3

The Distribution of Values of the Riemann Zeta Function, I

Probability tools	Arithmetic tools
Definition of convergence in law (Section B.3)	Dirichlet series (Section A.4)
Kolmogorov's Theorem for random series (Th. B.10.1)	Riemann zeta function (Section C.4)
Weyl Criterion and Kronecker's Theorem (Section B.6, Th. B.6.5)	Fundamental theorem of arithmetic
Menshov–Rademacher Theorem (Th. B.10.5)	Mean square of $\zeta(s)$ outside the critical line (Prop. C.4.1)
Lipschitz test functions (Prop. B.4.1)	Euler product (Lemma C.1.4)
Support of a random series (Prop. B.10.8)	Strong Mertens estimate and Prime Number Theorem (Cor. C.3.4)

3.1 Introduction

The Riemann zeta function is defined first for complex numbers s such that $\mathrm{Re}(s) > 1$, by means of the series

$$\zeta(s) = \sum_{n \geqslant 1} \frac{1}{n^s}.$$

It plays an important role in prime number theory, arising because of the famous Euler product formula, which expresses $\zeta(s)$ as a product over primes, in this region: we have

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1} \tag{3.1}$$

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if $\mathrm{Re}(s) > 1$ (see Corollary C.1.5). By standard properties of series of holomorphic functions (note that $s \mapsto n^s = e^{s \log n}$ is entire for any $n \geqslant 1$), the Riemann zeta function is holomorphic for $\mathrm{Re}(s) > 1$. It is of crucial importance however that it admits an analytic continuation to $\mathbf{C} - \{1\}$, with furthermore a simple pole at $s = 1$ with residue 1.

This analytic continuation can be performed simultaneously with the proof of the *functional equation*: the function defined by

$$\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$$

satisfies

$$\xi(1 - s) = \xi(s)$$

and has simple poles with residue 1 at $s = 0$ and $s = 1$. Since the inverse of the Gamma function is an entire function (Proposition A.3.2), the analytic continuation of the Riemann zeta function follows immediately.

However, for many purposes (including the results of this chapter), it is enough to know that $\zeta(s)$ has analytic continuation for $\mathrm{Re}(s) > 0$, and this can be checked quickly using the following computation, based on summation by parts (Lemma A.1.1): using the notation $\langle x \rangle$ for the fractional part of a real number x, namely, the unique real number in $[0, 1[$ such that $x - \langle x \rangle \in \mathbf{Z}$ for $\mathrm{Re}(s) > 1$,[1] we have

$$\begin{aligned}
\sum_{n \geqslant 1} \frac{1}{n^s} &= s \int_1^{+\infty} \Big(\sum_{1 \leqslant n \leqslant t} 1 \Big) t^{-s-1} dt \\
&= s \int_1^{+\infty} (t - \langle t \rangle) t^{-s-1} dt \\
&= s \int_1^{+\infty} t^{-s} dt - s \int_1^{+\infty} \langle t \rangle t^{-s-1} dt = \frac{s}{s-1} - s \int_1^{+\infty} \langle t \rangle t^{-s-1} dt.
\end{aligned}$$

The rational function $s \mapsto s/(s-1)$ has a simple pole at $s = 1$ with residue 1. Also, since $0 \leqslant \langle t \rangle \leqslant 1$, the integral defining the function

$$s \mapsto s \int_1^{+\infty} \langle t \rangle t^{-s-1} dt$$

is absolutely convergent, and therefore this function is holomorphic, for $\mathrm{Re}(s) > 0$. The expression above then shows that the Riemann zeta function is meromorphic, with a simple pole at $s = 1$ with residue 1, for $\mathrm{Re}(s) > 0$.

[1] A more standard notation would be $\{x\}$, but this clashes with the notation used for set constructions.

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Since $\zeta(s)$ is quite well behaved for $\mathrm{Re}(s) > 1$, and since the Gamma function is a very well-known function, the functional equation $\zeta(1-s) = \zeta(s)$ shows that one can understand the behavior of $\zeta(s)$ for s outside of the *critical strip*

$$\mathrm{S} = \{s \in \mathbf{C} \mid 0 \leqslant \mathrm{Re}(s) \leqslant 1\}.$$

The Riemann Hypothesis is a fundamental (still conjectural) statement about the Riemann zeta function in the critical strip: it states that if $s \in \mathrm{S}$ satisfies $\zeta(s) = 0$, then the real part of s must be $1/2$. Because holomorphic functions (with relatively slow growth, a property true for ζ, although this requires some argument to prove) are essentially characterized by their zeros (just like polynomials are!), the proof of this conjecture would enormously expand our understanding of the properties of the Riemann zeta function. Although it remains open, this should motivate our interest in the distribution of values of the zeta function. Another motivation is that it contains crucial information about primes, which will be very visible in Chapter 5.

We first focus our attention to a vertical line $\mathrm{Re}(s) = \tau$, where τ is a fixed real number such that $\tau \geqslant 1/2$ (the case $\tau \leqslant 1$ will be the most interesting, but some statements do not require this assumption). We consider real numbers $\mathrm{T} \geqslant 2$ and define the probability space $\Omega_{\mathrm{T}} = [-\mathrm{T}, \mathrm{T}]$ with the uniform probability measure $dt/(2\mathrm{T})$. We then view

$$t \mapsto \zeta(\tau + it)$$

as a random variable $\mathsf{Z}_{\tau,\mathrm{T}}$ on $\Omega_{\mathrm{T}} = [-\mathrm{T}, \mathrm{T}]$. These are arithmetically defined random variables. Do they have some specific, interesting, asymptotic behavior?

The answer to this question turns out to depend on τ, as the following first result of Bohr and Jessen reveals:

Theorem 3.1.1 (Bohr–Jessen) *Let $\tau > 1/2$ be a fixed real number. Define $\mathsf{Z}_{\tau,\mathrm{T}}$ as the random variable $t \mapsto \zeta(\tau + it)$ on Ω_{T}. There exists a probability measure μ_τ on $\mathbf{C}$ such that $\mathsf{Z}_{\tau,\mathrm{T}}$ converges in law to μ_τ as $\mathrm{T} \to +\infty$. Moreover, the support of μ_τ is compact if $\tau > 1$, and is equal to $\mathbf{C}$ if $1/2 < \tau \leqslant 1$.*

We will describe precisely the measure μ_τ in Section 3.2: it is a highly nongeneric probability distribution, whose definition (and hence properties) retains a significant amount of arithmetic, in contrast with the Erdős–Kac Theorem, where the limit is a very generic distribution.

Theorem 3.1.1 is in fact a direct consequence of a result due to Voronin [119] and Bagchi [4], which extends it in a very surprising direction. Instead of fixing $\tau \in]1/2, 1[$ and looking at the distribution of the single values $\zeta(\tau + it)$ as t varies, we consider for such τ some radius r such that the disc

$$\mathrm{D} = \{s \in \mathbf{C} \mid |s - \tau| \leqslant r\}$$

is contained in the interior of the critical strip, and we look for $t \in \mathbf{R}$ at the *functions*

$$\zeta_{\mathrm{D},t} : \begin{cases} \mathrm{D} & \longrightarrow & \mathbf{C}, \\ s & \mapsto & \zeta(s + it), \end{cases}$$

which are "vertical translates" of the Riemann zeta function restricted to D. For each $\mathrm{T} \geqslant 0$, we view $t \mapsto \zeta_{\mathrm{D},t}$ as a random variable (say, $\mathsf{Z}_{\mathrm{D},\mathrm{T}}$) on $([-\mathrm{T}, \mathrm{T}], dt/(2\mathrm{T}))$ with values in the space H(D) of functions which are holomorphic in the interior of D and continuous on its boundary. Bagchi's remarkable result is a convergence in law in this space, that is, a functional limit theorem: there exists a probability measure ν on H(D) such that the random variables $\mathsf{Z}_{\mathrm{D},\mathrm{T}}$ converge in law to ν as $\mathrm{T} \to +\infty$. Computing the support of ν (which is a nontrivial task) leads to a proof of Voronin's Universality Theorem: for any function $f \in \mathrm{H(D)}$ which does not vanish on D, and for any $\varepsilon > 0$, there exists $t \in \mathbf{R}$ such that

$$\|\zeta(\cdot + it) - f\|_{\infty} < \varepsilon,$$

where the norm is the supremum norm on D. In other words, up to arbitrarily small error, all holomorphic functions f (that do not vanish) can be seen by looking at some vertical translate of the Riemann zeta function!

