the fact that if $x^* \in c_0 \subset l^\infty$ and $\psi \in X^\perp$ then

$$||\psi + x^*|| = \text{Max}(||\psi||, ||x^*||).$$

If $\varphi \in X^{***}$, then $\varphi = \psi - x_0^*$ where $\psi \in X^\perp$ and $x_0^* \in l^\infty = X^*$. Proving that inequality (1) implies $\varphi \in X^\perp$ is equivalent to showing that

$$(2) \qquad ||\psi + x^*|| \geqq ||x_0^* + x^*|| \quad \text{for all } x^* \in X^*$$

implies $x_0^* = 0$. Assume inequality (2), and let n be a positive integer. If $x^* = ce_n^*$, where e_n^* is the usual basis element of c_0 and c is a complex number, then we have

$$||\psi + x^*|| = \text{Max}(||\psi||, |c|) \geqq ||x_0^* + ce_n^*|| \geqq |x_0^*(n) + c|.$$

Since this holds for every complex number c, $x_0^*(n)$ must be 0 for each n which completes the proof.

The James space $(J, ||\cdot||)$ ([**5**]) is defined to be the space consisting of all complex sequences $x = (x(n))$ such that

$$||x|| = \sup(\textstyle\sum_{j=1}^k |\sum_{n \in I_j} x(n)|^2)^{1/2} < +\infty$$

where the supremum is taken over all choices of disjoint finite intervals $I_1, I_2, \ldots, I_k$ of positive integers.

We are interested in an equivalent norm on this space, first defined by James [**6**]. The following is a slight modification of his norm.

$$|||x||| = \sup(\textstyle\sum_{j=1}^k |\sum_{n \in \hat{I}_j} x(n)|^2)^{1/2}$$

where $\hat{I}_j$ are either intervals or complements of intervals in the space of positive integers and the supremum is taken over all choices of disjoint $\hat{I}_1, \ldots, \hat{I}_k$. One easily sees that for $x \in J$,

$$||x|| \leqq |||x||| \leqq \sqrt{2}||x||.$$

1. $(J, ||\cdot||)$. In this section, J will always stand for $(J, ||\cdot||)$. The space J has the natural normalized basis $\{e_n\}$; for every $x \in J$ we have $x = \sum_{n=1}^\infty x(n)e_n$, where $e_n = (e_n(j)) = (\delta_{n,j})$ for $n, j = 1, 2, \ldots$. Let $\{e_n^*\}$ be the biorthogonal sequence with respect to $\{e_n\}$ and let Y be the closed linear span of $\{e_n^*\}$. Since $\{e_n\}$ is a boundedly complete monotone basis of J, J is isometrically isomorphic to the dual Y^* of Y by the canonical mapping (see p. 91 of [**2**]). We introduce the linear functionals φ_n on J by $\varphi_n(x) = \sum_{j=n}^\infty x(j)$ for $x \in J$ and $n = 1, 2, \ldots$.

As proved in [**5**], $\{\varphi_n\}$ forms a normalized basis of J^* and we have

$$J^* = [\varphi_1] \oplus Y.$$

We define $f_2 \in J^{**}$ as follows: $f_2(\varphi_1) = 1$ and f_2 annihilates Y. Then we have $J^{**} = [f_2] \oplus J$. More generally, denote $J^n (n = 0, 1, 2, \ldots)$ as the n'th dual of J, then we have

$$J = J^0 \subset J^2 \subset \ldots \subset J^{2n} \subset \ldots, \quad \text{and}$$
$$J^* = J^1 \subset J^3 \subset \ldots \subset J^{2n+1} \subset \ldots.$$

For $n \geqq 3$, $f_n \in J^n$ is defined as follows: $f_n(f_{n-1}) = 1$ and f_n annihilates J^{n-1}. Thus the one dimensional space $[f_n]$ is a norm 1 complement of J^{n-2} in J^n and we have

$$J^n = [f_n] \oplus J^{n-2} \quad \text{for} \quad n = 2, 3, \ldots.$$

For even $n \geqq 2$, $\{f_n, f_{n-2}, \ldots, f_4, f_2, e_1, e_2, \ldots\}$ forms a basis of J^n and for odd $n \geqq 3$, $\{f_n, f_{n-2}, \ldots, f_5, f_3, \varphi_1, e_1^*, e_2^*, \ldots\}$ forms a basis of J^n. The canonical bilinear functional defined on the product of $\bigcup_{n=0}^\infty J^{2n}$ and $\bigcup_{n=0}^\infty J^{2n+1}$ will be denoted by $\langle x, y\rangle$ for $x \in \bigcup_{n=0}^\infty J^{2n}$ and $y \in \bigcup_{n=0}^\infty J^{2n+1}$.

The following lemma will be used many times.

