APPROXIMATION OF UNIFORM TRANSPORT PROCESS ON A FINITE INTERVAL TO BROWNIAN MOTION

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§ 1. Introduction.

Let us consider a finite closed interval $[-a, a]$ which will be thought of as being a medium capable of transporting particles. These particles may move only to the right or to the left with the constant speed c, and each particle changes the moving-direction during the time Δ with probability $k\Delta + o(\Delta)$. If a right- (left-) moving particle hits the boundary point a $(-a)$, then either it turns to the left (right) with probability $1 - q_1(1 - q_{-1})$ or dies with probability $q_1(q_{-1})$. The particle, changed the moving-direction, starts afresh from that position. Now, let $x(t)$ be the coordinate of the particle at time t and let $\theta(t)$ be 1 or -1 according as the moving-direction at time t is right or left. Then $\mathbf{X}(t) = (x(t),\ \theta(t))$ can be considered as a Markov process over the state space $\boldsymbol{S} = \{(x, \theta)$ $-a \leqq x \leqq a,\ \theta = \pm 1\}$. For short, we shall call it *a uniform transport process* over the interval $[-a, a]$. we shall give the precise definition in § 2.

Let $\boldsymbol{T}_t$ be the semigroup corresponding to the Markov process $\boldsymbol{X}(t)$ $= (x(t),\ \theta(t))$. Then $u(t, x, \theta) = \boldsymbol{T}_t f(x, \theta)$ will be the solution of the following differential equation:

$$(\boldsymbol{T})\quad \begin{cases} \dfrac{\partial}{\partial t}u(t,x,1) = c\dfrac{\partial}{\partial x}u(t,x,1) - ku(t,x,1) + ku(t,x,-1) \\ \dfrac{\partial}{\partial t}u(t,x,-1) = -c\dfrac{\partial}{\partial x}u(t,x,-1) - ku(t,x,-1) + ku(t,x,1) \\ u(t,a,1) = (1-q_1)u(t,a,-1) \\ u(t,-a,-1) = (1-q_{-1})u(t,a,1) \\ u(t,x,\theta) \to f(x,\theta) \ \text{as}\ t \to 0 \end{cases}$$

At this stage, choose constants c, k, q_1, q_{-1} so that

$$(\boldsymbol{C}_1)\qquad c^2/k = 1$$

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and

$$(C_2) \qquad q_1 = c\sigma_1/(k + c\sigma_1), \quad q_{-1} = c\sigma_{-1}/k$$

hold. Under these conditions it is not hard to show that ($\boldsymbol{T}$) turns out to be

$$(\boldsymbol{T}') \quad \begin{cases} \dfrac{\partial}{\partial t}u(t,x,1) + \dfrac{1}{2c^2}\dfrac{\partial^2}{\partial t^2}u(t,x,1) = \dfrac{1}{2}\dfrac{\partial^2}{\partial x^2}u(t,x,1) \\ \dfrac{\partial}{\partial x}u(t,a,1) + \sigma_1 u(t,a,1) - \dfrac{1}{c}\dfrac{\partial}{\partial t}u(t,a,1) = 0 \\ \dfrac{\partial}{\partial x}u(t,-a,1) - \sigma_{-1}u(t,-a,1) - \dfrac{1}{c}\dfrac{\partial}{\partial t}u(t,-a,1) = 0 \\ u(t,x,1) \to f(x,1) \text{ as } t \to 0 \end{cases}$$

Here let us consider the limit of ($\boldsymbol{T}'$) formally when the speed c grows indefinitely. Then we have

$$(B) \quad \begin{cases} \dfrac{\partial}{\partial t}u(t,x,1) = \dfrac{1}{2}\dfrac{\partial^2}{\partial x^2}u(t,x,1) \\ \dfrac{\partial}{\partial x}u(t,a,1) + \sigma_1 u(t,a,1) = 0 \\ \dfrac{\partial}{\partial x}u(t,-a,1) - \sigma_{-1}u(t,-a,1) = 0 \\ u(t,x,1) \to f(x,1) \text{ as } t \to 0 \end{cases}$$

This suggests us that the solution $u(t,x,1)$ of ($\boldsymbol{T}$) converges to the solution $u^B(t,x,1)$ of ($\boldsymbol{B}$) as $c \to \infty$, that is, $u(t,x,1)$ approximates to $u^B(t,x,1)$. We can therefore propose also the convergence of the uniform transport process to Brownian motion on a finite interval, as is shown by Ikeda and Nomoto [4] in the case of the uniform transport process on the real line.

The purpose of this paper is to investigate the uniform transport process on a finite interval. Although the boundary condition must be present in our case, we shall show that the similar interpretations as in [4] can be given for weak convergence of the process with the boundary conditions.

In Section 2, we shall construct the uniform transport process on a finite interval by the similar way to the one in [3]. In Section 3, we shall find the weak infinitesimal operator for the transport process (Proposition 3. 3) and we shall further determine the explicit form of the resolvent (Theorem 3. 1). In Section 4, by using the result of § 3, we shall show that the resolvent converges to that of Brownian motion as $c \to \infty$ under the conditions (C_1) and (C_2) and prove that the corresponding semigroup also approximates to the semigroup of Brownian motion. In Section 5, we shall prove that the semigroup of our transport process can approach to that of the Brownian motion with absorbing barries by changing the condition (C_2). Finally we shall show that our method can be applied to the case where the state space of the transport process is the entire line.

Appendix is devoted to the statement of some properties of the infinitesimal operator for the transport process regarding as an operator on the Hilbert space L^2.

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§ 2. Preliminaries.

Let $\mathscr{S}_1$, $\mathscr{S}_{-1}$ be the product spaces $[-a, a) \times \{1\}$ and $(-a, a] \times \{-1\}$ respectively. Let S be the topological sum of $\mathscr{S}_1$, $\mathscr{S}_{-1}$ and ∂, where ∂ is an extra point. Let $\boldsymbol{x}^\theta = (x^\theta(t), \zeta^\theta, \Omega^\theta, P^\theta_{(x,\theta)}, R^1 \times \{\theta\})$ $(\theta = \pm 1)$ be the uniform motion on $R^1 \times \{\theta\}$ with the velocity $c\theta$ and the killing time ζ^θ subjecting to the exponential distribution:

$$P_{(x,\theta)}(\zeta^\theta > t) = \exp(-kt), \qquad k > 0. \tag{2. 1}$$

Let us denote by $\dot{\boldsymbol{x}}^\theta = (\dot{x}^\theta(t), \dot{\zeta}^\theta, \dot{\Omega}^\theta, \dot{P}^\theta_{(x,\theta)}, \mathscr{S}_\theta)$ the part-process of $\boldsymbol{x}^\theta$ on $\mathscr{S}_\theta$. Let $\dot{\boldsymbol{X}}^\partial = (\dot{x}^\partial(t), \dot{\zeta}^\partial, \dot{\Omega}^\partial, \dot{P}^\partial_\partial, \{\partial\})$ be the Markov process on a single point ∂, i.e.

$$\begin{cases} \dot{\Omega}^\partial = \{\dot{\omega}^\partial\} \\ \dot{\zeta}^\partial(\dot{\omega}^\partial) = +\infty \\ \dot{P}^\partial_\partial(\dot{x}^\partial(t) = \partial \quad \text{for all } t \geq 0) = 1 \end{cases} \tag{2. 2}$$