We illustrate this fact in Figure 3.1, which presents density plots of $|\zeta(s + it)|$ for various values of $t \in \mathbf{R}$, as functions of s in the square $[3/4 - 1/8, 3/4 + 1/8] \times [-1/8, 1/8]$. Voronin's Theorem implies that, for suitable t, such a picture will be indistinguishable from that associated to *any* holomorphic function on this square that never vanishes there.

We will prove the Bohr–Jessen–Bagchi theorems in the next section, and use in particular the computation of the support of Bagchi's limiting distribution for translates of the Riemann zeta function to prove Voronin's Universality Theorem in Section 3.3.

3.2 The Theorems of Bohr–Jessen and of Bagchi

We begin by stating a precise version of Bagchi's Theorem. In the remainder of this chapter, we denote by Ω_{T} the probability space $([-\mathrm{T}, \mathrm{T}], dt/(2\mathrm{T}))$ for $\mathrm{T} \geqslant 1$. We will often write $\mathbf{E}_{\mathrm{T}}(\cdot)$ and $\mathbf{P}_{\mathrm{T}}(\cdot)$ for the corresponding expectation and probability.

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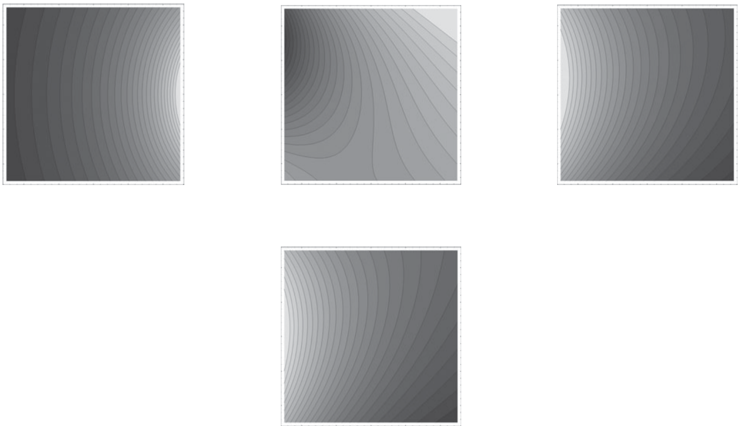

Figure 3.1 The modulus of $\zeta(s+it)$ for s in the square $[3/4-1/8, 3/4+1/8] \times [-1/8, 1/8]$, for $t = 0$, 21000, 58000, and 75000.

Theorem 3.2.1 (Bagchi [4]) *Let τ be such that $1/2 < \tau$. If $1/2 < \tau < 1$, let $r > 0$ be such that*

$$\mathrm{D} = \{s \in \mathbf{C} \mid |s-\tau| \leqslant r\} \subset \{s \in \mathbf{C} \mid 1/2 < \mathrm{Re}(s) < 1\},$$

and if $\tau \geqslant 1$, let D be any compact subset of $\{s \in \mathbf{C} \mid \mathrm{Re}(s) \geqslant 1\}$ such that $\tau \in \mathrm{D}$.

Consider the $\mathrm{H}(\mathrm{D})$-valued random variables $\mathsf{Z}_{\mathrm{D},\mathrm{T}}$ defined by

$$t \mapsto (s \mapsto \zeta(s+it))$$

on Ω_{T}. Let $(\mathrm{X}_p)_p$ be a sequence of independent random variables, indexed by the primes, which are identically distributed, with distribution uniform on the unit circle $\mathbf{S}^1 \subset \mathbf{C}^\times$.

Then we have convergence in law $\mathsf{Z}_{\mathrm{D},\mathrm{T}} \longrightarrow \mathrm{Z}_{\mathrm{D}}$ as $\mathrm{T} \to +\infty$, where Z_{D} is the random Euler product defined by

$$\mathrm{Z}_{\mathrm{D}}(s) = \prod_p (1 - p^{-s}\mathrm{X}_p)^{-1}.$$

In this theorem, the space $\mathrm{H}(\mathrm{D})$ is viewed as a Banach space (hence a metric space, so that convergence in law makes sense) with the norm

$$\|f\|_\infty = \sup_{z \in \mathrm{D}} |f(z)|.$$

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We can already see that Theorem 3.2.1 is (much) stronger than the convergence in law component of Theorem 3.1.1, which we now prove assuming this result:

Corollary 3.2.2 *Fix τ such that $1/2 < \tau$. As $\mathrm{T} \to +\infty$, the random variables $\mathsf{Z}_{\tau,\mathrm{T}}$ of Theorem 3.1.1 converge in law to the random variable $\mathrm{Z}_{\mathrm{D}}(\tau)$, where D is either a disc*

$$\mathrm{D} = \{s \in \mathbf{C} \mid |s - \tau| \leqslant r\}$$

contained in the interior of the critical strip, if $\tau < 1$, or any compact subset of $\{s \in \mathbf{C} \mid \mathrm{Re}(s) \geqslant 1\}$ such that $\tau \in \mathrm{D}$.

Proof Fix D as in the statement. Tautologically, we have

$$\mathsf{Z}_{\tau,\mathrm{T}} = \zeta_{\mathrm{D},\mathrm{T}}(\tau)$$

or $\mathsf{Z}_{\tau,\mathrm{T}} = e_\tau \circ \zeta_{\mathrm{D},\mathrm{T}}$, where

$$e_\tau \begin{cases} \mathrm{H}(\mathrm{D}) & \longrightarrow & \mathbf{C}, \\ f & \mapsto & f(\tau) \end{cases}$$

is the evaluation map. This map is continuous on H(D), so it follows by composition (Proposition B.3.2 in Appendix B) that the convergence in law $\mathsf{Z}_{\mathrm{D},\mathrm{T}} \longrightarrow \mathrm{Z}_{\mathrm{D}}$ of Bagchi's Theorem implies the convergence in law of $\mathsf{Z}_{\tau,\mathrm{T}}$ to the random variable $e_\tau \circ \mathrm{Z}_{\mathrm{D}}$, which is simply $\mathrm{Z}_{\mathrm{D}}(\tau)$. □

In order to prove the final part of Theorem 3.1.1, and to derive Voronin's Universality Theorem, we need to understand the support of the limit Z_{D} in Bagchi's Theorem. We will prove in Section 3.3:

Theorem 3.2.3 (Bagchi, Voronin) *Let τ be such that $1/2 < \tau < 1$, and r such that*

$$\mathrm{D} = \{s \in \mathbf{C} \mid |s - \tau| \leqslant r\} \subset \{s \in \mathbf{C} \mid 1/2 < \mathrm{Re}(s) < 1\}.$$

The support of Z_{D} contains

$$\mathrm{H}(\mathrm{D})^\times = \{f \in \mathrm{H}(\mathrm{D}) \mid f(z) \neq 0 \textit{ for all } z \in \mathrm{D}\}$$

and is equal to $\mathrm{H}(\mathrm{D})^\times \cup \{0\}$.