LEMMA 1. *For each* $n = 1, 2, \ldots,$

a) $\quad f_{2n} = w^* - \lim_{k\to\infty} \sum_{j=1}^{n-1} (-1)^{j+1} f_{2n-2j} + (-1)^{n+1} e_k$ *in* J^{2n},

b) $\quad f_{2n+1} = w^* - \lim_{k\to\infty} \sum_{j=1}^{n-1} (-1)^{j+1} f_{2n-2j+1} + (-1)^{n+1} \varphi_k$ *in* J^{2n+1}.

For each $k = 1, 2, \ldots$ *and* $n = 2, 3, \ldots,$

c) $\quad ||f_{2n} + x|| \leqq \sup_{k<k_1<k_2} ||(-1)^{n+1} e_{k_1} + (-1)^n e_{k_2} + x|| \quad (x \in J^{2n-2}),$

d) $\quad ||f_{2n+1} + y|| \leqq \sup_{k<k_1<k_2} ||(-1)^{n+1} \varphi_{k_1} + (-1)^n \varphi_{k_2} + y|| \quad (y \in J^{2n-1}).$

Proof. For a) and b) simply evaluate both sides of the equation on basis elements of J^{2n-1} and J^{2n} respectively.

c) Applying a) we have for $x \in J^{2n}$,

$$||f_{2n} + x|| \leqq \underline{\lim}_{k_1\to\infty} ||\textstyle\sum_{j=1}^{n-1} (-1)^{j+1} f_{2n-2j} + (-1)^{n+1} e_{k_1} + x||$$
$$= \underline{\lim}_{k_1\to\infty} ||f_{2n-2} + \textstyle\sum_{j=2}^{n-1} (-1)^{j+1} f_{2n-2j} + (-1)^{n+1} e_{k_1} + x||.$$

If $x \in J^{2n-2}$ we may apply a) again and we have

$$||f_{2n} + x|| \leqq \underline{\lim}_{k_1\to\infty} \underline{\lim}_{k_2\to\infty} ||(-1)^{n+1} e_{k_1} + (-1)^n e_{k_2} + x||$$
$$\leqq \sup_{k<k_1<k_2} ||(-1)^{n+1} e_{k_1} + (-1)^n e_{k_2} + x||.$$

d) is proven in a similar fashion to c) and we shall omit the proof.

LEMMA 2. a) $||f_{2n+1}|| = 1$ *for* $n = 1, 2, 3, \ldots,$

b) $||f_2|| = 1$ *and* $||f_{2n}|| = \sqrt{2}$ *for* $n = 2, 3, \ldots.$

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Proof. a) From Lemma 1d)

$$\|f_{2n+1}\| \leqq \sup_{k_1<k_2} \|\varphi_{k_1} - \varphi_{k_2}\| = 1.$$

On the other hand from standard duality arguments,

$$\|f_{2n+1}\|^{-1} = \inf_{x \in J^{2n-2}} \|f_{2n} + x\|.$$

Applying Lemma 1a), for all $x \in J^{2n-2}$

$$\|f_{2n} + x\| \leqq \underline{\lim}_{k\to\infty} \|\textstyle\sum_{j=1}^{n-1} (-1)^{j+1} f_{2n-2j} + (-1)^{n+1} e_k + x\|.$$

Setting $x = -\sum_{j=1}^{n-1} (-1)^{j+1} f_{2n-2j}$, we have

$$\|f_{2n} + x\| \leqq \underline{\lim}_{k\to\infty} \|(-1)^{n+1} e_k\| = 1.$$

Thus $\|f_{2n+1}\|^{-1} \leqq 1$ and the proof of a) is complete.

b) We have $\|f_2\|^{-1} = \inf_{y\in Y} \|\varphi_1 + y\|$, where $Y = [e_1^*, e_2^*, \ldots] \subset J^*$. For $y \in Y$,

$$\|\varphi_1 + y\| \geqq |\langle e_n, \varphi_1 + y\rangle| = |1 + \langle e_n, y\rangle| \to 1 \quad \text{as } n \to \infty.$$

Thus we have $\|f_2\|^{-1} = \inf_{y\in Y} \|\varphi_1 + y\| = 1$.