Now we define a process $\boldsymbol{X}^0 = (X^0(t), \zeta^0, \Omega^0, P^0_x, \boldsymbol{S})$ as follows:

$$\Omega^0 = \dot{\Omega}^1 \cup \dot{\Omega}^{-1} \cup \dot{\Omega}^\partial \tag{2. 3}$$

$$\zeta^0(\omega^0) = \begin{cases} \dot{\zeta}^\theta(\dot{\omega}^\theta) & \text{if } \omega^0 = \dot{\omega}^\theta \in \dot{\Omega}^\theta \\ \dot{\zeta}^\partial(\dot{\omega}^\partial) & \text{if } \omega^0 = \dot{\omega}^\partial \end{cases}$$

$$X^0(t, \omega^0) = \begin{cases} \dot{x}^\theta(t, \dot{\omega}^\theta) & \text{if } \omega^0 = \dot{\omega}^\theta \in \dot{\Omega}^\theta \text{ and } t < \dot{\zeta}^\theta(\dot{\omega}^\theta) \\ \dot{x}^\partial(t, \dot{\omega}^\partial) & \text{if } \omega^0 = \dot{\omega}^\partial \end{cases}$$

$$P^0_{\bar{x}}|\dot{\Omega}^\theta = \dot{P}^\theta_{\bar{x}},$$
$$P^0_{\bar{x}}|\dot{\Omega}^\partial = \dot{P}^\partial_{\bar{x}},$$

where $P^0_{\bar{x}}|A$ denotes the restriction of $P^0_{\bar{x}}$ to A.
Then we easily have

LEMMA 2. 1. *The process* $\boldsymbol{X}^0$ *is a strong Markov process on* $\boldsymbol{S}$.

Now, let $\bar{\boldsymbol{S}} = [-a, a]\times\{1\}\cup[-a, a,]\times\{-1\}\cup\{\partial\}$ and $\boldsymbol{B}(\bar{\boldsymbol{S}})$ be the collection of all of Borel subsets of $\bar{\boldsymbol{S}}$.

Let us define a kernel $\pi(\bar{x}, \bar{\Gamma})$ on $\bar{\boldsymbol{S}}\times\boldsymbol{B}(\bar{\boldsymbol{S}})$ by

$$\text{(2. 4)}\quad \pi(\bar{x},\bar{\Gamma}) = \begin{cases} \chi_{\bar{\Gamma}}((x, -\theta)) & \text{if } \bar{x} = (x,\theta) \neq (a\theta,\theta), -a\leqq x\leqq a, \theta = \pm 1 \\ (1-q_\theta)\chi_{\bar{\Gamma}}((a\theta, -\theta)) + q_\theta\chi_{\bar{\Gamma}}(\partial) & \text{if } \bar{x} = (a\theta,\theta) \\ \chi_{\bar{\Gamma}}(\partial) & \text{if } \bar{x} = \partial, \end{cases}$$

where $0\leqq q_1, q_{-1}\leqq 1$.
If we put for $\omega^0\in\Omega^0$, $\bar{\Gamma}\in\boldsymbol{B}(\bar{\boldsymbol{S}})$,

$$\text{(2. 5)}\qquad \mu(\omega^0,\bar{\Gamma}) = \pi(X^0(\zeta^0(\omega^0)-,\omega^0), \bar{\Gamma}\cap\boldsymbol{S})$$

where $X^0(\zeta^0(\omega^0)-,\omega^0) = \lim_{s\uparrow\zeta^0(\omega^0)} X^0(s,\omega^0)$, then we have

LEMMA 2. 2. *The kernel* μ *on* $\Omega^0\times\boldsymbol{B}(\boldsymbol{S})$, *with* $\boldsymbol{B}(\boldsymbol{S}) = \boldsymbol{B}(\bar{\boldsymbol{S}})|\boldsymbol{S}$, *satisfies the following conditions :*

(i) *for almost all* $\omega^0(P^0_{\bar{x}})$, $\mu(\omega^0,\cdot)$ *is a probability measure.*

(ii) *for any Markov time* $T(\omega^0)$ *of* $\boldsymbol{X}^0$ *and any* $\bar{\Gamma}\in\boldsymbol{B}(\boldsymbol{S})$, *it holds that*

$$P^0_{\bar{x}}[\mu(\omega^0,\bar{\Gamma}) = \mu(\theta_{T(\omega^0)}\omega^0,\bar{\Gamma}), T(\omega^0)<\zeta^0(\omega^0)] = P_{\bar{x}}(T(\omega^0)<\zeta^0(\omega^0)).$$

Proof is evident and is omitted.

Remark. Lemma 2. 2 means that the kernel μ is an instantaneous distribution in the sense of [3].

Now, we shall construct the uniform transport process from the process $\boldsymbol{X}^0$ and by the kernel μ. Let $\boldsymbol{X}^0_n = (X^0_n(t), \zeta^0_n, \Omega^0_n, P^0_{n\bar{x}}, \boldsymbol{S})$, $n = 1,2,\cdots$, be copies of the process $\boldsymbol{X}^0$ and these processes $\boldsymbol{X}^0_n$, $n = 1,2,\cdots$ are assumed to be mutually independent.
Define $\boldsymbol{X} = (X(t), \zeta, \Omega, P_{\bar{x}}, \boldsymbol{S})$ as follows :

$$
(2.6)\quad \begin{cases}
X(t,\omega)=\begin{cases}
\Omega=\prod_{j=1}^{\infty}\Omega_j^0, \qquad \omega=(\omega_1^0,\omega_2^0,\cdots),\quad \omega_j^0\in\Omega_j^0\\
\zeta(\omega)=\sum_{j=1}^{\infty}\zeta_j^0(\omega_j^0)\\
X_1^0(t,\omega_1^0) \qquad \text{if } t<\zeta_1^0(\omega_1^0)\\
X_2^0(t-\zeta_1^0(\omega_1^0),\omega_2^0), \qquad \text{if } \zeta_1^0(\omega_1^0)\leq t<\zeta_1^0(\omega_1^0)+\zeta_2^0(\omega_2^0)\\
\cdots\cdots\cdots\cdots\cdots\cdots\\
X_{n+1}^0(t-\sum_{j=1}^{n}\zeta_j^0(\omega_j^0),\omega_{n+1}^0), \qquad \text{if } \sum_{j=1}^{n}\zeta_j^0(\omega_j^0)\leq t<\sum_{j=1}^{n+1}\zeta_j^0(\omega_j^0)\\
\cdots\cdots\cdots\cdots\cdots\cdots
\end{cases}\\
P_{\bar{x}}(A)=\int_{A_1}\int_{S}P_{1,\bar{x}}^0(d\omega_1^0)\mu(\omega_1^0,d\bar{y}_1)\int_{A_2}\int_{S}P_{2,\bar{y}_1}(d\omega_2^0)\mu(\omega_2^0,d\bar{y}_2)\\
\qquad\cdots\int_{A_n}\int_{S}P_{n,y_{n-1}}^0(d\omega_n^0)\mu(\omega_n^0,S)\\
(A=\prod_{j=1}^{n}A_j\times\prod_{n+1}^{\infty}\Omega_j^0\in\boldsymbol{B}(\Omega)).
\end{cases}
$$

Then we have

THEOREM 2.1. *The process* $\boldsymbol{X}$ *is a strong Markov process on* $\boldsymbol{S}$. *Furthermore, it holds that , for any* $\bar{\Gamma}\in\boldsymbol{B}(\boldsymbol{S})$,

(i) $$P_{\bar{x}}(X(\tau)\in\bar{\Gamma}/X(\tau-))=\pi(X(\tau-),\bar{\Gamma}) \quad \text{(a.e. } P_{\bar{x}}),$$

where τ *is the first jumping time, i.e.*

$$\tau(\omega)=\mathit{inf}\{t>0: X(t,\omega)\neq X(t-,\omega)\}^{1)}$$

(ii) *the subprocess of* $\boldsymbol{X}$, *killed at time* τ, *is stochastically equivalent to* $\boldsymbol{X}^0$.

