ON THE PROPERTIES OF AN ENTIRE FUNCTION OF TWO COMPLEX VARIABLES

ARUN KUMAR AGARWAL

1. Let

$$(1.1) \qquad f(z_1, z_2) = \sum_{m_1, m_2 = 0}^{\infty} a_{m_1 m_2} z_1^{m_1} z_2^{m_2}$$

be an entire function of two complex variables z_1 and z_2, holomorphic in the closed polydisk $P\ \{|z_j| \leqslant r_j,\ j = 1, 2\}$. Let

$$M(r_1, r_2) = M(r_1, r_2; f) = \max_{|z_j| \leqslant r_j} |f(z_1, z_2)|, \qquad j = 1, 2.$$

Following S. K. Bose (**1**, pp. 214–215), $\mu(r_1, r_2; f)$ denotes the maximum term in the double series (1.1) for given values of r_1 and r_2 and $\nu_1(m_2; r_1, r_2)$ or $\nu_1(r_1, r_2)$, r_2 fixed, $\nu_2(m_1; r_1, r_2)$ or $\nu_2(r_1, r_2)$, r_1 fixed and $\nu(r_1, r_2)$ denote the ranks of the maximum term of the double series (1.1). Let us write

$$(1.2) \qquad I_\delta(r_1, r_2; f) = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} |f(r_1 e^{i\theta_1}, r_2 e^{i\theta_2})|^\delta d\theta_1\, d\theta_2$$

and

$$(1.3) \qquad \mathfrak{M}_{\delta, k_1, k_2}(r_1, r_2; f) = \frac{1}{r_1^{k_1+1} r_2^{k_2+1}} \int_0^{r_1} \int_0^{r_2} M_\delta(x_1, x_2; f) x_1^{k_1} x_2^{k_2} dx_1\, dx_2$$

where $\delta \geqslant 1$ and $-1 < k_1, k_2 < \infty$, and

$$M_\delta(r_1, r_2; f) = \{I_\delta(r_1, r_2; f)\}^{1/\delta}.$$

Then $I_\delta(r_1, r_2; f)$ is an increasing function of r_1 and r_2 when one remains fixed and the other increases or both increase. The finite order ρ of an entire function $f(z_1, z_2)$ is defined as (**1**, p. 219)

$$(1.4) \qquad \limsup_{r_1, r_2 \to \infty} \left\{ \frac{\log \log M(r_1, r_2; f)}{\log (r_1 r_2)} \right\} = \rho.$$

Similarly, we can define the lower order λ of $f(z_1, z_2)$ as

$$(1.5) \qquad \liminf_{r_1, r_2 \to \infty} \left\{ \frac{\log \log M(r_1, r_2; f)}{\log (r_1 r_2)} \right\} = \lambda.$$

In this paper we have deduced an asymptotic relation between the two mean values defined by (1.2) and (1.3) and a number of results connecting $M(r_1, r_2; f)$ and $\mu(r_1, r_2; f)$ and $\nu(r_1, r_2; f)$ and the coefficients $a_{m_1 m_2}$.

Received June 1, 1966. This research has been supported by a Junior Research Fellowship, an award of the Council of Scientific and Industrial Research, New Delhi, India. The author is grateful to the referee for his helpful suggestions.

2. THEOREM 1. *Let $f(z_1, z_2)$ be an entire function of finite non-zero order ρ and lower order λ $(2\lambda < \rho)$*; then

$$\log \mathfrak{M}_{\delta,k_1,k_2}(r_1, r_2; f) \sim \log M_\delta(r_1, r_2; f), \tag{2.1}$$

where $\delta \geqslant 1$ and $-1 < k_1, k_2 < \infty$, and $f(0,0) \neq 0$.