Applying Lemma 1c), we have

$$\|f_{2n}\| \leqq \sup_{k_1<k_2} \|e_{k_1} - e_{k_2}\| = \sqrt{2}.$$

From standard duality arguments, we have

$$\|f_{2n}\|^{-1} = \inf_{y\in J^{2n-3}} \|f_{2n-1} + y\|.$$

Applying Lemma 1b), for $y \in J^{2n-3}$

$$\|f_{2n-1} + y\| \leqq \underline{\lim}_{k_1\to\infty} \|\textstyle\sum_{j=1}^{n-2} (-1)^{j+1} f_{2n-2j-1} + (-1)^n \varphi_{k_1} + y\|.$$

Setting $y = -\frac{1}{2}(\sum_{j=1}^{n-2} (-1)^{j+1} f_{2n-2j-1} + (-1)^n \varphi_1)$ and applying Lemma 1b) again, we have

$$\begin{aligned} \|f_{2n-1} + y\| &\leqq \underline{\lim}_{k_1\to\infty} \|\tfrac{1}{2}\textstyle\sum_{j=1}^{n-2} (-1)^{j+1} f_{2n-2j-1} + (-1)^n \varphi_{k_1} - \tfrac{1}{2}(-1)^n \varphi_1\| \\ &\leqq \underline{\lim}_{k_1\to\infty} \underline{\lim}_{k_2\to\infty} \| -\tfrac{1}{2}(-1)^n \varphi_1 + (-1)^n \varphi_{k_1} + \tfrac{1}{2}(-1)^{n-1} \varphi_{k_2}\| \\ &\leqq \sup_{1<k_1<k_2} \tfrac{1}{2}\|(-\varphi_1 + \varphi_{k_1}) + (\varphi_{k_1} - \varphi_{k_2})\| \leqq \sqrt{2}/2. \end{aligned}$$

The last inequality is true because for $x \in J$,

$$\begin{aligned} |\langle x, -\varphi_1 + \varphi_{k_1} + \varphi_{k_1} - \varphi_{k_2}\rangle| &= |-\textstyle\sum_1^{k_1-1} x(j) + \sum_{k_1}^{k_2-1} x(j)| \\ &\leqq \sqrt{2}(|\textstyle\sum_1^{k_1-1} x(j)|^2 + |\sum_{k_1}^{k_2-1} x(j)|^2)^{1/2} \leqq \sqrt{2}\|x\|. \end{aligned}$$

This completes the proof of the Lemma.

As a consequence of Lemma 2a), we have $\|f_{2n} + x\| \geqq 1$ for all $x \in J^{2n-2}$ and $n = 1, 2, \ldots$ which implies the following:

COROLLARY 1.

$$\|\textstyle\sum_{k=1}^n \beta_k f_{2k} + \sum_{j=1}^\infty \alpha_j e_j\| \geqq \sup_{k,j} \{|\beta_k|, |\alpha_j|\}.$$

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THEOREM 1. 1) *$(J^n, \|\cdot\|)$ has a unique isometric predual for all* $n = 0, 1, 2, \ldots$.
2) *Y, the unique predual of J, does not have any isometric predual.*

Proof. 1a). *J^{2n-1} has a unique isometric predual for all* $n = 1, 2, \ldots$.

To show this it suffices to show that $[f_{2n+1}]$ is the unique norm 1 complement of J^{2n-1} in J^{2n+1}. In this case, this is equivalent to showing that if $y_0 \in J^{2n-1}$ and

(3) $\quad \|f_{2n+1} + y\| \geqq \|y_0 + y\| \quad$ for all $y \in J^{2n-1}$

then $y_0 = 0$. Let $y_0 = \sum_1^{n-1} \beta_k f_{2k+1} + \beta\varphi_1 + \sum_1^\infty \alpha_j e_j^*$ and assume inequality (3). Given c, a complex number and l, a positive integer, set $y = ce_l^*$. Then by Lemma 1d), we have

$$\|f_{2n+1} + ce_l^*\| \leqq \sup_{l<k_1<k_2} \|ce_l^* + (-1)^{n+1}\varphi_{k_1} + (-1)^n\varphi_{k_2}\| \leqq \sqrt{|c|^2 + 1}$$

(use Schwartz's inequality as in the end of the proof of Lemma 2b)).
On the other hand

$$\|y_0 + ce_l^*\| \geqq |\langle e_l, y_0 + ce_l^*\rangle| = |\beta + \alpha_l + c|.$$

Thus $|\beta + \alpha_l + c| \leqq \sqrt{|c|^2 + 1}$ for all complex numbers c which implies $\beta + \alpha_l = 0$ for all l and we have $\beta = \alpha_l = 0$ for all l. Suppose $\beta_k = 0$ for $1 \leqq k < l \leqq n - 1$. Applying Lemma 1d) two times, we have for each complex number c

$$\begin{aligned}\|f_{2n+1} + cf_{2l+1}\| &\leqq \sup_{k_1<k_2} \|(-1)^{n+1}\varphi_{k_1} + (-1)^n\varphi_{k_2} + cf_{2l+1}\| \\ &\leqq \sup_{k_1<k_2<k_3<k_4} \|(-1)^{n+1}\varphi_{k_1} + (-1)^n\varphi_{k_2} + c((-1)^{l+1}\varphi_{k_3} \\ &\qquad + (-1)^l\varphi_{k_4})\| \\ &\leqq \sqrt{|c|^2 + 1}.\end{aligned}$$