Proof. It can be proved by the similar way to that of [3] and hence we omit the proof.

DEFINITION 2.1. The process $\boldsymbol{X}$ is called the uniform transport process on $[-a,a]$.

§ 3. Semigroup, Infinitesimal operator and Resolvent.

We first introduce various spaces of functions $f(x,\theta)$ [2] on $\boldsymbol{S}$. Let $\boldsymbol{C}^*(\boldsymbol{S})$ be the totality of bounded functions $f(x,\theta)$ each of which is right or

1) inf $\phi=\infty$, where ϕ denotes the empty set.
2) In the following, we always assume that $f(\partial)=0$

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left continuous in $x(-a \leqq x \leqq a)$ according as $\theta=1$ or -1. Then $\boldsymbol{C}^*(\boldsymbol{S})$ is a Banach space with the uniform norm $\|f\| = \sup_{(x,\theta)\in \boldsymbol{S}} |f(x,\theta)|$.

Let $\boldsymbol{C}(\boldsymbol{S})$ be the subspace of $\boldsymbol{C}^*(\boldsymbol{S})$ of bounded continuous functions and $\boldsymbol{C}_k(\boldsymbol{S})$ be its subspace of continuous functions with support in $(-a,a)\times\{1\} \cup (-a,a)\times\{-1\}$. Denote by $\boldsymbol{C}_1^*(\boldsymbol{S})$ and $\boldsymbol{C}_1(\boldsymbol{S})$ the toality of functions in $\boldsymbol{C}(\boldsymbol{S})$ such that $\frac{d\pm}{dx}f(x,\theta)$ exist and belong to $\boldsymbol{C}^*(\boldsymbol{S})$ and $\boldsymbol{C}(\boldsymbol{S})$ respectively, where $\frac{d\pm}{dx}f(x,\theta)$ denotes the right or left derivatives of $f(x,\theta)$ in x according as $\theta=1$ or -1.

Let $\boldsymbol{T}_t$ be the semigroup corresponding to the uniform transport process, i.e.

$$\boldsymbol{T}_t f(x,\theta) = \boldsymbol{E}_{(x,\theta)}[f(X_t)] = \int_\Omega f(X(t,\omega))P_{(x,\theta)}(d\omega), \tag{3. 1}$$

where $f(x,\theta)$ is a bounded Borel measurable function on $\boldsymbol{S}$. Then we have

PROPOSITION 3. 1.

(i)
$$\begin{aligned}\boldsymbol{T}_t f(x,\theta) &= f(x+c\theta t,\theta)e^{-kt}\chi\left(t<\left|\frac{a\theta-x}{c}\right|\right)^{3)}\\ &+\int_0^{t\wedge\left|\frac{a\theta-x}{c}\right|}\boldsymbol{T}_{t-s}f(x+c\theta s,-\theta)ke^{-ks}ds\\ &+(1-q_\theta)\boldsymbol{T}_{t-|(a\theta-x)|/c|}f(a\theta,-\theta)\exp(-k|(a\theta-x)/c|)\\ &\times\chi\left(\left|\frac{a\theta-x}{c}\right|\leq t\right)\end{aligned}$$

(ii) $$\lim_{t\downarrow 0}\boldsymbol{T}_t f(x,\theta)=f(x,\theta) \qquad f\in \boldsymbol{C}^*(\boldsymbol{S}),\quad (x,\theta)\in\boldsymbol{S}$$

(iii) $$T_t f\in \boldsymbol{C}^*(\boldsymbol{S}) \qquad f\in\boldsymbol{C}^*(\boldsymbol{S})$$

(iv) $$s-\lim_{t\downarrow 0}\boldsymbol{T}_t f = f^{3)} \qquad f\in\boldsymbol{C}_k(\boldsymbol{S})$$

Proof. From the method of constructing the process $\boldsymbol{X}$, it follows that

$$\begin{aligned}P_{(x,\theta)}(X(\tau-)\in\bar{\Gamma},\tau\in ds)\\ &=\chi_{\bar{\Gamma}}((x+c\theta s,\theta))\chi_{[0,|(a\theta-x)/c|)}(s)ke^{-ks}ds\\ &+\chi_{\bar{\Gamma}}((a\theta,\theta))\delta_{\left|\frac{a\theta-x}{c}\right|}(ds)e^{-ks}\end{aligned} \tag{3. 2}$$

---

3) $\chi\left(t<\left|\frac{a\theta-x}{c}\right|\right)=\chi_A(t,x,\theta)$ where $A=\left\{(t,x,\theta)\,;\,t<\left|\frac{a\theta-x}{c}\right|,\ t\geq 0,\ (x,\theta)\in\boldsymbol{S}\right\}$.

4) $s-\lim_{n\to\infty} f_n = f$ means that $\|f_n - f\|\to 0$ as $n\to\infty$.

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Therefore, by (3. 2) and the strong Markov property, we have

$$
\begin{aligned}
\boldsymbol{T}_t f(x,\theta) &= \boldsymbol{E}_{(x,\theta)}[f(X(t));\ t<\tau] \\
&\quad + \boldsymbol{E}_{(x,\theta)}[\boldsymbol{E}_{x(\tau)} f(X(t-\tau))\ ;\ t\geq\tau] \\
&= \boldsymbol{E}_{(x,\theta)}[f(X(t));\ t<\tau] \\
&\quad + \boldsymbol{E}_{(x,\theta)}[\boldsymbol{E}_{x(\tau)} f(X(t-\tau))\ ;\ t\geq\tau, X(\tau-)\in \boldsymbol{S}] \\
&\quad + \boldsymbol{E}_{(x,\theta)}[\boldsymbol{E}_{x(\tau)} f(X(t-\tau))\ ;\ t\geq\tau, X(\tau-)=(a\theta,\theta)] \\
&= \text{the right hand side of (i)}.
\end{aligned}
\tag{3. 3}
$$

Thus (i) has been proved.

The expression of $\boldsymbol{T}_t f$ in (i) clearly implies (ii).

Let us show that $\boldsymbol{T}_t f(x,\theta)$ is right-continuous on $[-a,a)\times\{1\}$. Since $f\in \boldsymbol{C}^*(\boldsymbol{S})$ and $\chi(t<|(a\theta-x)/c|), \chi(t\geq|(a\theta-x)|/c)$ are both right-continuous, the first and the third term in the right hand side of (i) are bounded and right-continuous in (x,θ). So we need to consider only the second term in (i). Let $x_0<x_1$.