Proof. We have

$$\mathfrak{M}_{\delta,k_1,k_2}(r_1, r_2; f) = \frac{1}{r_1^{k_1+1} r_2^{k_2+1}} \int_0^{r_1} \int_0^{r_2} M_\delta(x_1, x_2; f) x_1^{k_1} x_2^{k_2} dx_1\, dx_2 \leqslant \frac{M_\delta(r_1, r_2; f)}{(k_1+1)(k_2+1)}. \tag{2.2}$$

Further, for $0 < r_i < R_i < 2^{1/(\rho+\epsilon)} r_i$ $(i = 1, 2)$,

$$\mathfrak{M}_{\delta,k_1,k_2}(R_1, R_2; f) > \frac{1}{R_1^{k_1+1} R_2^{k_2+1}} \int_{r_1}^{R_1} \int_{r_2}^{R_2} M_\delta(x_1, x_2; f) x_1^{k_1} x_2^{k_2} dx_1\, dx_2 \geqslant \frac{M_\delta(r_1, r_2; f)}{(k_1+1)(k_2+1)} \frac{(R_1^{k_1+1} - r_1^{k_1+1})}{R_1^{k_1+1}} \frac{(R_2^{k_2+1} - r_2^{k_2+1})}{R_2^{k_2+1}}. \tag{2.3}$$

From (2.2) and (2.3), we get that

$$\mathfrak{M}_{\delta,k_1,k_2}(r_1, r_2; f) \leqslant \frac{M_\delta(r_1, r_2; f)}{(k_1+1)(k_2+1)} \leqslant \mathfrak{M}_{\delta,k_1,k_2}(R_1, R_2; f) \frac{R_1^{k_1+1}}{(R_1^{k_1+1} - r_1^{k_1+1})} \frac{R_2^{k_2+1}}{(R_2^{k_2+1} - r_2^{k_2+1})}. \tag{2.4}$$

Since

$$\frac{\partial^2}{\partial R_1\, \partial R_2} \{\log(R_1^{k_1+1} R_2^{k_2+1} \mathfrak{M}_{\delta,k_1,k_2}(R_1, R_2; f))\} = \frac{\partial}{\partial R_1} \left\{ \frac{R_2^{k_2} \int_0^{R_1} x_1^{k_1} M_\delta(x_1, R_2; f) dx_1}{R_1^{k_1+1} R_2^{k_2+1} \mathfrak{M}_{\delta,k_1,k_2}(R_1, R_2; f)} \right\}$$

$$= \frac{R_1^{k_1} R_2^{k_2} M_\delta(R_1, R_2; f) \{R_1^{k_1+1} R_2^{k_2+1} \mathfrak{M}_{\delta,k_1,k_2}(R_1, R_2; f)\}}{\{R_1^{k_1+1} R_2^{k_2+1} \mathfrak{M}_{\delta,k_1,k_2}(R_1, R_2; f)\}^2} - \frac{\left\{R_1^{k_1} \int_0^{R_2} x_2^{k_2} M_\delta(R_1, x_2; f) dx_2\right\} \left\{R_2^{k_2} \int_0^{R_1} x_1^{k_1} M_\delta(x_1, R_2; f) dx_1\right\}}{\{R_1^{k_1+1} R_2^{k_2+1} \mathfrak{M}_{\delta,k_1,k_2}(R_1, R_2; f)\}^2}$$

$$\leqslant \frac{M_\delta(R_1, R_2; f)}{\mathfrak{M}_{\delta,k_1,k_2}(R_1, R_2; f)} \frac{1}{R_1 R_2},$$

we have

$$\log\{R_1^{k_1+1} R_2^{k_2+1} \mathfrak{M}_{\delta,k_1,k_2}(R_1, R_2; f)\} \leqslant \int_0^{R_1} \int_0^{R_2} \frac{M_\delta(x_1, x_2; f)}{\mathfrak{M}_{\delta,k_1,k_2}(x_1, x_2; f)} \frac{dx_1}{x_1} \frac{dx_2}{x_2}$$