On the other hand, since $[f_{2j+1}]$ is a norm 1 complement of J^{2j-1} in J^{2j+1} for all $j = 1, 2, \ldots$, we have

$$\begin{aligned}\|y_0 + cf_{2l+1}\| &= \|\textstyle\sum_{k=l}^{n-1} \beta_k f_{2k+1} + cf_{2l+1}\| \geqq \|(\beta_l + c)f_{2l+1}\| \\ &= |\beta_l + c|\,\|f_{2l+1}\| = |\beta_l + c| \quad \text{(Lemma 2a))}.\end{aligned}$$

Thus $|\beta_l + c| \leqq \sqrt{|c|^2 + 1}$ for all complex numbers c which implies that $\beta_l = 0$. This completes the proof that $y_0 = 0$.

1b) *J has a unique isometric predual.*

In the beginning of this section, we noticed that J has an isometric predual Y (this fact can be proven directly by showing that $\|f_2 + x\| \geqq \|x\|$ for all $x \in J$ in a manner similar to the proof of Lemma 3 in Section 2). Let $x_0 = \sum_1^\infty \alpha_j e_j$ and assume

$$\|f_2 + x\| \geqq \|x_0 + x\| \quad \text{for all } x \in J.$$

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Given l, then $\alpha_l = e^{i\theta}|\alpha_l|$ and we set $x = ce_l - e_{l+1}$ with $c > 0$. Then applying Lemma 1a) we have

$$\begin{aligned} ||e^{-i\theta}f_2 + ce_l - e_{l+1}|| &\leqq \underline{\lim}_{k\to\infty} ||ce_l - e_{l+1} + e^{-i\theta}e_k|| \\ &\leqq \sqrt{c^2+2} \quad \text{(Definition of the norm in } J). \end{aligned}$$

On the other hand, we have

$$||e^{-i\theta}x_0 + ce_l - e_{l+1}|| \geqq |\langle e^{-i\theta}x_0 + ce_l - e_{l+1}, e_l^*\rangle| = |\alpha_l| + c.$$

Thus $|\alpha_l| + c \leqq \sqrt{c^2+2}$ for all $c > 0$ which implies that $\alpha_l = 0$.

1c) *J^{2n} has a unique isometric predual for all $n = 1, 2, \ldots$.*

Let $x_0 = \sum_{k=1}^{n} \beta_k f_{2k} + \sum_{j=1}^{\infty} \alpha_j e_j$ and assume

$$||f_{2n+2} + x|| \geqq ||x_0 + x|| \quad \text{for all } x \text{ in } J^{2n}.$$

i) We claim $\alpha_j = 0$ for all $j = 1, 2, \ldots$.

As in 1b), we set $\alpha_l = e^{i\theta}|\alpha_l|$ and $x = ce_l - e_{l+1}$ with $c > 0$. Lemma 1c) implies that

$$\begin{aligned} ||e^{-i\theta}f_{2n+2} + ce_l - e_{l+1}|| &\leqq \sup_{l+1<k_1<k_2} ||ce_l - e_{l+1} + e^{-i\theta}((-1)^{n+2}e_{k_1} \\ &+ (-1)^{n+1}e_{k_2})|| \leqq \sqrt{c^2+5} \quad \text{(Definition of the norm in} J). \end{aligned}$$

On the other hand, we have

$$||e^{-i\theta}x_0 + ce_l - e_{l+1}|| \geqq |\alpha_l| + c.$$

Thus $|\alpha_l| + c \leqq \sqrt{c^2+5}$ for all $c > 0$ which implies that $\alpha_l = 0$.

ii) We claim $\beta_k = 0$ for all $k = 1, 2, \ldots, n$.

Suppose $\beta_1 = \beta_2 = \ldots = \beta_{l-1} = 0$ for $1 \leqq l \leqq n$. Set

$$x = -\sum_{j=1}^{n} (-1)^{j+1} f_{2n-2j+2}.$$

Then by Lemma 1a)

$$||f_{2n+2} + x|| \leqq \lim_{k\to\infty} ||(-1)^{n+2}e_k|| = 1.$$

On the other hand the corollary to Lemma 2 implies that

$$||x_0 + x|| \geqq |\beta_l + (-1)^{n-l+1}|.$$

Consequently, we have

$$(4) \qquad |\beta_l + (-1)^{n-l+1}| \leqq 1.$$

If $l = 1$, set $x = (-1)^{n+1}ce_1$ with $c > 0$. Then by Lemma 1c), we have

$$\begin{aligned} ||f_{2n+2} + (-1)^{n+1}ce_1|| &\leqq \sup_{1<k_1<k_2} ||(-1)^{n+1}ce_1 + (-1)^{n+2}e_{k_1} \\ &\qquad + (-1)^{n+1}e_{k_2}|| \\ &= \sup_{1<k_1<k_2} ||ce_1 - e_{k_1} + e_{k_2}|| \leqq \sqrt{c^2+2}, \end{aligned}$$

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and we have

$$\left\|\sum_{k=1}^{n} \beta_k f_{2k} + (-1)^{n+1} c e_1\right\| \geqq \left|\left\langle \sum_{k=1}^{n} \beta_k f_{2k} + (-1)^{n+1} c e_1, \varphi_1 \right\rangle\right| = |\beta_1 + (-1)^{n+1} c|.$$

Thus we have

(5) $\quad |\beta_1 + (-1)^{n+1} c| \leqq \sqrt{c^2 + 2} \quad$ for all $c > 0$.