Then we have

$$
\begin{aligned}
&\int_0^{t\wedge(a-x_1)/c} \boldsymbol{T}_{t-s} f(x_1+cs,-1)ke^{-ks}ds - \int_0^{t\wedge(a-x_0)/c} \boldsymbol{T}_{t-s} f(x_0+cs,-1)ke^{-ks}ds \\
&= \int_0^{t\wedge(a-x_1)/c} \boldsymbol{T}_{t-s} f(x_0+c(s+(x_1-x_0)/c),-1)ke^{-ks}ds \\
&- \int_0^{t\wedge(a-x_0)/c} \boldsymbol{T}_{t-s} f(x_0+cs,-1)ke^{-ks}ds = \Big\{\int_{t\wedge(a-x_0)/c}^{[t\wedge(a-x_1)/c]+(x_1-x_0)/c} \\
&- \int_0^{(x_1-x_0)/c}\Big\}(\boldsymbol{T}_{t-s+(x_1-x_0)/c} f(x_0+cs,-1)k[\exp\{-k(s-(x_1-x_0)/c)\}]ds \\
&+ \int_0^{t\wedge(a-x_0)/c} \boldsymbol{T}_{t-s+(x_1-x_0)/c} f(x_0+cs,-1)\times k[\exp -k(s-(x_1-x_0)/c) - \exp(-ks)]ds \\
&+ \int_0^{t\wedge(a-x_0)/c} \boldsymbol{T}_{t-s}[\boldsymbol{T}_{(x_1-x_0)/c} f(x_0+cs,-1) - f(x_0+cs,-1)]ke^{-ks}ds
\end{aligned}
\tag{3. 4}
$$

Now (ii) and Lebesgue's convergence theorem show that the last term in the right hand side of (3. 4) converges to zero as $x_1\to x_0$. Clearly, the other terms are of order of $0(1)\times\left|\dfrac{x_1-x_0}{c}\right|$. Therefore, the second term in (i) is also right-continuous. Thus (iii) has been proved.

Since $f\in \boldsymbol{C}_k(\boldsymbol{S})$, there exist x_1 and x_2 such that $f(x,\theta)$ is zero on $\boldsymbol{S}-([-a,x_2]\times\{1\}\cup[x_1,a]\times\{-1\})$, and hence we can prove (iv) by use of the expression (i).

Thus we have proved the proposition.

Let $C^*(S)_0 = \{f \in C^*(S) : w - \lim_{t \downarrow 0} T_t f = f\}$ [5]. Then we have

COROLLARY.

$$C^*(S)_0 \supset C_k(S)$$

On account of Proposition 3. 1 (iii), T_t may be considered to be a semigroup on $C^*(S)$, and accordingly we can define the resolvent and the infinitesimal operator of T_t as follows.

$$R_\lambda f(x,\theta) = \int_0^\infty e^{-\lambda t} T_t f(x,\theta) dt \qquad f \in C^*(S), \quad \lambda > 0 \tag{3. 5}$$

$$Af = w - \lim_{t \downarrow 0} \frac{T_t f - f}{t} \qquad f \in C^*(S), \tag{3. 6}$$

if the right-hand side exists and belongs to $C^*(S)$. Consider the set

$$D(A) = \{f \in C^*(S) : Af \text{ exists}\} \tag{3. 7}$$

Then, from the general theory on Markov process [6], we have

PROPOSITION 3. 2 *The operator* $\lambda I - A (\lambda > 0)$ *is one-to-one transformation from* $D(A)$ *onto* $C^*(S)_0$ *and* $(\lambda I - A) R_\lambda = I$, *where* I *denotes an identity operator.*

PROPOSITION 3. 3.

(i) *If* $f \in D(A)$, *then both* $f(a-0,1) = \lim_{x \uparrow a} f(x,1)$ *and* $f(-a+0,-1) = \lim_{x \downarrow -a} f(x,-1)$ *exist and they satisfy*

$$\begin{cases} f(a-0,\,1) = (1-q_1) f(a,-1) \\ f(-a+0,-1) = (1-q_{-1}) f(-a,\,1) \end{cases} \tag{3. 8}$$

(ii) *If a function* f *in* $C_1^*(S)$ *satisfies the boundary conditions* (3. 8), *then* f *belongs to* $D(A)$ *and it holds that*

$$Af(x,\theta) = c\theta \frac{d^\pm}{dx} f(x,\theta) - kf(x,\theta) + kf(x,-\theta) \tag{3. 9}$$

where

$$\frac{d^\pm}{dx} f(x,\theta) = \begin{cases} \dfrac{d^+}{dx} f(x,\theta) & \text{if } \theta = 1 \\ \dfrac{d^-}{dx} f(x,\theta) & \text{if } \theta = -1 \end{cases}$$

---

5) $w - \lim_{n \to \infty} f_n = f$ means that $\lim_{n \to \infty} f_n(x,\theta) = f(x,\theta)$ for any (x,θ) and $\sup_n \| f_n - f \| < +\infty$.

6) cf. [1], Theorem 1.7.

Proof. According to Proposition 3. 2., it is enough to prove that (i) holds for $\boldsymbol{R}_\lambda f(x,\theta)$, where $f \in \boldsymbol{C}^*(\boldsymbol{S})_0$. First assume that $\theta = 1$. If $f \in \boldsymbol{C}^*(\boldsymbol{S})$, then Proposition 3. 1 implies that

$$(3.\ 10)\qquad \begin{aligned} \boldsymbol{R}_\lambda f(x,1) &= \int_0^\infty e^{-\lambda t} \boldsymbol{T}_t f(x,1) dt \\ &= \int_0^{(a-x)/c} e^{-\lambda t} f(x+ct,1) e^{-kt} dt \\ &\quad + \int_0^\infty e^{-\lambda t} dt \int_0^{t\wedge(a-x)/c} \boldsymbol{T}_{t-s} f(x+cs,-1) k e^{-ks} ds \\ &\quad + (1-q_1) \int_{(a-x)/c}^\infty e^{-\lambda t} \boldsymbol{T}_{t-(a-x)/c} f(a,1) \exp(-k(a-x)/c) dt. \end{aligned}$$

The first and the second terms in the above expression tend to zero as $x \to a$ and the third term converges to $(1-q_1)\int_0^\infty e^{-\lambda t} \boldsymbol{T}_t f(a,-1) dt$ as $x \to a$. So, $\boldsymbol{R}_\lambda f(a-0,\,1)$ exists and is equal to $(1-q_1)\boldsymbol{R}_\lambda f(a,-1)$ for $f \in \boldsymbol{C}^*(\boldsymbol{S})_0 \subset \boldsymbol{C}^*(\boldsymbol{S})$. When $\theta = -1$, we can similarly show that $\boldsymbol{R}_\lambda f(-a+0,-1)$ exists and $\boldsymbol{R}_\lambda f(-a+0,-1) = (1-q_{-1})\boldsymbol{R}_\lambda f(-a,\,1)$, $f \in \boldsymbol{C}^*(S)$.

Noting the expression of $\boldsymbol{T}_t f$ in Proposition 3. 1 (i), we can show (ii) by the similar way to Prop 3. 1 (ii) and (iii).
Thus the proposition is proved.