In particular, for any function $f \in \mathrm{H}(\mathrm{D})^\times$, and for any $\varepsilon > 0$, there exists $t \in \mathbf{R}$ such that

$$\sup_{s \in \mathrm{D}} |\zeta(s + it) - f(s)| < \varepsilon. \tag{3.2}$$

It is then obvious that if $1/2 < \tau < 1$, the support of the Bohr–Jessen random variable $\mathrm{Z}_{\mathrm{D}}(\tau)$ is equal to $\mathbf{C}$.

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Exercise 3.2.4 Prove that the support of the Bohr–Jessen random variable $Z_D(1)$ is also equal to $\mathbf{C}$.

We now begin the proof of Theorem 3.2.1 by giving some intuition for the result and in particular for the shape of the limiting distribution. Indeed, this very elementary argument will suffice to prove Bagchi's Theorem in the case $\tau > 1$. This turns out to be similar to the intuition behind the Erdős–Kac Theorem. We begin with the Euler product

$$\zeta(s+it) = \prod_p (1-p^{-s-it})^{-1},$$

which is valid for $\mathrm{Re}(s) > 1$. We can express this also (formally, we "compute the logarithm"; see Proposition A.2.2 (2)) in the form

$$\zeta(s+it) = \exp\Big(-\sum_p \log(1-p^{-s-it})\Big). \tag{3.3}$$

This displays the Riemann zeta function on Ω_T as the exponential of a sum involving the sequence (indexed by primes) of random variables $(\mathsf{X}_{p,T})_p$ such that

$$\mathsf{X}_{p,T}(t) = p^{-it},$$

each taking value in the unit circle $\mathbf{S}^1$. To understand how the zeta function will behave statistically on Ω_T, the first step is to understand the limiting behavior of this sequence.

This has a very simple answer:

Proposition 3.2.5 *For* $T \geqslant 0$*, let* $\mathsf{X}_T = (\mathsf{X}_{p,T})_p$ *be the sequence of random variables on* Ω_T *given by*

$$t \mapsto (p^{-it})_p.$$

Then X_T *converges in law as* $T \to +\infty$ *to a sequence* $X = (X_p)_p$ *of independent random variables, each of which is uniformly distributed on* $\mathbf{S}^1$.

Bagchi's Theorem is therefore to be understood as saying that we can "pass to the limit" in the formula (3.3) to obtain a convergence in law of $\zeta(s+it)$, for $s \in D$, to

$$\exp\Big(-\sum_p \log(1-p^{-s}X_p)\Big),$$

viewed as a random function.

This sketch is of course incomplete in general, the foremost objection being that we are especially interested in the zeta function outside of the region of absolute convergence, so the meaning of (3.3) is unclear. But we will see that nevertheless enough connections remain to carry the argument through.

We isolate the crucial part of the proof of Proposition 3.2.5 as a lemma, since we will also use it in the proof of Selberg's Theorem in the next chapter (see Section 4.2).

Lemma 3.2.6 *Let $r > 0$ be a real number. We have*

$$|\,\mathbf{E}_{\mathrm{T}}(r^{-it})| \leqslant \min\left(1, \frac{1}{\mathrm{T}|\log r|}\right). \tag{3.4}$$

In particular, if $r = n_1/n_2$ for some positive integers $n_1 \neq n_2$, then we have

$$\mathbf{E}_{\mathrm{T}}(r^{-it}) \ll \min\left(1, \frac{\sqrt{n_1 n_2}}{\mathrm{T}}\right), \tag{3.5}$$

where the implied constant is absolute.

Proof of Lemma 3.2.6 Since $|r^{-it}| = 1$, we see that the expectation is always $\leqslant 1$. If $r \neq 1$, then we get

$$\mathbf{E}_{\mathrm{T}}(r^{-it}) = \frac{1}{2\mathrm{T}}\left[\frac{i}{\log r} r^{-it}\right]_{-\mathrm{T}}^{\mathrm{T}} = \frac{i(r^{i\mathrm{T}} - r^{-i\mathrm{T}})}{2\mathrm{T}(\log r)},$$

which has modulus at most $|\log r|^{-1}\mathrm{T}^{-1}$, hence the first bound holds.

Assume now that $r = n_1/n_2$ with $n_1 \neq n_2$ positive integers. Assume that $n_2 > n_1 \geqslant 1$. Then $n_2 \geqslant n_1 + 1$, and hence

$$\left|\log \frac{n_1}{n_2}\right| \geqslant \left|\log\left(1 + \frac{1}{n_1}\right)\right| \gg \frac{1}{n_1} \geqslant \frac{1}{\sqrt{n_1 n_2}}.$$

If $n_2 < n_1$, we exchange the role of n_1 and n_2, and since both sides of the bound (3.5) are symmetric in terms of n_1 and n_2, the result follows. □

Proof of Proposition 3.2.5 It is convenient here to view the sequences $(\mathsf{X}_{p,\mathrm{T}})_p$ and $(\mathrm{X}_p)_p$ as two random variables on Ω_{T}, taking value in the infinite product

$$\widehat{\mathbf{S}}^1 = \prod_p \mathbf{S}^1$$

of copies of the unit circle indexed by primes. Note that $\widehat{\mathbf{S}}^1$ is a compact abelian group (with componentwise product).

In this interpretation, the limit (or more precisely the law of (X_p)) is simply the probability Haar measure on the group $\widehat{\mathbf{S}}^1$ (see Section B.6). This allows us to prove convergence in law using the well-known Weyl Criterion: the statement of the proposition is equivalent with the property that

$$\lim_{\mathrm{T}\to+\infty} \mathbf{E}_{\mathrm{T}}(\chi(\mathsf{X}_{p,\mathrm{T}})) = 0 \tag{3.6}$$

for any nontrivial continuous unitary character $\chi : \widehat{\mathbf{S}}^1 \longrightarrow \mathbf{S}^1$. An elementary property of compact groups shows that for any such character there exists a finite nonempty subset S of primes, and for each $p \in \mathrm{S}$ some integer $m_p \in \mathbf{Z} - \{0\}$, such that

$$\chi(z) = \prod_{p\in\mathrm{S}} z_p^{m_p}$$

for any $z = (z_p)_p \in \widehat{\mathbf{S}}^1$ (see Example B.6.2(2)). We then have by definition

$$\mathbf{E}_{\mathrm{T}}(\chi(\mathsf{X}_{p,\mathrm{T}})) = \frac{1}{2\mathrm{T}}\int_{-\mathrm{T}}^{\mathrm{T}} \prod_{p\in\mathrm{S}} p^{-itm_p}\,dt = \frac{1}{2\mathrm{T}}\int_{-\mathrm{T}}^{\mathrm{T}} r^{-it}\,dt,$$

where $r > 0$ is the rational number given by

$$r = \prod_{p\in\mathrm{S}} p^{m_p}.$$

Since we have $r \neq 1$ (because S is not empty and $m_p \neq 0$), we obtain $\mathbf{E}_{\mathrm{T}}(\chi(\mathsf{X}_{p,\mathrm{T}})) \to 0$ as $\mathrm{T} \to +\infty$ from (3.4). ☐

As a corollary, Bagchi's Theorem follows formally for $\tau > 1$ and D contained in the set of complex numbers with real part > 1. This is once more a very simple fact which is often not specifically discussed, but which gives an indication and a motivation for the more difficult study in the critical strip.