If $l > 1$, we set $x = (-1)^{n-l} c \sum_{j=1}^{l-1} (-1)^{j+1} f_{2l-2j}$ with $c > 0$. Then by Lemma 1c), we have

$$\begin{aligned} \|f_{2n+2} + x\| &\leqq \sup_{k_1 < k_2} \|(-1)^{n+2} e_{k_1} + (-1)^{n+1} e_{k_2} + x\| \\ &= \sup_{k_1 < k_2} \|(-1)^{n-l} c f_{2l-2} + (-1)^{n-l} c \sum_{j=2}^{l-1} (-1)^{j+1} f_{2l-2j} \\ &\qquad + (-1)^{n+2} e_{k_1} + (-1)^{n+1} e_{k_2}\|. \end{aligned}$$

Applying Lemma 1a), we have

$$\begin{aligned} \|f_{2n+2} + x\| &\leqq \sup_{k_1 < k_2 < k_3} \|(-1)^{n+2} e_{k_1} + (-1)^{n+1} e_{k_2} + (-1)^{n-l+l} c e_{k_3}\| \\ &= \sup_{k_1 < k_2 < k_3} \|e_{k_1} - e_{k_2} + c e_{k_3}\| \leqq \sqrt{c^2 + 2}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \|x_0 + x\| &= \left\|\sum_{k=l}^{n} \beta_k f_{2k} + (-1)^{n-l} c \sum_{j=1}^{l-1} (-1)^{j+1} f_{2l-2j}\right\| \\ &\geqq |\langle x_0 + x, f_{2l-1}\rangle| = |\beta_l + (-1)^{n-l} c|. \end{aligned}$$

Note that we used the fact that $\|f_{2l-1}\| = 1$, which was proved in Lemma 2a). Thus we have

(6) $\quad |\beta_l + (-1)^{n-l} c| \leqq \sqrt{c^2 + 2} \quad$ for all $c > 0$.

The inequalities (4), (5) and (6) say that for all l with $1 \leqq l \leqq n$, we have

$$|\beta_l + (-1)^{n-l+1}| \leqq 1 \quad \text{and} \quad |\beta_l + (-1)^{n-l} c| \leqq \sqrt{c^2 + 2} \text{ for all } c > 0.$$

Therefore if $n - l$ is even, then $|\beta_l - 1| \leqq 1$ and $|\beta_l + c| \leqq \sqrt{c^2 + 2}$ for all $c > 0$ which implies $|\beta_l - 1| \leqq 1$ and $\operatorname{Re} \beta_l \leqq 0$. Thus we have $\beta_l = 0$. A similar argument shows that when $n - l$ is odd, $\beta_l = 0$. This completes the proof of 1).

2) To prove that Y has no isometric predual, we show that Y is not norm 1 complemented in J^*. That is, if $y_0 = \sum_{j=1}^{\infty} \alpha_j e_j^* \in Y$, then the condition

(7) $\quad \|\varphi_1 + y_0 + y\| \geqq \|y\| \quad$ for all $y \in Y$

leads to a contradiction. Assuming (7) and setting $y = -y_0 - 2e_n^*$, we have

$$\|\varphi_1 - 2e_n^*\| \geqq \|-y_0 - 2e_n^*\| \quad \text{for each } n.$$

If $x = \sum_{j=1}^{\infty} \beta_j e_j \in J$, then

$$\begin{aligned} |\langle x, \varphi_1 - 2e_n^*\rangle| &= \left|\sum_{1}^{n-1} \beta_j - \beta_n + \sum_{n+1}^{\infty} \beta_j\right| \\ &\leqq \sqrt{3}\left(\left|\sum_{1}^{n-1} \beta_j\right|^2 + |\beta_n|^2 + \left|\sum_{n+1}^{\infty} \beta_j\right|^2\right)^{1/2} \leqq \sqrt{3}\|x\|. \end{aligned}$$

Thus $\|\varphi_1 - 2e_n^*\| \leqq \sqrt{3}$ for each n. On the other hand, we have

$$\|-y_0 - 2e_n^*\| \geqq |\langle e_n, -y_0 - 2e_n^*\rangle| = |-\alpha_n - 2|.$$

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Thus we have $\sqrt{3} \geqq |\alpha_n + 2|$ for each n which is a contradiction since α_n approaches 0 as $n \to \infty$. This completes the proof of Theorem 1.