For any given $f \in \boldsymbol{C}^*(\boldsymbol{S})_0$, it follows from Proposition 3. 2 that $u(x,\theta,\lambda) = \boldsymbol{R}_\lambda f(x,\theta)$ is the unique solution of

$$(3.\ 11)\qquad \begin{cases} (\lambda \boldsymbol{I} - \boldsymbol{A})u = f \\ u \in \boldsymbol{D}(\boldsymbol{A}) \end{cases}$$

So, by use of Proposition 3. 3., we can see under the assumption $\boldsymbol{R}_\lambda f \in \boldsymbol{C}_1(\boldsymbol{S})$ that the equation (3. 11) can be expressed as follows:

$$(3.\ 12)\qquad \begin{cases} \lambda u(x,1,\lambda) - c\dfrac{d}{dx} u(x,1,\lambda) + ku(x,1,\lambda) - ku(x,\,-1,\lambda) = f(x,1) \\ \lambda u(x,-1,\lambda) + c\dfrac{d}{dx} u(x,-1,\lambda) + ku(x,-1,\lambda) - ku(x,1,\lambda) = f(x,-1) \\ u(a,1,\lambda) = (1-q_1)u(a,-1,\lambda)\ ^{7)} \\ u(-a,-1,\lambda) = (1-q_{-1})u(-a,1,\lambda) \end{cases}$$

Moreover, if we assume that the conditions $(\boldsymbol{C}_1)$ and $(\boldsymbol{C}_2)$ in § 1 are satisfied

---

7) Hereafter we shall denote $u(a-0,1,\lambda)$ and $u(-a+0,\,-1,\lambda)$ by $u(a,1,\lambda)$ and $u(-a,-1,\lambda)$ respectively.

and $f(x,\theta)$ is in $C_1(S)$, then it is not hard to show that $u(x,1,\lambda)$ in (3. 12) is the solution of the following boundary value problem :

(3. 13)
$$\begin{cases} \dfrac{d^2}{dx^2}u(x,1,\lambda)-(2\lambda+\lambda^2/c^2)u(x,1,\lambda) \\ = -\left\{f(x,1)+f(x,-1)+\dfrac{1}{c}\dfrac{d}{dx}f(x,1)+\dfrac{\lambda}{c^2}f(x,1)\right\} \\ \dfrac{d}{dx}u(a,1,\lambda)+\left(\sigma_1-\dfrac{\lambda}{c}\right)u(a,1,\lambda)=-\dfrac{1}{c}f(a,1) \\ \dfrac{d}{dx}u(-a,1,\lambda)-\left(\sigma_{-1}+\dfrac{\lambda}{c}\right)u(-a,1,\lambda)=-\dfrac{1}{c}f(a,1) \end{cases}$$

Therefore, we have the following Theorem.

THEOREM 3. 1. *Assume that the conditions* (C_1) *and* (C_2) *in* §1 *hold. Then for* $f\in C(S)$ *we have*

(3. 14)
$$\begin{aligned} R_\lambda f(x,1) &= A_1(\lambda)e^{\beta x}+A_2(\lambda)e^{-\beta x} \\ &-\frac{1}{2\beta}\left[\int_{-a}^{x}e^{-\beta y}\left\{f(y,1)+f(y,-1)+\frac{\beta}{c}f(y,1)+\frac{\lambda}{c^2}f(y,1)\right\}dy\right]e^{\beta x} \\ &+\frac{1}{2\beta}\left[\int_{-a}^{x}e^{\beta y}\left\{f(y,1)+f(y,-1)-\frac{\beta}{c}f(y,1)+\frac{\lambda}{c^2}f(y,1)\right\}dy\right]e^{-\beta x} \\ &+\frac{1}{2c\beta}f(-a,1)[e^{\beta(x+a)}-e^{-\beta(x+a)}], \end{aligned}$$

where $\beta=(2\lambda+\lambda^2/c^2)^{1/2}$ *and* $A_1(\lambda)$, $A_2(\lambda)$ *are determined by*

(3. 15)
$$\begin{aligned} &\left(\sigma_1+\beta-\frac{\lambda}{c}\right)e^{\beta a}A_1(\lambda)+\left(\sigma_1-\beta-\frac{\lambda}{c}\right)e^{-\beta a}A_2(\lambda) \\ &=\frac{1}{2}\left(1+\frac{\sigma_1}{\beta}-\frac{\lambda}{c\beta}\right)e^{\beta a} \\ &\left[\int_{-a}^{a}e^{-\beta y}\left\{f(y,1)+f(y,-1)+\frac{\beta}{c}f(y,1)+\frac{\lambda}{c^2}f(y,1)\right\}dy\right. \\ &\quad\left.+\left\{e^{-\beta a}f(a,1)-e^{\beta a}f(-a,1)\right\}/c\right]+\frac{1}{2}\left(1-\frac{\sigma_1}{\beta}+\frac{\lambda}{c\beta}\right)e^{-\beta a} \\ &\left[\int_{-a}^{a}e^{\beta y}\left\{f(y,1)+f(y,-1)-\frac{\beta}{c}f(y,1)+\frac{\lambda}{c^2}f(y,1)\right\}dy\right. \\ &\quad\left.+\left\{e^{\beta a}f(a,1)-e^{-\beta a}f(-a,1)\right\}/c\right]-\frac{1}{c}f(a,1). \\ &\left(\sigma_{-1}-\beta+\frac{\lambda}{c}\right)e^{-\beta a}A_1(\lambda)+\left(\sigma_{-1}+\beta+\frac{\lambda}{c}\right)e^{\beta a}A_2(\lambda)=\frac{1}{c}f(a,1). \end{aligned}$$

(3. 16)
$$R_\lambda f(x,1)=\frac{1}{c^2}\left\{(\lambda+c^2)R_\lambda f(x,1)-c\frac{d}{dx}R_\lambda f(x,1)-f(x,1)\right\}.$$

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Proof. Assume that $f \in \boldsymbol{C}_1(\boldsymbol{S})$. Let us denote by $u_0(x, 1, \lambda)$ the solution of (3. 13). Then we can show by elementary calculations that $u_0(x, 1, \lambda)$ is given by the right hand side of (3. 14). Put $u_0(x, -1, \lambda) = \frac{1}{c^2}\Big\{(\lambda + c^2)u_0(x, 1, \lambda) - c\frac{d}{dx}u_0(x, 1, \lambda) - f(x, 1)\Big\}$.

Then $u_0 \in \boldsymbol{C}_1(\boldsymbol{S})$ and is the solution of (3. 12). Therefore, it follows from Prop. 3. 3. (ii) that $u_0 \in \boldsymbol{D}(\boldsymbol{A})$ and $(\lambda - \boldsymbol{A})u_0 = f$. Hence $\boldsymbol{R}_\lambda f(x, \theta) = u_0(x, \lambda, \theta)$.

For $f \in \boldsymbol{C}(\boldsymbol{S})$, we choose a sequence $\{f_n\}$ in $\boldsymbol{C}_1(\boldsymbol{S})$ such that $s - \lim_{n\to\infty} f_n = f$. Denote by u_n the solution of (3. 12) for f_n, then the expression (3. 14) of u_n shows that $u = s\text{-}\lim_{n\to\infty} u_n$ exists and u can be expressed as the same form. On the other hand, since $\boldsymbol{R}_\lambda$ is a bounded operator, $\boldsymbol{R}_\lambda f = s - \lim_{n\to\infty} \boldsymbol{R}_\lambda f_n$. Therefore $\boldsymbol{R}_\lambda f = s - \lim_{n\to\infty} u_n$. This completes the proof.

§ 4. Approximation.