$$= \int_0^{r_1} \int_0^{r_2} \frac{M_\delta(x_1, x_2; f)}{\mathfrak{M}_{\delta,k_1,k_2}(x_1, x_2; f)} \frac{dx_1}{x_1} \frac{dx_2}{x_2} + \left[\int_0^{r_1} \int_{r_2}^{R_2} + \int_{r_1}^{R_1} \int_0^{r_2} + \int_{r_1}^{R_1} \int_{r_2}^{R_2} \right] \frac{M_\delta(x_1, x_2; f)}{\mathfrak{M}_{\delta,k_1,k_2}(x_1, x_2; f)} \frac{dx_1}{x_1} \frac{dx_2}{x_2},$$

or

$$(2.5)\quad \log\left\{\frac{R_1^{k_1+1}R_2^{k_2+1}\mathfrak{M}_{\delta,k_1,k_2}(R_1, R_2; f)}{r_1^{k_1+1}r_2^{k_2+1}\mathfrak{M}_{\delta,k_1,k_2}(r_1, r_2; f)}\right\}$$

$$\leqslant \left[\int_0^{r_1}\int_{r_2}^{R_2} + \int_{r_1}^{R_1}\int_0^{r_2} + \int_{r_1}^{R_1}\int_{r_2}^{R_2}\right] \frac{M_\delta(x_1, x_2; f)}{\mathfrak{M}_{\delta,k_1,k_2}(x_1, x_2; f)} \frac{dx_1}{x_1}\frac{dx_2}{x_2}$$

$$= J_1 + J_2 + J_3, \quad \text{say.}$$

For any positive number ϵ, from **(2)**, we have

$$(2.6)\qquad \frac{M_\delta(r_1, r_2; f)}{\mathfrak{M}_{\delta,k_1,k_2}(r_1, r_2; f)} < (r_1 r_2)^{\rho+\epsilon},$$

for sufficiently large r_1 and r_2. Also

$$(2.7)\qquad \log\left(\frac{R_i}{r_i}\right) = \log\left(1 + \frac{R_i - r_i}{r_i}\right) < \frac{R_i - r_i}{r_i} \qquad \text{for } i = 1, 2.$$

Let us choose R_i such that

$$(2.8)\qquad (R_i - r_i)/r_i = r_i^{-\rho-\epsilon} \qquad \text{for } i = 1, 2.$$

Then

$$J_1 = \int_0^{r_1}\int_{r_2}^{R_2} \frac{M_\delta(x_1, x_2; f)}{\mathfrak{M}_{\delta,k_1,k_2}(x_1, x_2; f)} \frac{dx_1}{x_1}\frac{dx_2}{x_2} < \log\left(\frac{R_2}{r_2}\right)\int_0^{r_1}(x_1 R_2)^{\rho+\epsilon}\frac{dx_1}{x_1}$$

from (2.6)

$$< \left(\frac{R_2}{r_2}\right)^{\rho+\epsilon}\frac{r_1^{\rho+\epsilon}}{(\rho+\epsilon)}$$

$$< 2\frac{r_1^{\rho+\epsilon}}{(\rho+\epsilon)},$$

since $r_i < R_i < 2^{1/(\rho+\epsilon)}r_i$ for $i = 1, 2$. Similarly,

$$J_2 < 2r_2^{\rho+\epsilon}/(\rho+\epsilon),$$

and

$$J_3 = \int_{r_1}^{R_1}\int_{r_2}^{R_2} \frac{M_\delta(x_1, x_2; f)}{\mathfrak{M}_{\delta,k_1,k_2}(x_1, x_2; f)} \frac{dx_1}{x_1}\frac{dx_2}{x_2}$$

$$< (R_1 R_2)^{\rho+\epsilon}\left(\frac{R_1 - r_1}{r_1}\right)\left(\frac{R_2 - r_2}{r_2}\right) < 4,$$