Special case of Theorem 3.2.1 for $\tau > 1$ Assume that $\tau > 1$ and that D is a compact subset containing τ contained in $\{s \in \mathbf{C} \mid \mathrm{Re}(s) > 1\}$. We view $\mathsf{X}_{\mathrm{T}} = (\mathsf{X}_{p,\mathrm{T}})$ as random variables with values in the topological space $\widehat{\mathbf{S}}^1$, as before. This is also (as a countable product of metric spaces) a metric space. We claim that the map

$$\varphi \quad \begin{cases} \widehat{\mathbf{S}}^1 & \longrightarrow & \mathrm{H}(\mathrm{D}), \\ (x_p) & \mapsto & \Big(s \mapsto -\sum_p \log(1 - x_p p^{-s})\Big) \end{cases}$$

is continuous (see Definition A.2.1 again for the definition of the logarithm here). If this is so, then the composition principle (see Proposition B.3.2) and Proposition 3.2.5 imply that $\varphi(\mathsf{X}_{\mathrm{T}})$ converges in law to the H(D)-valued random variable $\varphi(\mathrm{X})$, where $\mathrm{X} = (\mathrm{X}_p)$ with the X_p uniform and independent on $\mathbf{S}^1$. But this is exactly the statement of Bagchi's Theorem for D.

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Now we check the claim. Fix $\varepsilon > 0$. Let $\mathrm{T} > 0$ be some parameter to be chosen later in terms of ε. For any $x = (x_p)$ and $y = (y_p)$ in $\widehat{\mathbf{S}}^1$, we have

$$\|\varphi(x) - \varphi(y)\|_\infty \leqslant \sum_{p \leqslant \mathrm{T}} \|\log(1 - x_p p^{-s}) - \log(1 - y_p p^{-s})\|_\infty \\ + \sum_{p > \mathrm{T}} \|\log(1 - x_p p^{-s})\|_\infty + \sum_{p > \mathrm{T}} \|\log(1 - y_p p^{-s})\|_\infty.$$

Because D is compact in the half-plane $\mathrm{Re}(s) > 1$, the minimum of the real part of $s \in \mathrm{D}$ is some real number $\sigma_0 > 1$. Since $|x_p| = |y_p| = 1$ for all primes, and since

$$|\log(1 - z)| \leqslant 2|z|$$

for $|z| \leqslant 1/2$ (Proposition A.2.2 (3)), it follows that

$$\sum_{p > \mathrm{T}} \|\log(1 - x_p p^{-s})\|_\infty + \sum_{p > \mathrm{T}} \|\log(1 - y_p p^{-s})\|_\infty \leqslant 4 \sum_{p > \mathrm{T}} p^{-\sigma_0} \ll \mathrm{T}^{1-\sigma_0}.$$

We fix T so that $\mathrm{T}^{1-\sigma_0} < \varepsilon/2$. Now the map

$$(x_p)_{p \leqslant \mathrm{T}} \mapsto \sum_{p \leqslant \mathrm{T}} \|\log(1 - x_p p^{-s}) - \log(1 - y_p p^{-s})\|_\infty$$

is obviously continuous, and therefore uniformly continuous since the domain is a compact set. This function has value 0 when $x_p = y_p$ for $p \leqslant \mathrm{T}$, so there exists $\delta > 0$ such that

$$\sum_{p \leqslant \mathrm{T}} |\log(1 - x_p p^{-s}) - \log(1 - y_p p^{-s})| < \frac{\varepsilon}{2}$$

if $|x_p - y_p| \leqslant \delta$ for $p \leqslant \mathrm{T}$. Therefore, provided that

$$\max_{p \leqslant \mathrm{T}} |x_p - y_p| \leqslant \delta,$$

we have

$$\|\varphi(x) - \varphi(y)\|_\infty \leqslant \varepsilon.$$

This proves the (uniform) continuity of φ. ☐

We now begin the proof of Bagchi's Theorem in the critical strip. The argument follows partly his original proof [4], which is quite different from the Bohr–Jessen approach, with some simplifications. Here are the main steps of the proof:

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- we prove convergence almost surely of the random Euler product, and of its formal Dirichlet series expansion; this also shows that they define random *holomorphic* functions;
- we prove that both the Riemann zeta function and the limiting Dirichlet series are, in suitable mean sense, limits of smoothed partial sums of their respective Dirichlet series;
- we then use an elementary argument to conclude using Proposition 3.2.5.

We fix from now on a sequence $(\mathrm{X}_p)_p$ of independent random variables all uniformly distributed on $\mathbf{S}^1$. We often view the sequence (X_p) as an $\widehat{\mathbf{S}}^1$-valued random variable, as in the proof of Proposition 3.2.5. Furthermore, for any positive integer $n \geqslant 1$, we define

$$\mathrm{X}_n = \prod_{p \mid n} \mathrm{X}_p^{v_p(n)}, \tag{3.7}$$

where $v_p(n)$ is the p-adic valuation of n. Thus (X_n) is a sequence of $\mathbf{S}^1$-valued random variables.

Exercise 3.2.7 Prove that the sequence $(\mathrm{X}_n)_{n \geqslant 1}$ is neither independent nor symmetric.

Exercise 3.2.8 The following exercise provides the starting point of recent probabilistic approaches to the problem of estimating the so-called *pseudo-moments* of the Riemann zeta function (see the thesis of M. Gerspach [46]), although it is often proved using different approaches, such as the ergodic theorem for flows.

For any real numbers $q \geqslant 0$ and $x \geqslant 1$ and any sequence of complex numbers (a_n), prove that the limit

$$\lim_{\mathrm{T} \to +\infty} \frac{1}{\mathrm{T}} \int_{\mathrm{T}}^{2\mathrm{T}} \Big| \sum_{n \leqslant x} a_n n^{-it} \Big|^q \, dt$$

exists and that it is equal to

$$\mathbf{E}\Big(\Big| \sum_{n \leqslant x} a_n \mathrm{X}_n \Big|^q \Big).$$

We first show that the limiting random functions are indeed well defined as $\mathrm{H(D)}$-valued random variables.

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Proposition 3.2.9 *Let* $\tau \in]1/2, 1[$, *and let* $\mathrm{U}_\tau = \{s \in \mathbf{C} \mid \mathrm{Re}(s) > \tau\}$.

(1) *The random Euler product defined by*

$$\mathrm{Z}(s) = \prod_p (1 - \mathrm{X}_p p^{-s})^{-1}$$

converges almost surely for any $s \in \mathrm{U}_\tau$. *For any compact subset* $\mathrm{K} \subset \mathrm{U}_\tau$, *the random function*

$$\mathrm{Z}_\mathrm{K} \colon \begin{cases} \mathrm{K} & \longrightarrow \mathbf{C}, \\ s & \mapsto \mathrm{Z}(s) \end{cases}$$

is an $\mathrm{H}(\mathrm{K})$*-valued random variable.*

(2) *The random Dirichlet series defined by*

$$\tilde{\mathrm{Z}} = \sum_{n \geqslant 1} \mathrm{X}_n n^{-s}$$

converges almost surely for any $s \in \mathrm{U}_\tau$. *For any compact subset* $\mathrm{K} \subset \mathrm{U}_\tau$, *the random function* $\tilde{\mathrm{Z}}_\mathrm{K} \colon s \mapsto \tilde{\mathrm{Z}}(s)$ *on* K *is an* $\mathrm{H}(\mathrm{K})$*-valued random variable.*

(3) *We have* $\tilde{\mathrm{Z}}_\mathrm{K} = \mathrm{Z}_\mathrm{K}$ *almost surely.*

Proof (1) For $\mathrm{N} \geqslant 1$ and $s \in \mathrm{K}$, we have by definition

$$\sum_{p \leqslant \mathrm{N}} \log(1 - \mathrm{X}_p p^{-s})^{-1} = \sum_{p \leqslant \mathrm{N}} \frac{\mathrm{X}_p}{p^s} + \sum_{k \geqslant 2} \sum_{p \leqslant \mathrm{N}} \frac{\mathrm{X}_{p^k}}{p^{ks}}.$$

Since $\mathrm{Re}(s) > 1/2$ for $s \in \mathrm{K}$, the series

$$\sum_{k \geqslant 2} \sum_p \frac{\mathrm{X}_{p^k}}{p^{ks}}$$

converges absolutely for $s \in \mathrm{U}_\tau$. By Lemma A.4.1, its sum is therefore an $\mathrm{H}(\mathrm{K})$-valued random variable for any compact subset K of U_τ.