Let $\boldsymbol{X}^B = (X^B(t), \zeta^B, \Omega^B, P^B_{\tilde{x}}, \boldsymbol{S})$ be a Markov process on $\boldsymbol{S}$, whose semigroup $\boldsymbol{T}^B_t f(x, \theta) = u(t, x, \theta)$ is determined by the following equation:

$$(4.\ 1)\qquad \begin{cases} \dfrac{\partial}{\partial t} u(t, x, \theta) = \dfrac{1}{2}\dfrac{\partial^2}{\partial x^2} u(t, x, \theta), & (x, \theta) \in \boldsymbol{S} \\ \dfrac{\partial}{\partial x} u(t, a, \theta) + \sigma_1 u(t, a, \theta) = 0 \\ \dfrac{\partial}{\partial x} u(t, -a, \theta) - \sigma_{-1} u(t, -a, \theta) = 0 \\ \qquad u(t, x, \theta) \to f(x, \theta) \text{ as } t \to 0 \end{cases}$$

In the sequel, we call $\boldsymbol{X}^B$ *the Brownian motion on* $\boldsymbol{S}$. Let $\boldsymbol{R}^B_\lambda$ be the resolvent of the semigroup $\boldsymbol{T}^B_t$. Then, by the similar computation as Theorem 3. 1., we have

PROPOSITION 4. 1. *It holds, for* $f \in \boldsymbol{C}(\boldsymbol{S})$,

$$(4.\ 2)\qquad \boldsymbol{R}_\lambda f(x, \theta) = A^{B(\lambda)}_{1,\theta} e^{\sqrt{2\lambda}\, x} + A^B_{2,\theta}(\lambda) e^{-\sqrt{2\lambda}\, x} - \frac{1}{2\sqrt{2\lambda}}\Big[\int_{-a}^{x} e^{-\sqrt{2\lambda}\, y}\{2f(y, \theta)\}dy\Big]e^{\sqrt{2\lambda}\, x} + \frac{1}{2\sqrt{2\lambda}}\Big[\int_{-a}^{x} e^{\sqrt{2\lambda}\, y}\{2f(y, \theta)\}dy\Big]e^{-\sqrt{2\lambda}\, x}$$

where $A^B_{1,\theta}(\lambda)$ *and* $A^B_{2,\theta}(\lambda)$ *are the solutions of*

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$$
(4.3)\quad \begin{cases} (\sigma_1+\sqrt{2\lambda})e^{\sqrt{2\lambda}\,a}A^B_{1,\theta}(\lambda)+(\sigma_1-\sqrt{2\lambda})e^{-\sqrt{2\lambda}\,a}A^B_{2,\theta}(\lambda) \\ =\frac{1}{2}\Big(1+\frac{\sigma_1}{\sqrt{2\lambda}}\Big)e^{\sqrt{2\lambda}\,a}\int_{-a}^{a}e^{-\sqrt{2\lambda}\,y}\{2f(y,1)\}dy \\ +\frac{1}{2}\Big(1-\frac{\sigma_1}{\sqrt{2\lambda}}\Big)e^{-\sqrt{2\lambda}\,a}\int_{-a}^{a}e^{\sqrt{2\lambda}\,y}\{2f(y,1)\}dy \\ (\sigma_{-1}-\sqrt{2\lambda})e^{-\sqrt{2\lambda}\,a}A^B_{1,\theta}(\lambda)+(\sigma_{-1}+\sqrt{2\lambda})e^{\sqrt{2\lambda}\,a}A^B_{2,\theta}(\lambda)=0 \end{cases}
$$

Let $\boldsymbol{T}_t^c$ and $\boldsymbol{R}_\lambda^c$ denote the semigroup and the resolvent operator corresponding to the uniform transport process with speed c which satisfies the conditions $(\boldsymbol{C}_1)$ and $(\boldsymbol{C}_2)$. Then we have

MAIN THEOREM. *Assume that the conditions $(\boldsymbol{C}_1)$ and $(\boldsymbol{C}_2)$ hold. Then we have*

$$
s-\lim_{c\to\infty}\boldsymbol{T}_t^c f=\boldsymbol{T}_t^B f
$$

for all f in $\boldsymbol{C}(\boldsymbol{S})$ such that $f(x,\theta)=f(x,-\theta)$.

We prepare a few lemmas to prove the main theorem.

LEMMA 4. 1. *It holds that*

$$
s-\lim_{c\to\infty}\boldsymbol{R}_\lambda^c f=\boldsymbol{R}_\lambda^B f
$$

for all f in $\boldsymbol{C}(\boldsymbol{S})$ such that $f(x,\theta)=f(x,-\theta)$.

Proof. Since $f(x,1)+f(x,-1)=2f(x,\theta)$ and $\beta=(2\lambda+\lambda^2/c^2)^{1/2}$ tends to $\sqrt{2\lambda}$ as $c\to\infty$, therefore the coefficients and the constant terms in the right handside of the equation (3. 15) converge to the coefficients and the constant terms of the equation (4. 3), respectively, as $c\to\infty$. Moreover, by using (3. 14), (3. 16) and (4. 2), $\boldsymbol{R}_\lambda^c f(x,\theta)$ converges uniformly to $\boldsymbol{R}_\lambda^B f(x,\theta)$ as $c\to\infty$. This proves the lemma.

LEMMA 4. 2. (*Trotter*) *Let $\boldsymbol{T}(t)$ be a strongly continuous*[8] *contraction semigroups on a Banach space $\boldsymbol{L}$ and $\boldsymbol{R}(\lambda)$ $(\lambda>0)$ be its resolvent. Let also $\mathscr{G}$ be the strong infinitesimal operator of $\boldsymbol{T}(t)$ i.e. $\mathscr{G}=\lambda-(\boldsymbol{R}(\lambda))^{-1}$, $\lambda>0$ $(\boldsymbol{D}(\mathscr{G})=\boldsymbol{R}(\lambda)(\boldsymbol{L}))$*[9][10]. *Then we have, for $f=\boldsymbol{R}(\lambda)^2 g$ $(g\in\boldsymbol{L})$,*

[8] A strong continuity of $\boldsymbol{T}(t)$ on $\boldsymbol{L}$ means that
$$\|\boldsymbol{T}_t f-f\|\to 0 \text{ as } t\to 0 \text{ for all } f\in\boldsymbol{L}.$$

[9] $\mathscr{G}$ and $D(\mathscr{G})$ are independent of λ.

[10] $\boldsymbol{R}(\lambda)(\boldsymbol{L})=\{\boldsymbol{R}(\lambda)f: f\in\boldsymbol{L}\}$.

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(i) $\| \mathscr{G} f \| \leq \frac{2}{\lambda} \| g \|$

(ii) $\| \mathscr{G}^2 f \| \leq 4 \| g \|$

(iii) $\| (r\boldsymbol{R}(r))^{[rt]} f - \boldsymbol{T}(t) f \| \leq \frac{2}{r} (\| \mathscr{G} f \| + t \| \mathscr{G}^2 f \|)$. [11]

Proof is found in [5, Theorem 4. 1. and 5. 1.].