from (2.8) and for $r_i < R_i < 2^{1/(\rho+\epsilon)}r_i$ $(i = 1, 2)$. Hence (2.5) becomes

$$(2.9)\quad \log\left\{\frac{R_1^{k_1+1}R_2^{k_2+1}\mathfrak{M}_{\delta,k_1,k_2}(R_1, R_2; f)}{r_1^{k_1+1}r_2^{k_2+1}\mathfrak{M}_{\delta,k_1,k_2}(r_1, r_2; f)}\right\} < 2\frac{r_1^{\rho+\epsilon}}{(\rho+\epsilon)} + 2\frac{r_2^{\rho+\epsilon}}{(\rho+\epsilon)} + 4.$$

Using (2.9) in (2.4), we get

$$\log M_\delta(r_1, r_2; f) - \log(k_1 + 1) - \log(k_2 + 1)$$

$$< \log\left\{\frac{1}{(R_1^{k_1+1} - r_1^{k_1+1})(R_2^{k_2+1} - r_2^{k_2+1})}\right\}$$

$$+ \log\{r_1^{k_1+1}r_2^{k_2+1}\mathfrak{M}_{\delta,k_1,k_2}(r_1, r_2; f)\} + 2\frac{r_1^{\rho+\epsilon}}{(\rho+\epsilon)} + 2\frac{r_2^{\rho+\epsilon}}{(\rho+\epsilon)} + 4$$

$$= (1 + o(1))\log \mathfrak{M}_{\delta,k_1,k_2}(r_1, r_2; f),$$

for large r_1, r_2 and on using **(2)**

$$\limsup_{r_1, r_2 \to \infty} \left\{ \frac{\log \log \mathfrak{M}_{\delta, k_1, k_2}(r_1, r_2; f)}{\log (r_1 r_2)} \right\} = \rho.$$

Taking the limit, we get

$$\lim_{r_1, r_2 \to \infty} \log M_\delta(r_1, r_2; f) / \log \mathfrak{M}_{\delta, k_1, k_2}(r_1, r_2; f) \leqslant 1,$$

and from (2.4)

$$\lim_{r_1, r_2 \to \infty} \log M_\delta(r_1, r_2; f) / \log \mathfrak{M}_{\delta, k_1, k_2}(r_1, r_2; f) \geqslant 1,$$

which gives the result.

3. THEOREM 2. *Let $f(z_1, z_2)$ be an entire function of order zero and suppose that*

$$\lim_{k \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \left[\frac{\log k}{\log \left\{ \frac{1}{k} \log \frac{1}{|a_{k_1 k_2}|} \right\}} \right] = \begin{matrix} A_2 \\ a_2 \end{matrix}, \tag{3.1}$$

$$\lim_{r_1, r_2 \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \left\{ \frac{\log \log M(r_1, r_2; f)}{\log \log (r_1 r_2)} \right\} = \begin{matrix} P_2 \\ p_2 \end{matrix}, \tag{3.2}$$

then if $1 < A_2 < l < \infty$,

$$P_2 \leqslant \{(l+1)/l\} A_2 \tag{3.3}$$

and a fortiori

$$p_2 < (l+1) \tag{3.4}$$

where $k = k_1 + k_2$.

Proof. We know **(1**, p. 218**)** that

$$M(r_1, r_2) < \mu(r_1, r_2)[3\nu(2r_1, 2r_2) + 3].$$

Taking the logarithms of both the sides, we get

$$\begin{aligned} \log M(r_1, r_2) &< \log \mu(r_1, r_2) + \log \nu(2r_1, 2r_2) + O(1) \\ &= \log \mu(r_1, r_2) + \log \log \mu(2r_1, 2r_2) + o(\log (r_1 r_2)), \end{aligned}$$

from **(1**, (5.1), p. 218 and (5.2), p. 219**)**, for large values of r_1 and r_2. This gives