Fix now $\tau_1 < \tau$ such that $\tau_1 > \frac{1}{2}$. We can apply Kolmogorov's Theorem, B.10.1, to the independent random variables $(\mathrm{X}_p p^{-\tau_1})$, since

$$\sum_p \mathbf{V}(p^{-\tau_1} \mathrm{X}_p) = \sum_p \frac{1}{p^{2\tau_1}} < +\infty.$$

Thus the series

$$\sum_p \frac{\mathrm{X}_p}{p^{\tau_1}}$$

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converges almost surely. By Lemma A.4.1 again, it follows that

$$\mathrm{P}(s) = \sum_p \frac{\mathrm{X}_p}{p^s}$$

converges almost surely for all $s \in \mathrm{U}_\tau$, and is holomorphic on U_τ. By restriction, its sum is an H(K)-valued random variable for any K compact in U_τ.

These facts show that the sequence of partial sums

$$\sum_{p \leqslant \mathrm{N}} \log(1 - \mathrm{X}_p p^{-s})^{-1}$$

converges almost surely as $\mathrm{N} \to +\infty$ to a random holomorphic function on K. Taking the exponential, we obtain the almost sure convergence of the random Euler product to a random holomorphic function $\mathrm{Z_K}$ on K.

(2) The argument is similar, except that the sequence $(\mathrm{X}_n)_{n \geqslant 1}$ is not independent. However, it is orthonormal: if $n \neq m$, we have

$$\mathbf{E}(\mathrm{X}_n \overline{\mathrm{X}}_m) = 0 \quad \text{and} \quad \mathbf{E}(|\mathrm{X}_n|^2) = 1$$

(indeed X_n and X_m may be viewed as characters of $\widehat{\mathbf{S}}^1$, and they are distinct if $n \neq m$, so that this is the orthogonality property of characters of compact groups). We can then apply the Menshov–Rademacher Theorem, B.10.5, to (X_n) and $a_n = n^{-\tau_1}$: since

$$\sum_{n \geqslant 1} |a_n|^2 (\log n)^2 = \sum_{n \geqslant 1} \frac{(\log n)^2}{n^{2\tau_1}} < +\infty,$$

the series $\sum \mathrm{X}_n n^{-\tau_1}$ converges almost surely, and Lemma A.4.1 shows that $\tilde{\mathrm{Z}}$ converges almost surely on U_τ, and defines a holomorphic function there. Restricting to K leads to $\tilde{\mathrm{Z}}_\mathrm{K}$ as H(K)-valued random variable.

Finally, to prove that $\mathrm{Z_K} = \tilde{\mathrm{Z}}_\mathrm{K}$ almost surely, we may replace K by the compact subset

$$\mathrm{K}_1 = \{s \in \mathbf{C} \mid \tau_1 \leqslant \sigma \leqslant \mathrm{A}, \quad |t| \leqslant \mathrm{B}\},$$

with $\mathrm{A} \geqslant 2$ and B chosen large enough to ensure that $\mathrm{K} \subset \mathrm{K}_1$. The previous argument shows that the random Euler product and Dirichlet series converge almost surely on K_1. But K_1 contains the open set

$$\mathrm{V} = \{s \in \mathbf{C} \mid 1 < \mathrm{Re}(s) < 2, \quad |t| < \mathrm{B}\},$$

where the Euler product and Dirichlet series converge absolutely, so that Lemma C.1.4 proves that the random holomorphic functions $\mathrm{Z}_{\mathrm{K}_1}$ and $\tilde{\mathrm{Z}}_{\mathrm{K}_1}$ are equal when restricted to V. By analytic continuation (and continuity), they are equal also on K_1, hence *a posteriori* on K. □

We will prove Bagchi's Theorem using the random Dirichlet series, which is easier to handle than the Euler product. However, we will still denote it $\mathrm{Z}(s)$, which is justified by the last part of the proposition.

Some additional properties of this random Dirichlet series are now needed. Most importantly, we need to find a finite approximation that also applies to the Riemann zeta function. This will be done using *smooth partial sums*.

First we need to check that $\mathrm{Z}(s)$ is of polynomial growth on average on vertical strips.

Lemma 3.2.10 *Let* $\mathrm{Z}(s)$ *be the random Dirichlet series* $\sum \mathrm{X}_n n^{-s}$ *defined and holomorphic almost surely for* $\mathrm{Re}(s) > 1/2$*. For any* $\sigma_1 > 1/2$*, we have*

$$\mathbf{E}(|\mathrm{Z}(s)|) \ll 1 + |s|$$

uniformly for all s *such that* $\mathrm{Re}(s) \geqslant \sigma_1$.

Proof The series

$$\sum_{n \geqslant 1} \frac{\mathrm{X}_n}{n^{\sigma_1}}$$

converges almost surely. Therefore the partial sums

$$\mathrm{S}_u = \sum_{n \leqslant u} \frac{\mathrm{X}_n}{n^{\sigma_1}}$$

are bounded almost surely.

By summation by parts (Lemma A.1.1), it follows that for any s with real part $\sigma > \sigma_1$, we have

$$\mathrm{Z}(s) = (s - \sigma_1) \int_1^{+\infty} \frac{\mathrm{S}_u}{u^{s-\sigma_1+1}} du,$$

where the integral converges almost surely. Hence

$$|\mathrm{Z}(s)| \leqslant (1 + |s|) \int_1^{+\infty} \frac{|\mathrm{S}_u|}{u^{\sigma-\sigma_1+1}} du.$$

Fubini's Theorem (for nonnegative functions) and the Cauchy–Schwarz inequality then imply

$$\begin{aligned}
\mathbf{E}(|\mathrm{Z}(s)|) &\leqslant (1 + |s|) \int_1^{+\infty} \mathbf{E}(|\mathrm{S}_u|) \frac{du}{u^{\sigma-\sigma_1+1}} \\
&\leqslant (1 + |s|) \int_1^{+\infty} \mathbf{E}(|\mathrm{S}_u|^2)^{1/2} \frac{du}{u^{\sigma-\sigma_1+1}} \\
&= (1 + |s|) \int_1^{+\infty} \left(\sum_{n \leqslant u} \frac{1}{n^{2\sigma_1}} \right)^{1/2} \frac{du}{u^{\sigma-\sigma_1+1}}
\end{aligned}$$

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using the orthonormality of the variables X_n. The integrand is $\ll u^{-\frac{1}{2}-\sigma}$, hence the integral converges uniformly for $\sigma \geqslant \sigma_1$. □

We can then deduce a good result on average approximation by partial sums. We refer to Section A.3 for the definition and properties of the Mellin transform.

Proposition 3.2.11 *Let* $\varphi\colon [0,+\infty[\longrightarrow [0,1]$ *be a smooth function with compact support such that* $\varphi(0)=1$. *Let* $\widehat{\varphi}$ *denote its Mellin transform. For* $N \geqslant 1$, *define the* $H(D)$*-valued random variable*

$$Z_{D,N} = \sum_{n\geqslant 1} X_n \varphi\left(\frac{n}{N}\right) n^{-s}.$$

There exists $\delta > 0$ *such that*

$$\mathbf{E}(\|Z_D - Z_{D,N}\|_\infty) \ll N^{-\delta}$$

for $N \geqslant 1$.