LEMMA 4. 3. *Let $\boldsymbol{T}(t)$, $\boldsymbol{T}_n(t)$ $(n = 1, 2, \cdots)$ be semigroups on a Banach space $\boldsymbol{L}$ and $\boldsymbol{R}(\lambda)$, $\boldsymbol{R}_n(\lambda)$ be the resolvents of $\boldsymbol{T}(t)$ and $\boldsymbol{T}_n(t)$ respectively. Assume that there exists a subspace $\boldsymbol{M}$ such that*

(i) $s - \lim_{t \downarrow 0} \boldsymbol{T}_n(t) f = f$ *for* $f \in \boldsymbol{M}$

(ii) $s - \lim_{t \downarrow 0} \boldsymbol{T}(t) f = f$ *for* $f \in \boldsymbol{M}$

(iii) $s - \lim_{n \to \infty} R_n(\lambda) f = \boldsymbol{R}(\lambda) f$ *for* $f \in \boldsymbol{M} \cup \boldsymbol{R}(\lambda)(\boldsymbol{M}) \cup \boldsymbol{R}^2(\lambda)(\boldsymbol{M})$

Then it holds that

$$s - \lim_{n \to \infty} \boldsymbol{T}_n(t) f = \boldsymbol{T}(t) f \qquad \textit{for } f \in \overline{\boldsymbol{R}(\lambda)^2(\boldsymbol{M})}$$ [12]

Proof. Set

(4. 4) $$\boldsymbol{L}_0 = \{ f \in \boldsymbol{L} : s - \lim_{t \downarrow 0} \boldsymbol{T}(t) f = f \}$$

and

(4. 5) $$\boldsymbol{L}_n = \{ f \in \boldsymbol{L} : s - \lim_{t \downarrow 0} \boldsymbol{T}_n(t) f = f \} \qquad n = 1, \cdots.$$

It is clear from (i) and (ii) that

(4. 6) $$\boldsymbol{M} \subset \boldsymbol{L}_0 \text{ and } \boldsymbol{M} \subset \boldsymbol{L}_n, \qquad n = 1, 2, \cdots.$$

Also, let $\mathscr{G}_0$ and $\mathscr{G}_n$ $(n = 1, 2, \cdots)$ be the strong infinitesimal operators of $\boldsymbol{T}(t)$ and $\boldsymbol{T}_n(t)$ $(n = 1, 2, \cdots)$ respectively. More precisely, let

(4. 7) $$\mathscr{G}_0 = \lambda - (\boldsymbol{R}(\lambda))^{-1}, \quad \boldsymbol{D}(\mathscr{G}_0) = \boldsymbol{R}(\lambda)(\boldsymbol{L}_0)$$

and

(4. 8) $$\mathscr{G}_n = \lambda - (\boldsymbol{R}_n(\lambda))^{-1}, \quad \boldsymbol{D}(\mathscr{G}_n) = \boldsymbol{R}_n(\lambda) \ (\boldsymbol{L}_n).$$

---

[11] [ ] is the Gauss symbol.

[12] $\overline{\boldsymbol{R}(\lambda)^2(\boldsymbol{M})}$ is a closure of a set $\{ f : f = \boldsymbol{R}(\lambda)^2 g, \ g \in \boldsymbol{M} \}$.

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Now, we put $f=\boldsymbol{R}(\lambda)^2 g$ and $f_n=\boldsymbol{R}_n(\lambda)^2 g$ for an arbitrary but fixed $g\in\boldsymbol{M}$. Then we have by Lemma 4. 2.,

(4. 9)
$$\begin{aligned}
&\|\boldsymbol{T}(t)f-\boldsymbol{T}_n(t)f\|\\
&\leq\|\boldsymbol{T}(t)f-r\boldsymbol{R}(r)^{[rt]}f\|+\|(r\boldsymbol{R}(r))^{[rt]}f-(r\boldsymbol{R}_n(r))^{[rt]}f\|\\
&+\|(r\boldsymbol{R}_n(r))^{[rt]}f-(r\boldsymbol{R}_n(r))^{[rt]}f_n\|+\|(r\boldsymbol{R}_n(r))^{[rt]}f_n-\boldsymbol{T}_n(t)f_n\|\\
&+\|\boldsymbol{T}_n(t)f_n-\boldsymbol{T}_n(t)f\|\\
&\leq\frac{2}{r}(\|\mathscr{G}_0 f\|+t\|\mathscr{G}_0{}^2 f\|)+\|(r\boldsymbol{R}(r))^{[rt]}f-(r\boldsymbol{R}_n(r))^{[rt]}f\|\\
&+\|f-f_n\|+\frac{2}{r}(\|\mathscr{G}_n f_n\|+t\|\mathscr{G}_n^2 f_n\|)\\
&+\|f_n-f\|\\
&\leq\frac{2}{r}\Big(\frac{2}{\lambda}\|g\|+4t\|g\|\Big)+\|(r\boldsymbol{R}(r))^{[rt]}f-(r\boldsymbol{R}_n(r))^{[rt]}f\|\\
&+\|f_n-f\|+\frac{2}{r}\Big(\frac{2}{\lambda}\|g\|+4t\|g\|\Big)\\
&+\|f_n-f\|
\end{aligned}$$

By the condition (iii), the second, the third and the last terms go to the zero as $n\to\infty$. Hence

(4. 10)
$$\lim_{n\to\infty}-\sup\|\boldsymbol{T}(t)f-\boldsymbol{T}_n(t)f\|\leq\frac{4}{r}\Big(\frac{2}{\lambda}+4t\Big)\|g\|.$$

so that, letting $r\to\infty$, we get

(4. 11)
$$\boldsymbol{T}(t)f=s-\lim_{n\to\infty}\boldsymbol{T}_n(t)f.$$

Since $\boldsymbol{T}(t)$ and $\boldsymbol{T}_n(t)$ are both continuous operators, we can now easily conclude that (4. 11) holds for any f in $\overline{(\boldsymbol{R}(\lambda))^2(\boldsymbol{M})}$.
Thus the lemma has been proved.

Proof of Main Theorem. Denote by $\boldsymbol{P}$ the operaor :

(4. 12)
$$\boldsymbol{P}f(x,\theta)=\frac{1}{2}\{f(x,\theta)+f(x,-\theta)\}.$$

Take $\boldsymbol{P}(\boldsymbol{C}^*(\boldsymbol{S}))$, $\boldsymbol{P}(\boldsymbol{C}_k(\boldsymbol{S}))$, $\boldsymbol{T}_t^c$ and $\boldsymbol{T}_t^B$ for $\boldsymbol{L},\boldsymbol{M},\boldsymbol{T}_n(t)$ and $\boldsymbol{T}(t)$ in Lemma 4. 3. respectively. Noting the fact that $(\boldsymbol{R}_\lambda^B)^2(\boldsymbol{C}_k(\boldsymbol{S}))$ is dense in $\boldsymbol{C}(\boldsymbol{S})$, we can see by Proposition 3. 1. (iv), Proposition 4. 1. and the well-known properties of Brownian motion that the hypotheses (i), (ii) and (iii) in Lemma 4. 3. hold. So Lemma 4. 3. implies Main Theorem.

§ 5. Supplements.

We now replace the condition (C_2) by

$$(C_2') \qquad q_1 = 1 \qquad q_{-1} = 1$$

After an analogous consideration, we have

THEOREM 5. 1. *Assume the conditions* (C_1) *and* (C_2'). *Then it holds that, for* $f \in C(S)$ *such that* $f(x,\theta) = f(x,-\theta)$,

$$s - \lim_{c \to \infty} T_t^c f = T_t^{B_0} f$$

where $T_t^{B_0}$ *is the semigroup of the Brownian motion on* S *with absorbing barriers at* a *and* $-a$.

Let us consider transport process on the real line $R^1 = (-\infty, \infty)$. which has been studied in [4]. By the similar considerations as in § 4, we can obtain, for $f \in C_k(S) \cap C_1(S)$,

$$(5.\ 1) \qquad R_\lambda^c f(x,1) = \frac{1}{2\beta}\int_{-\infty}^{\infty} e^{-\beta|x-y|}\Big\{f(y,1) + f(y,-1) + \frac{\lambda}{k} f(y,1) - \frac{\theta}{c}\frac{d}{dy} f(y,\theta)\Big\}dy$$

where $$S = (-\infty,\infty) \times \{1\} \cup (-\infty,\infty) \times \{-1\}$$

and $$\beta = (2\lambda + \lambda^2/c^2)^{1/2}$$

On the other hand, the resolvent of Brownian motion on S is represented by

$$(5.\ 2) \qquad R_\lambda^B f(x,\theta) = \frac{1}{2\sqrt{2\lambda}}\int_{-\infty}^{\infty} e^{-\sqrt{2\lambda}|x-y|}\{2f(y,\theta)\}dy.$$

Therefore we can get a new proof of Theorem 4. 1. in [4].