COROLLARY 2. *If $n \neq m \geqq -1$, then J^n is not isometrically isomorphic to J^m where $J^{-1} = Y$, the unique predual of J. Furthermore, J is not isometrically isomorphic to the second dual X^{**} of any Banach space X.*

2. $(J, |||\cdot|||)$. In this section, J^n will always stand for the n'th dual of $(J, |||\cdot|||)$ and we use the same basis as introduced in Section 1 for J^n.

LEMMA 3. $|||f_2 - e_1 + x||| \geqq |||x|||$ *for all* $x \in J$.

Proof. Without loss of generality, we may assume that the given $x \in J$ has a finite expansion, $x = \sum_1^n \alpha_j e_j$. Then given $\epsilon > 0$, there exists a finite set of disjoint $\hat{I}_k$, $k = 1, 2, \ldots, m$ such that

$$(\textstyle\sum_{k=1}^m |\sum_{j\in\hat{I}_k} \alpha_j|^2)^{1/2} \geqq |||x||| - \epsilon.$$

Since x has a finite expansion, we may assume $\hat{I}_1 = \{1, 2, \ldots, l_1\} \cup \{l_2, l_2 + 1, \ldots\}$ with $l_1 < l_2$ and $\hat{I}_2, \ldots, \hat{I}_m$ are the usual finite intervals. Choose β_k, $k = 1, 2, \ldots, m$ such that

$$\textstyle\sum_1^m |\beta_k|^2 = 1 \quad \text{and} \quad \sum_1^m \beta_k(\sum_{j\in\hat{I}_k} \alpha_j) = (\sum_1^m |\sum_{j\in\hat{I}_k} \alpha_j|^2)^{1/2}$$

and define

$$y = \textstyle\sum_2^m \beta_k(\sum_{j\in\hat{I}_k} e_j^*) + \beta_1(\varphi_1 - \sum_{l_1<j<l_2} e_j^*).$$

Then for any $z = \sum_1^\infty \gamma_j e_j \in J$, we have

$$\begin{aligned}|\langle z, y\rangle| &= |\textstyle\sum_{k=2}^m \beta_k(\sum_{j\in\hat{I}_k} \gamma_j) + \beta_1(\sum_1^{l_1} \gamma_j + \sum_{l_2}^\infty \gamma_j)| \\ &\leqq (\textstyle\sum_1^m |\beta_k|^2)^{1/2}(\sum_{k=2}^m |\sum_{j\in\hat{I}_k} \gamma_j|^2 + |\sum_{j\in\hat{I}_1} \gamma_j|^2)^{1/2} \\ &\leqq 1 \cdot |||z|||.\end{aligned}$$

Thus $|||y||| \leqq 1$. It is clear that $\langle f - e_1, y\rangle = 0$. Thus we have

$$\begin{aligned}|||x||| - \epsilon &< (\textstyle\sum_{k=1}^m |\sum_{j\in\hat{I}_k} \alpha_j|^2)^{1/2} = \langle x, y\rangle = \langle f_2 - e_1 + x, y\rangle \\ &\leqq |||f_2 - e_1 + x|||\ |||y||| \leqq |||f_2 - e_1 + x|||.\end{aligned}$$

Hence we can conclude that $|||x||| \leqq |||f_2 - e_1 + x|||$.

Since the annihilator of $f_2 - e_1$ in J^* is clearly the closed linear span $Z = [\varphi_1, e_2^*, e_3^*, \ldots]$, we have

COROLLARY 3. $(J, |||\cdot|||)$ *has an isometric predual, namely,* $(Z, |||\cdot|||)$.

THEOREM 2. $(J, |||\cdot|||)$ *and* $(J^*, |||\cdot|||)$ *have unique isometric preduals.*

Proof. a) *J has a unique isometric predual.*

We have observed in Corollary 3 that Z is a predual of J. We shall show that

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$[f_2 - e_1]$ is the only norm 1 complement of J in J^2. Let $x_0 = \sum_1^\infty \alpha_j e_j \in J$, and assume

$$(8) \quad |||f_2 + x||| \geqq |||x_0 + x||| \quad \text{for all } x \in J.$$

We claim $x_0 = e_1$. For $n > 1$, choose θ so that $\alpha_n = e^{i\theta}|\alpha_n|$. Set $x = -e_1 + ce_n - e_{n+1}$ with $c > 0$. Then by Lemma 1a),

$$|||e^{-i\theta}f_2 + x||| \leqq \underline{\lim}_{k\to\infty} |||-e_1 + ce_n - e_{n+1} + e^{-i\theta}e_k|||.$$