We recall that the norm $\|\cdot\|_\infty$ refers to the sup norm on the compact set D.

Proof The first step is to apply the smoothing process of Proposition A.4.3 in Appendix A. The random Dirichlet series

$$Z(s) = \sum_{n\geqslant 1} X_n n^{-s}$$

converges almost surely for $\mathrm{Re}(s) > 1/2$. For $\sigma > 1/2$ and any $\delta > 0$ such that

$$-\delta + \sigma \geqslant 1/2,$$

we have therefore almost surely the representation

$$Z_D(s) - Z_{D,N}(s) = -\frac{1}{2i\pi}\int_{(-\delta)} Z(s+w)\widehat{\varphi}(w)N^w\,dw \tag{3.8}$$

for $s \in D$. (Figure 3.2 may help understand the location of the regions involved in the proof.)

Note that here and below, it is important that the "almost surely" property holds for *all* s; this is simply because we work with random variables taking values in $H(D)$, and not with particular evaluations of these random functions at a specific $s \in D$.

We need to control the supremum norm on D, since this is the norm on the space $H(D)$. For this purpose, we use Cauchy's integral formula.

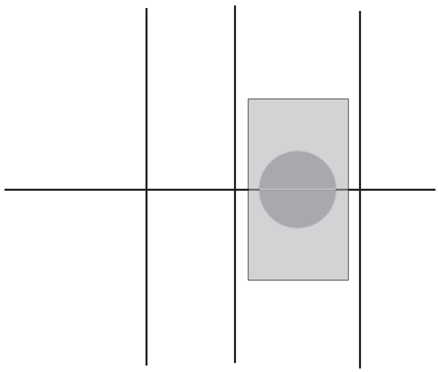

Figure 3.2 Regions and contours in the proof of Proposition 3.2.11.

Let S be a compact segment in $]1/2,1[$ such that the fixed rectangle $\mathrm{R} = \mathrm{S} \times [-1/2, 1/2] \subset \mathbf{C}$ contains D in its interior. Then, almost surely, for any v in D, Cauchy's Theorem gives

$$\mathrm{Z_D}(v) - \mathrm{Z_{D,N}}(v) = \frac{1}{2i\pi} \int_{\partial\mathrm{R}} (\mathrm{Z_D}(s) - \mathrm{Z_{D,N}}(s)) \frac{ds}{s-v},$$

where the boundary of R is oriented counterclockwise. The definition of R ensures that $|s-v|^{-1} \gg 1$ for $v \in \mathrm{D}$ and $s \in \partial\mathrm{R}$, so that the random variable $\|\mathrm{Z_D} - \mathrm{Z_{D,N}}\|_\infty$ satisfies

$$\|\mathrm{Z_D} - \mathrm{Z_{D,N}}\|_\infty \ll \int_{\partial\mathrm{R}} |\mathrm{Z_D}(s) - \mathrm{Z_{D,N}}(s)|\,|ds|.$$

Using (3.8) and writing $w = -\delta + iu$ with $u \in \mathbf{R}$, we obtain

$$\|\mathrm{Z_D} - \mathrm{Z_{D,N}}\|_\infty \ll \mathrm{N}^{-\delta} \int_{\partial\mathrm{R}} \int_{\mathbf{R}} |\mathrm{Z}(-\delta + \sigma + i(t+u))|\,|\widehat{\varphi}(-\delta + iu)||ds|du.$$

Therefore, taking the expectation, and using Fubini's Theorem (for nonnegative functions), we get

$$\begin{aligned}
&\mathbf{E}(\|\mathrm{Z_D} - \mathrm{Z_{D,N}}\|_\infty) \\
&\quad \ll \mathrm{N}^{-\delta} \int_{\partial\mathrm{R}} \int_{\mathbf{R}} \mathbf{E}\left(|\mathrm{Z}(-\delta + \sigma + i(t+u))|\right)\ |\widehat{\varphi}(-\delta + iu)||ds|du \\
&\quad \ll \mathrm{N}^{-\delta} \sup_{s=\sigma+it \in \mathrm{R}} \int_{\mathbf{R}} \mathbf{E}\left(|\mathrm{Z}(-\delta + \sigma + i(t+u))|\right)\ |\widehat{\varphi}(-\delta + iu)|du.
\end{aligned}$$

We therefore need to bound

$$\int_{\mathbf{R}} \mathbf{E}\left(|\mathrm{Z}(-\delta + \sigma + i(t+u))|\right)\ |\widehat{\varphi}(-\delta + iu)|du$$

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for some fixed $\sigma + it$ in the compact rectangle R. We take

$$\delta = \frac{1}{2}(\min \mathrm{S} - 1/2),$$

which is > 0 since S is compact in $]1/2, 1[$, so that

$$-\delta + \sigma > 1/2 \quad \text{and} \quad 0 < \delta < 1.$$

Since $\widehat{\varphi}$ decays faster than any polynomial at infinity in vertical strips, and

$$\mathbf{E}(|\mathrm{Z}(s)|) \ll 1 + |s|$$

uniformly for $s \in \mathrm{R}$ by Lemma 3.2.10, we have

$$\int_{\mathbf{R}} \mathbf{E}\left(|\mathrm{Z}(-\delta + \sigma + i(t+u))|\right) \; |\widehat{\varphi}(-\delta + iu)| du \ll 1$$

uniformly for $s = \sigma + it \in \mathrm{R}$, and the result follows. ☐

The last preliminary result is a similar approximation result for the translates of the Riemann zeta function by smooth partial sums of its Dirichlet series.

Proposition 3.2.12 *Let* $\varphi\colon [0, +\infty[\longrightarrow [0,1]$ *be a smooth function with compact support such that* $\varphi(0) = 1$. *Let* $\widehat{\varphi}$ *denote its Mellin transform. For* $\mathrm{N} \geqslant 1$, *define*[2]

$$\zeta_{\mathrm{N}}(s) = \sum_{n \geqslant 1} \varphi\left(\frac{n}{\mathrm{N}}\right) n^{-s},$$

and define $\mathsf{Z}_{\mathrm{N,T}}$ *to be the* H(D)*-valued random variable*

$$t \mapsto (s \mapsto \zeta_{\mathrm{N}}(s + it)).$$

There exists $\delta > 0$ *such that*

$$\mathbf{E}_{\mathrm{T}}(\|\mathsf{Z}_{\mathrm{D,T}} - \mathsf{Z}_{\mathrm{N,T}}\|_\infty) \ll \mathrm{N}^{-\delta} + \mathrm{N}\mathrm{T}^{-1}$$

for $\mathrm{N} \geqslant 1$ *and* $\mathrm{T} \geqslant 1$.

Note that ζ_{N} is an entire function, since φ has compact support, so that the range of the sum is in fact finite. The meaning of the statement is that the smoothed partial sums ζ_{N} give very uniform and strong approximations to the vertical translates of the Riemann zeta function.

Proof We will write Z_{T} for $\mathsf{Z}_{\mathrm{D,T}}$ for simplicity. We begin by applying the smoothing process of Proposition A.4.3 in Appendix A in the case $a_n = 1$.

[2] There should be no confusion with $\mathsf{Z}_{\mathrm{D,T}}$.