THEOREM 5. 2. *Assume that* $c^2/k = 1$. *Then it holds that*

$$s - \lim_{c \to \infty} T_t^c f = T_t^B f$$

for all f *in* $C_k(S)$ *such that* $f(x,\theta) = f(x,-\theta)$.

APPENDIX

I. Semigroup.

Let S be the space $S = [-a,a] \times \{1\} \cup [-a,a] \times \{-1\}$ and let $L^2(S)$

be the Hilbert space consisting of all complex-valued functions on S with the inner product :

$$(f, g) = \sum_{\theta=1,-1} \int_{-a}^{a} f(x,\theta)\overline{g(x,\theta)}dx.$$

Denote by $D(A)$ the set of all functions $u(x,\theta)$ with the following properties :

(i) $u \in L^2(S) \cap C(S)$ [1]

(ii) $\frac{d}{dx}u(x,\theta)$ [2] exists and belongs to $L^2(S)$

(iii) $u(a,1) = (1-q_1)u(a,-1),\ u(-a,-1) = (1-q_{-1})u(-a,1),\ 0 \leq q_1,\ q_{-1} \leq 1$

Define the operator A with the domain $D(A)$ by the formula :

(I. 1) $$Au(x,\theta) = c\theta\frac{d}{dx}u(x,\theta) - ku(x,\theta) + ku(x,-\theta), \quad u \in D(A),$$

where k and c are positive constants.

Then we have (cf. [6] chap. 8)

(i) The set $D(A)$ is dense in $L^2(S)$

(ii) The operator A is closed.

(iii) There exists a solution $u \in D(A)$ of $(\lambda - A)u = f$ for any $f \in L^2(S)$.

(iv) $\|(\lambda - A)u\| \geq \|\lambda u\|$ for any $u \in D(A)$, $\lambda > 0$.

Therefore, following the Hille-Yosida's theorem, we obtain

THEOREM A. 1. *The operator A generates a strongly continuous semigroup T_t such that* $Af = s - \lim_{t \downarrow 0} \frac{T_t f - f}{t}$ *for* $f \in D(A)$.

II. Spectrum and Resolvent set

Assume that both the conditions (C_1) and (C_2) hold (cf. § 1). Then the method used in the proof of Theorem 3. 1. can be applied to the eigenvalue problem :

(II. 1) $$(\lambda - A)u = 0, \quad u \in D(A)$$

and to the equation:

---

1) $C(S)$ is the set of all continuous function on S.

2) $\frac{d}{dx}u(x,\theta)$ denotes the Radon-Nikodym derivative of $u(x,\theta)$.

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(II. 2) $$(\lambda - A)u = f, \quad u \in D(A) \text{ for } f \in L^2(S).$$

Thus we can determine the point spectrum $P_\sigma(A)$ and the resolvent set $\rho(A)$ as follows.

1° *Point spectrum* $P_\sigma(A)$ (Eigenvalue)

Case (a). $$2\lambda + \frac{\lambda^2}{c^2} \neq 0:$$

Eigenvalue λ : λ is the solution of

$$(\sigma_1 + \beta - \lambda/c)(\sigma_{-1} + \beta + \lambda/c)e^{2\beta a}$$
$$= (\sigma_1 - \beta - \lambda/c)(\sigma_{-1} - \beta + \lambda/c)e^{-2\beta a}$$

where $$\beta = (2\lambda + \lambda^2/c^2)^{1/2}.$$

Eingenfunction φ_λ *for* λ :
$$\begin{cases} \varphi_\lambda(x, 1) = A_1(\lambda)e^{\beta x} + A_2(\lambda)e^{-\beta x} \\ \varphi_\lambda(x, -1) = \dfrac{1}{k}(\lambda + k - c\beta)A_1(\lambda)e^{\beta x} \\ \qquad\qquad + \dfrac{1}{k}(\lambda + k + c\beta)A_2(\lambda)e^{-\beta x} \end{cases}$$

Case (b). $$2\lambda + \lambda^2/c^2 = 0 \text{ and } \sigma_1 = \sigma_{-1} = 0:$$

Eigenvalue λ : $$\lambda = 0$$

Eigenfunction φ_λ : $$\varphi_\lambda(x, \theta) = A_1(\lambda)$$

Case (c). $$2\lambda + \lambda^2/c^2 = 0 \text{ and}$$
$$(-2c + \sigma_{-1})(2ca + a\sigma_1 + 1) = (2c + \sigma_1)(2ca - a\sigma_{-1} + 1)$$

Eigenvalue λ : $$\lambda = -2k$$

Eigenfunction φ_λ :
$$\begin{cases} \varphi_\lambda(x, 1) = A_1(\lambda)x + A_2(\lambda) \\ \varphi_\lambda(x, 1) = -\left\{A_1(\lambda)x + A_2(\lambda) + \dfrac{c}{k}A_1(\lambda)\right\}. \end{cases}$$

The coefficients $A_1(\lambda)$ and $A_2(\lambda)$ are uniquely determined by the boundary condition and the normalizing condition in each case.

2° *Resolvent set* $\rho(A)$

The equation (II, 2) can be solved for any $\lambda \notin P_\sigma(A)$ and its solution is given by the form (3. 14) in Theorem 3. 1.

Thus we have

THEOREM A. 2. *A spectral plane of A consists of two sets; the point spectrum $P_\sigma(A)$ and the resolvent set $\rho(A)$.*

III. Adjoint of A and Eigenfunction expansion of T_t

Let A^* be the adjoint operator of A. Then

$$A^*u(x,\theta) = -c\,\theta \frac{d}{dx} u(x,\theta) - ku(x,\theta) + ku(x,-\theta).$$

Setting $u^*(x,\theta) = \overline{u(x,-\theta)}$ and observing the expression of φ_λ in II. 1°, we have

PROPOSITION A. 3. *If λ is the eigenvalue of A, then φ_λ is the eigenfunction of A^* for eigenvalue $\bar{\lambda}$.*
Moreover,

$$\begin{aligned} (\varphi_\lambda, \varphi_\mu^*) &= 0 \quad \text{if } \lambda \neq \mu \\ &\neq 0 \quad \text{if } \lambda = \mu \end{aligned}$$

We normalize φ_λ so that $(\varphi_\lambda, \varphi_\lambda^*) = 1$ holds and again we denote it by the same symbol φ_λ. Then we have

PROPOSITION A. 4. *If $f = \sum_{\lambda \in P_\sigma(A)} C_\lambda \varphi_\lambda$ in $L^2(S)$, C_λ being constants, then*

$$T_t f = \sum_{\lambda \in P_\sigma(A)} C_\lambda e^{\lambda t} \varphi_\lambda.$$

Finally, it is desirable to express it in the form

$$T_t f = \sum_{\lambda \in P_\sigma(A)} (f, \varphi_\lambda^*) e^{\lambda t} \varphi_\lambda$$

for all f in $L^2(S)$, but it remains unsolved.

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