$$\log M(r_1, r_2) < \log \mu(r_1, r_2) + \log \log \mu(2r_1, 2r_2) + o(\log \mu(r_1, r_2));$$

hence

$$\log M(r_1, r_2) < (1 + o(1)) \log \mu(r_1, r_2). \tag{3.5}$$

For any positive ϵ, there must exist a sufficiently large positive number $k_0(A_2)$ such that

$$|a_{k_1 k_2}| < \exp(-k^{1+1/(A_2+\epsilon)}), \tag{3.6}$$

for every $k > k_0(A_2)$. Let us suppose that

(3.7) $$r_{1,k_1} = \exp\{k_1^{1/A_2} + k_1^{1/l}\}$$

and

(3.7a) $$r_{2,k_2} = \exp\{k_2^{1/A_2} + k_2^{1/l}\},$$

where $A_2 < l < \infty$ and $r_{1,k_1} \leqslant r_1 < r_{1,k_1+1}$, $r_{2,k_2} \leqslant r_2 < r_{2,k_2+1}$. Now, for $r_1 < r_{1,k_1+1}$ and $r_2 < r_{2,k_2+1}$,

(3.7b) $$\begin{aligned} \log \mu(r_1, r_2) &= \log\{|a_{k_1k_2}|r_1^{k_1}r_2^{k_2}\} \\ &< \log\{|a_{k_1k_2}|r_{1,k_1+1}^{k_1}r_{2,k_2+1}^{k_2}\}, \end{aligned}$$

where the maximum term $\mu(r_1, r_2)$ has the position (k_1, k_2) in the square representation of the double series (1.1) of $f(z_1, z_2)$. We obtain

$$\begin{aligned} \log \mu(r_1, r_2) < -(k_1 + k_2)^{1+1/(A_2+\epsilon)} &+ k_1\{(k_1 + 1)^{1/A_2} + (k_1 + 1)^{1/l}\} \\ &+ k_2\{(k_2 + 1)^{1/A_2} + (k_2 + 1)^{1/l}\}, \end{aligned}$$

from (3.6), (3.7), and (3.7a). Hence,

$$\log \mu(r_1, r_2) < (1 + o(1))(k_1 + k_2)^{1+1/l},$$

for k_1, k_2 sufficiently large and $A_2 < l$. Hence for sufficiently large k_1 and k_2,

(3.8) $$\log \log \mu(r_1, r_2) < (1 + o(1))(1 + 1/l)\log k.$$

Also for $r_{1,k_1} \leqslant r_1$ and $r_{2,k_2} \leqslant r_2$,

(3.9) $$\begin{aligned} \log \log(r_1 r_2) &\geqslant \log \log(r_{1,k_1} r_{2,k_2}) \\ &= \log\{k_1^{1/A_2} + k_1^{1/l} + k_2^{1/A_2} + k_2^{1/l}\} \quad \text{from (3.7) and (3.7a)} \\ &\sim \log\{(k_1 + k_2)^{1/A_2}\} \quad \text{for } 1 < A_2 < l \\ &\sim \frac{1}{A_2}\log k. \end{aligned}$$

Hence from (3.5), (3.8), and (3.9), we have

$$\limsup_{r_1, r_2 \to \infty}\left\{\frac{\log \log M(r_1, r_2; f)}{\log \log(r_1 r_2)}\right\} \leqslant \left(\frac{l+1}{l}\right)A.$$

i.e.

$$P_2 \leqslant \left(\frac{l+1}{l}\right)A_2,$$

and *a fortiori*

$$p_2 < (l + 1).$$

THEOREM 3. *Let $f(z_1, z_2)$ be an entire function of order zero and suppose that*