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For $\sigma > 1/2$ and any $\delta > 0$ such that $-\delta + \sigma \geqslant 1/2$, we have (as in the previous proof) the representation

$$\zeta(s) - \zeta_{\mathrm{N}}(s) = -\frac{1}{2i\pi}\int_{(-\delta)} \zeta(s+w)\widehat{\varphi}(w)\mathrm{N}^{w}\,dw - \mathrm{N}^{1-s}\widehat{\varphi}(1-s), \quad (3.9)$$

where the second term on the right-hand side comes from the fact that the Riemann zeta function has a pole at $s = 1$ with residue 1.

As before, let S be a compact segment in $]1/2, 1[$ such that the fixed rectangle $\mathrm{R} = \mathrm{S} \times [-1/2, 1/2] \subset \mathbf{C}$ contains D in its interior. Then for any v with $\mathrm{Re}(v) > 1/2$ and $t \in \mathbf{R}$, Cauchy's Theorem gives

$$\zeta(v+it) - \zeta_{\mathrm{N}}(v+it) = \frac{1}{2i\pi}\int_{\partial\mathrm{R}} (\zeta(s+it) - \zeta_{\mathrm{N}}(s+it))\frac{ds}{s-v},$$

where the boundary of R is oriented counterclockwise; using $|s-v|^{-1} \gg 1$ for $v \in \mathrm{D}$ and $s \in \partial\mathrm{R}$, we deduce that the random variable $\|\mathsf{Z}_{\mathrm{T}} - \mathsf{Z}_{\mathrm{N,T}}\|_{\infty}$, which takes the value

$$\sup_{s\in\mathrm{D}} |\zeta(s+it) - \zeta_{\mathrm{N}}(s+it)|$$

at $t \in \Omega_{\mathrm{T}}$, satisfies

$$\|\mathsf{Z}_{\mathrm{T}} - \mathsf{Z}_{\mathrm{N,T}}\|_{\infty} \ll \int_{\partial\mathrm{R}} |\zeta(s+it) - \zeta_{\mathrm{N}}(s+it)||ds|$$

for $t \in \Omega_{\mathrm{T}}$. Taking the expectation with respect to t (i.e., integrating over $t \in [-\mathrm{T}, \mathrm{T}]$) and applying Fubini's Theorem for nonnegative functions leads to

$$\begin{aligned}\mathbf{E}_{\mathrm{T}}\left(\|\mathsf{Z}_{\mathrm{T}} - \mathsf{Z}_{\mathrm{N,T}}\|_{\infty}\right) &\ll \int_{\partial\mathrm{R}} \mathbf{E}_{\mathrm{T}}\left(|\zeta(s+it) - \zeta_{\mathrm{N}}(s+it)|\right)|ds| \\ &\ll \sup_{s\in\partial\mathrm{R}} \mathbf{E}_{\mathrm{T}}\left(|\zeta(s+it) - \zeta_{\mathrm{N}}(s+it)|\right). \qquad (3.10)\end{aligned}$$

We take again $\delta = \frac{1}{2}(\min \mathrm{S} - 1/2) > 0$, so that $0 < \delta < 1$. For any fixed $s = \sigma + it \in \partial\mathrm{R}$, we have

$$-\delta + \sigma \geqslant \frac{1}{2} + \delta > \frac{1}{2}.$$

Applying (3.9) and using again Fubini's Theorem, we obtain

$$\begin{aligned}&\mathbf{E}_{\mathrm{T}}\left(|\zeta(s+it) - \zeta_{\mathrm{N}}(s+it)|\right) \\ &\quad\ll \mathrm{N}^{-\delta}\int_{\mathbf{R}} |\widehat{\varphi}(-\delta+iu)|\,\mathbf{E}_{\mathrm{T}}\left(|\zeta(\sigma-\delta+i(t+u))|\right)du \\ &\qquad + \mathrm{N}^{1-\sigma}\,\mathbf{E}_{\mathrm{T}}(|\widehat{\varphi}(1-s-it)|).\end{aligned}$$

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The rapid decay of $\widehat{\varphi}$ on vertical strips shows that the second term (arising from the pole) is $\ll \mathrm{NT}^{-1}$. In the first term, since $\sigma - \delta \geqslant \min(\mathrm{S}) - \delta \geqslant \frac{1}{2} + \delta$, we have

$$\mathbf{E}_{\mathrm{T}}\left(|\zeta(\sigma-\delta+i(t+u))|\right) = \frac{1}{2\mathrm{T}}\int_{-\mathrm{T}}^{\mathrm{T}} |\zeta(\sigma-\delta+i(t+u))|dt$$
$$\ll 1 + \frac{|u|}{\mathrm{T}} \ll 1 + |u| \qquad (3.11)$$

by Proposition C.4.1 in Appendix C. Hence

$$\mathbf{E}_{\mathrm{T}}\left(|\zeta(s+it) - \zeta_{\mathrm{N}}(s+it)|\right) \ll \mathrm{N}^{-\delta}\int_{\mathbf{R}} |\widehat{\varphi}(-\delta+iu)|(1+|u|)du + \mathrm{NT}^{-1}. \qquad (3.12)$$

Now the fast decay of $\widehat{\varphi}(s)$ on the vertical line $\mathrm{Re}(s) = -\delta$ shows that the last integral is bounded, and we conclude from (3.10) that

$$\mathbf{E}_{\mathrm{T}}\left(\|\mathsf{Z}_{\mathrm{T}} - \mathsf{Z}_{\mathrm{N,T}}\|_{\infty}\right) \ll \mathrm{N}^{-\delta} + \mathrm{NT}^{-1},$$

as claimed. □

Finally we can prove Theorem 3.2.1:

Proof of Bagchi's Theorem By Proposition B.4.1, it is enough to prove that for any bounded and Lipschitz function $f : \mathrm{H(D)} \longrightarrow \mathbf{C}$, we have

$$\mathbf{E}_{\mathrm{T}}(f(\mathsf{Z}_{\mathrm{D,T}})) \longrightarrow \mathbf{E}(f(\mathrm{Z}_{\mathrm{D}}))$$

as $\mathrm{T} \to +\infty$. We may use the Dirichlet series expansion of Z_{D} according to Proposition 3.2.9, (2).

Since D is fixed, we omit it from the notation for simplicity, denoting $\mathsf{Z}_{\mathrm{T}} = \mathsf{Z}_{\mathrm{D,T}}$ and $\mathrm{Z} = \mathrm{Z}_{\mathrm{D}}$. Fix some integer $\mathrm{N} \geqslant 1$ to be chosen later. We denote

$$\mathsf{Z}_{\mathrm{T,N}} = \sum_{n\geqslant 1} n^{-s-it}\varphi\left(\frac{n}{\mathrm{N}}\right)$$

(viewed as a random variable defined for $t \in [-\mathrm{T}, \mathrm{T}]$) and

$$\mathrm{Z}_{\mathrm{N}} = \sum_{n\geqslant 1} \mathrm{X}_n n^{-s}\varphi\left(\frac{n}{\mathrm{N}}\right)$$

the smoothed partial sums of the Dirichlet series as in Propositions 3.2.12 and 3.2.11.