However for $k > n + 1$,

$$|||-e_1 + ce_n - e_{n+1} + e^{-i\theta}e_k||| \leqq \sqrt{c^2 + 9} \quad \text{and}$$
$$|||e^{-i\theta}x_0 + x||| = |||e^{-i\theta}\textstyle\sum_1^\infty \alpha_j e_j - e_1 + ce_n - e_{n+1}||| \geqq ||\alpha_n| + c|.$$

Thus we have $|\alpha_n| + c \leqq \sqrt{c^2 + 9}$ for all $c > 0$ which implies $\alpha_n = 0$, and we have $x_0 = \alpha_1 e_1$. Thus inequality (8) becomes

$$(9) \quad |||f_2 + x||| \geqq |||\alpha_1 e_1 + x||| \quad \text{for all } x \in J.$$

If $y = \beta\varphi_1 + \sum_1^\infty \beta_j e_j^*$ with $|||y||| \leqq 1$, then

$$|\beta + \beta_n| = |\langle e_n, y\rangle| \leqq |||e_n|||\ |||y||| \leqq 1 \quad \text{for all } n = 1, 2, \ldots.$$

This implies that $|\langle f_2, y\rangle| = |\beta| = \lim_{n\to\infty} |\beta + \beta_n| \leqq 1$. Since $|||f_2||| \geqq ||f_2|| = 1$ (Lemma 2b)), we have $|||f_2||| = 1$. Thus inequality (9) implies $1 = |||f_2||| \geqq |||\alpha_1 e_1||| = |\alpha_1|$. Set $x = -e_1 + e_2 - ce_3$ with $c > 0$. Then by Lemma 1a),

$$|||f_2 + x||| \leqq \underline{\lim}_{k\to\infty} |||-e_1 + e_2 - ce_3 + e_k||| \leqq \sqrt{c^2 + 3}, \quad \text{and}$$
$$|||\alpha_1 e_1 + x||| = |||(\alpha_1 - 1)e_1 + e_2 - ce_3||| \geqq |\alpha_1 - 1 - c|.$$

Hence (9) implies that $|\alpha - 1 - c| \leqq \sqrt{c^2 + 3}$ for all $c > 0$, and we have $\text{Re}(\alpha_1 - 1) \geqq 0$. Since $|\alpha_1| \leqq 1$, we can conclude that $\alpha_1 = 1$ which completes the proof of a).

b) *J^* has a unique isometric predual.*

We shall show that $[f_3]$ is the only norm 1 complement of J^* in J^3. Suppose $y_0 = \beta\varphi_1 + \sum_1^\infty \beta_j e_j^* \in J^*$ and

$$(13) \quad |||f_3 + y||| \geqq |||y_0 + y||| \quad \text{for all } y \in J^*.$$

We claim $y_0 = 0$. If $y = ce_n^*$ where c is any complex number, Lemma 1b) implies

$$|||f_3 + ce_n^*||| \leqq \underline{\lim}_{k\to\infty} |||ce_n^* + \varphi_k||| \leqq \sqrt{|c|^2 + 1}.$$

On the other hand,

$$|||y_0 + ce_n^*||| \geqq |\langle e_n, y_0 + ce_n^*\rangle| = |\beta + \beta_n + c|.$$

Thus, from (10) we have $|\beta + \beta_n + c| \leqq \sqrt{|c|^2 + 1}$ for all complex numbers c which implies $\beta + \beta_n = 0$ for each n. Thus $\beta = \beta_n = 0$ for all n since $\beta_n \to 0$ as $n \to \infty$. This completes the proof of the theorem.

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Our norm $|||\cdot|||$ is a slight variant of the norm introduced in James [**6**]. We present the following proof of his theorem for completeness.

THEOREM 3. (James) *If* $x = \sum_1^\infty \alpha_j e_j \in J$ *and* $Sx = \alpha_1 f_2 + \sum_1^\infty \alpha_{j+1} e_j \in J^{**}$, *then* S *is an isometry from* J *onto* J^{**}.

Proof. It is sufficient to show that $|||Sx||| = |||x|||$ for any $x = \sum_1^n \alpha_j e_j$ with finite expansion. For such an x, set $z_k = \alpha_2 e_1 + \alpha_3 e_2 + \ldots + \alpha_n e_{n-1} + \alpha_1 e_k$ for $k \geqq n$. Then, we have $|||z_k||| = |||x|||$ for all $k \geqq n$ and by Lemma 1a) $w^* - \lim_{k\to\infty} z_k = Sx$ in J^{**}. Thus $|||Sx||| \leqq \underline{\lim}_{k\to\infty} |||z_k||| = |||x|||$.