(3.10) $$\lim_{r_1, r_2 \to \infty} \begin{matrix}\sup\\ \inf\end{matrix}\left\{\frac{\log \nu(r_1, r_2; f)}{\log \log(r_1 r_2)}\right\} = \begin{matrix} r \\ \eta \end{matrix},$$

if $1 < \alpha_2 < \beta < \infty$ *(α_2 as defined in Theorem* 2); *then*

$$\gamma \leqslant \{(\beta + 1)/\beta\}\alpha_2 \tag{3.11}$$

and

$$\eta < (\beta + 1). \tag{3.12}$$

Proof. For any positive ϵ, there must exist a sufficiently large $k_0(\alpha_2)$ such that

$$|a_{k_1k_2}| < \exp\{-k^{1+1/(\alpha_2-\epsilon)}\} \quad \text{for every } k > k_0(\alpha_2). \tag{3.13}$$

Let us suppose that

$$r_{1,k_1} = \exp\{k_1^{1/\alpha_2} + k_1^{1/\beta}\} \tag{3.14}$$

and

$$r_{2,k_2} = \exp\{k_2^{1/\alpha_2} + k_2^{1/\beta}\}, \tag{3.14a}$$

where $\alpha_2 < \beta < \infty$ and $r_{1,k_1} \leqslant r_1 < r_{1,k_1+1}$ and $r_{2,k_2} \leqslant r_2 < r_{2,k_2+1}$. From **(1**, (5.4), p. 220**)**, we have

$$\log \nu(r_1, r_2) \leqslant (1 + o(1)) \log \log M(r_1, r_2), \tag{3.14b}$$

for r_1 and r_2 sufficiently large. Using (3.5) in (3.14b), we get

$$\log \nu(r_1, r_2) < (1 + o(1)) \log \log \mu(r_1, r_2). \tag{3.15}$$

Now using (3.13), (3.14) and (3.14a) in (3.7b), we obtain

$$\begin{aligned}\log \mu(r_1, r_2) &< -(k_1 + k_2)^{1+1/(\alpha_2-\epsilon)} + k_1\{(k_1 + 1)^{1/\alpha_2} + (k_1 + 1)^{1/\beta}\} \\ &\qquad + k_2\{(k_2 + 1)^{1/\alpha_2} + (k_2 + 1)^{1/\beta}\} \\ &< (1 + o(1))(k_1 + k_2)^{1+1/\beta},\end{aligned}$$

for $1 < \alpha_2 < \beta$ and k_1, k_2 sufficiently large. Hence

$$\log \log \mu(r_1, r_2) < (1 + o(1))\left(\frac{\beta + 1}{\beta}\right) \log k. \tag{3.16}$$

Also, for $r_{1,k_1} \leqslant r_1$ and $r_{2,k_2} \leqslant r_2$

$$\begin{aligned}\log \log(r_1 r_2) &\geqslant \log \log(r_{1,k_1} r_{2,k_2}) \\ &= \log\{k_1^{1/\alpha_2} + k_1^{1/\beta} + k_2^{1/\alpha_2} + k_2^{1/\beta}\} && \text{from (3.14) and (3.14a)} \\ &\sim (1/\alpha_2)\log k && \text{for } 1 < \alpha_2 < \beta.\end{aligned} \tag{3.17}$$

Hence from (3.15), (3.16), and (3.17), we have

$$\limsup_{r_1, r_2 \to \infty}\left\{\frac{\log \nu(r_1, r_2; f)}{\log \log(r_1 r_2)}\right\} \leqslant \left(\frac{\beta + 1}{\beta}\right) \alpha_2,$$

from which the required result follows.

I take this opportunity to express my thanks to Dr. S. K. Bose for suggesting the problem and for guidance in the preparation of this paper.

References

1. S. K. Bose and Devendra Sharma, *Integral functions of two complex variables*, Compositio Math., *15* (1963), 210–226.

2. G. P. Dikshit and A. K. Agarwal, *On the means of entire functions of several complex variables* (submitted for publication).

3. B. A. Fuks, *Theory of analytic functions of several complex variables*, (Moscow, 1963).

Department of Mathematics and Astronomy,
Lucknow University, Lucknow, India