Choose $y_0 = \beta\varphi_1 + \sum_1^\infty \beta_j e_j^* \in J^*$ with $|||y_0||| = 1$ such that $\langle z_n, y_0\rangle = |||z_n||| = |||x|||$. If $\hat{y}_0 = \beta\varphi_1 + \sum_1^{n-1} \beta_j e_j^* + \beta_n \varphi_n$, we can see that

$$\text{(11)} \quad \langle z, \hat{y}_0\rangle = \langle \tilde{z}, y_0\rangle \quad \text{for all } z = \sum_1^\infty \gamma_j e_j \in J,$$

where $\tilde{z} = \sum_1^{n-1} \gamma_j e_j + (\sum_n^\infty \gamma_j) e_n$. Thus we have

$$\langle Sx, \hat{y}_0\rangle = \lim_{k\to\infty} \langle z_k, \hat{y}_0\rangle = \lim_{k\to\infty} \langle \tilde{z}_k, y_0\rangle = \lim_{k\to\infty} \langle z_n, y_0\rangle$$
$$= \langle z_n, y_0\rangle = |||z_n||| = |||x|||,$$

which implies $|||x||| \leqq |||Sx|||\, |||\hat{y}_0|||$. However, since $|||\tilde{z}||| \leqq |||z|||$ for all $z \in J$, identity (11) implies $|||\hat{y}_0||| \leqq |||y_0||| = 1$, and we have $|||x||| \leqq |||Sx|||$. This completes the proof of theorem.

Note that Corollary 3 can be shown by using Theorem 3.

The following is a consequence of Theorems 2 and 3.

COROLLARY 4. *For each positive integer* n, *the* n*'th isometric predual of* $J = (J, |||\cdot|||)$ *is uniquely defined (denoted by* J^{-n}*). In fact for each integer* n *(positive or negative)* J^n *is isometrically isomorphic fo* J *(if* n *is even) or* J^* *(if* n *is odd).*

Note that R. C. James [**7**] proved that J and J^* are not even isomorphic.

3. Remarks. 1. If a Banach space X has a unique isometric predual, then it is not necessarily true that X^* has a unique isometric predual. Such an example is l^∞. Since $(l^\infty)^* = l^1 \oplus (c_0)^\perp$ (the l^1-direct sum), the l^1-direct sum of l^1 and $(l^\infty)^*$ is isometrically isomorphic to $(l^\infty)^*$. Therefore $l^\infty \oplus c_0$ (the l^∞-direct sum) is an isometric predual of $(l^\infty)^*$. J. Lindenstrauss [**8**] shows that any infinite dimensional complemented subspace of l^∞ is isomorphic to l^∞. Consequently $l^\infty \oplus c_0$ is not even isomorphic to l^∞.

2. As we have noticed in the introduction, a Banach space X has an isometric predual if and only if X has a w^*-closed norm 1 complement in X^{**}. Suppose A is such a complement then the annihilator $A_\perp$ of A in X^* is an isometric predual of X. Thus, in order to prove that X has a unique isometric predual it is necessary to show that if B is another w^*-closed norm 1 complement of X in X^{**}, then $A_\perp$ is isometrically isomorphic to $B_\perp$. Therefore uniqueness of w^*-closed norm 1 complement of X in X^{**} appears to be stronger than unique-

ness of isometric preduals of X. Every known proof of uniqueness of isometric preduals, as well as ours, shows this stronger condition.

Addendum. In a recent paper, G. Godefroy (Espaces de Banach: Existence et unicité de certains préduax, Ann. Inst. Fourier, Grenoble, *28*, no. 3 (1978), 87–105) showed that if the dual X^* of a Banach space X does not contain isomorphic copy of l^1 then X is the unique isometric predual of X^*. Thus any quasireflexive Banach space X (dim $X^{**}/X < +\infty$) has a unique isometric predual or no isometric predual. Our proofs for J^n in this paper are direct and use elementary properties of the norm of J. They also suggest a new notion of unique predual which will be expanded upon in a subsequent paper.

References

1. T. Ando, *Uniqueness of predual of H^∞* (to appear).

2. M. M. Day, *Normed linear spaces* (third ed., New York, 1973).

3. A. Grothendieck, *Une caracterisation vectorielle-metrique de espaces L^1*, Can. J. Math. *7* (1955), 552–561.

4. T. Ito, *On Banach spaces with unique isometric preduals*, Mich. Math. J. *24* (1977), 321–324.

5. R. C. James, *Bases and reflexivity of Banach spaces*, Ann. of Math. *52* (1950), 518–527.

6. ——— *A non-reflexive Banach space isometric with its second conjugate space*, Proc. Nat. Acad. Sci. U.S.A. *37* (1951), 174–177.

7. ——— *Banach spaces quasi-reflexive of order one*, Studia Math. *60* (1977), 157–177.

8. J. Lindenstrauss, *On complemented subspace of m*, Israel J. Math. *5* (1967), 153–156.

9. S. Sakai, *C*-algebras and W*-algebras* (Springer-Verlag, Berlin, 1971).

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