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We then write

$$\begin{aligned}|\mathbf{E}_{\mathrm{T}}(f(\mathsf{Z}_{\mathrm{T}}))-\mathbf{E}(f(\mathrm{Z}))| \leqslant{} & |\mathbf{E}_{\mathrm{T}}(f(\mathsf{Z}_{\mathrm{T}})-f(\mathsf{Z}_{\mathrm{T,N}}))| \\ & +|\mathbf{E}_{\mathrm{T}}(f(\mathsf{Z}_{\mathrm{T,N}}))-\mathbf{E}(f(\mathrm{Z}_{\mathrm{N}}))| \\ & +|\mathbf{E}(f(\mathrm{Z}_{\mathrm{N}})-f(\mathrm{Z}))|.\end{aligned}$$

Since f is a Lipschitz function on H(D), there exists a constant $\mathrm{C}\geqslant 0$ such that

$$|f(x)-f(y)|\leqslant \mathrm{C}\|x-y\|_\infty$$

for all $x, y\in \mathrm{H(D)}$. Hence we have

$$\begin{aligned}|\mathbf{E}_{\mathrm{T}}(f(\mathsf{Z}_{\mathrm{T}}))-\mathbf{E}(f(\mathrm{Z}))| \leqslant{} & \mathrm{C}\,\mathbf{E}_{\mathrm{T}}(\|\mathsf{Z}_{\mathrm{T}}-\mathsf{Z}_{\mathrm{T,N}}\|_\infty) \\ & +|\mathbf{E}_{\mathrm{T}}(f(\mathsf{Z}_{\mathrm{T,N}}))-\mathbf{E}(f(\mathrm{Z}_{\mathrm{N}}))|+\mathrm{C}\,\mathbf{E}(\|\mathrm{Z}_{\mathrm{N}}-\mathrm{Z}\|_\infty).\end{aligned}$$

Fix $\varepsilon>0$. Propositions 3.2.12 and 3.2.11 together show that there exists some $\mathrm{N}\geqslant 1$ and some constant $\mathrm{C}_1\geqslant 0$ such that

$$\mathbf{E}_{\mathrm{T}}(\|\mathsf{Z}_{\mathrm{T}}-\mathsf{Z}_{\mathrm{T,N}}\|_\infty)<\varepsilon+\frac{\mathrm{C}_1\mathrm{N}}{\mathrm{T}}$$

for all $\mathrm{T}\geqslant 1$ and

$$\mathbf{E}(\|\mathrm{Z}_{\mathrm{N}}-\mathrm{Z}\|_\infty)<\varepsilon.$$

We *fix* such a value of N. By Proposition 3.2.5 and composition, the random variables $\mathsf{Z}_{\mathrm{T,N}}$ (which are Dirichlet polynomials) converge in law to Z_{N} as $\mathrm{T}\to+\infty$. Since $\mathrm{N}/\mathrm{T}\to 0$ also for $\mathrm{T}\to+\infty$, we deduce that for all T large enough, we have

$$|\mathbf{E}_{\mathrm{T}}(f(\mathsf{Z}_{\mathrm{T}}))-\mathbf{E}(f(\mathrm{Z}))|<4\varepsilon.$$

This finishes the proof. □

Exercise 3.2.13 Prove that if $\sigma>1/2$ is fixed, then we have almost surely

$$\lim_{\mathrm{T}\to+\infty}\frac{1}{2\mathrm{T}}\int_{-\mathrm{T}}^{\mathrm{T}}|\mathrm{Z}(\sigma+it)|^2dt=\zeta(2\sigma).$$

[**Hint**: Use the Birkhoff–Khintchine pointwise ergodic theorem for flows; see, e.g., [30, §8.6.1].]

Before we continue toward the computation of the support of Bagchi's measure, and hence the proof of Voronin's Theorem, we can use the current available information to obtain bounds on the probability that the Riemann zeta function is "large" on the subset D. More precisely, it is natural to discuss the probability that the logarithm of the modulus of translates of the zeta function is large, since this will also detect how close it might approach zero.

Proposition 3.2.14 *Let σ_0 be the infimum of the real part of s for $s \in \mathrm{D}$. There exists a positive constant $c > 0$, depending on* D*, such that for any $\mathrm{A} > 0$, we have*

$$\limsup_{\mathrm{T}\to+\infty} \mathbf{P}_{\mathrm{T}}(\| \log |\mathsf{Z}_{\mathrm{D,T}}| \, \|_\infty > \mathrm{A}) \leqslant c \exp\left(-c^{-1}\mathrm{A}^{1/(1-\sigma_0)}(\log \mathrm{A})^{1/(2(1-\sigma_0))}\right).$$

Proof Convergence in law implies that

$$\limsup_{\mathrm{T}\to+\infty} \mathbf{P}_{\mathrm{T}}(\| \log |\mathsf{Z}_{\mathrm{D,T}}| \, \|_\infty > \mathrm{A}) \leqslant \mathbf{P}_{\mathrm{T}}(\| \log |\mathrm{Z}_{\mathrm{D}}| \, \|_\infty > \mathrm{A})$$

and

$$\log |\mathrm{Z}_{\mathrm{D}}| = \sum_p \mathrm{Re}\left(\frac{\mathrm{X}_p}{p^s}\right) + \mathrm{O}(1),$$

where the implied constant depends on D. In addition, we have

$$\mathbf{P}_{\mathrm{T}}\left(\left\|\sum_p \mathrm{Re}\left(\frac{\mathrm{X}_p}{p^s}\right)\right\|_\infty > \mathrm{A}\right) \leqslant \mathbf{P}_{\mathrm{T}}\left(\left\|\sum_p \frac{\mathrm{X}_p}{p^s}\right\|_\infty > \mathrm{A}\right).$$

Since $\sigma_0 > \frac{1}{2}$ and the random variables (X_p) are independent and bounded by 1, we can therefore estimate the right-hand side of this last inequality using the variant of Proposition B.11.13 discussed in Remark B.11.14 (2) for the Banach space H(D), and hence conclude the proof. □

Remark 3.2.15 It is also possible to obtain lower bounds for these probabilities, by evaluating at a fixed element of D (see Theorem 6.3.1 for a similar argument, although the shape of the lower bound is different).

3.3 The Support of Bagchi's Measure

Our goal in this section is to explain the proof of Theorem 3.2.3, which is due to Bagchi [4, Ch. 5]. Since it involves results of complex analysis that are quite far from the main interest of this book, we will only treat in detail the part of the proof that involves arithmetic, giving references for the other results that are used.

The support is easiest to compute using the random Euler interpretation of the random Dirichet series, because it is essentially a sum of independent random variables. To be precise, define

$$\mathrm{P}(s) = \sum_p \frac{\mathrm{X}_p}{p^s} \quad \text{and} \quad \tilde{\mathrm{P}}(s) = \sum_p \sum_{k \geqslant 1} \frac{\mathrm{X}_p^k}{p^{ks}}$$

(see the proof of Proposition 3.2.9). The series converge almost surely for $\mathrm{Re}(s) > 1/2$. We claim that the support of the distribution of $\tilde{\mathrm{P}}$, when viewed as an H(D)-valued random variable, is equal to H(D). Let us first assume this.

Since $\mathrm{Z} = \exp(\tilde{\mathrm{P}})$, we deduce by composition (see Lemma B.2.1) that the support of Z is the closure of the set of functions of the form e^g, where $g \in \mathrm{H(D)}$. But this last set is precisely $\mathrm{H(D)}^\times$, and Lemma A.5.5 in Appendix A shows that its closure in H(D) is $\mathrm{H(D)}^\times \cup \{0\}$.

Finally, to prove the approximation property (3.2), which is the original version of Voronin's Universality Theorem, we simply apply Lemma B.3.3 to the family of random variables Z_T, which gives the much stronger statement that for any $\varepsilon > 0$, we have

$$\liminf_{\mathrm{T} \to +\infty} \lambda \left(\{ t \in [-\mathrm{T}, \mathrm{T}] \mid \sup_{s \in \mathrm{D}} |\zeta(s + it) - f(s)| < \varepsilon \} \right) > 0,$$

where λ denotes Lebesgue measure.

From Proposition B.10.8 in Appendix B, the following proposition will imply that the support of the random Dirichlet series P is H(D). The statement is slightly more general to help with the last step afterward.

Proposition 3.3.1 *Let τ be such that $1/2 < \tau < 1$. Let $r > 0$ be such that*

$$\mathrm{D} = \{ s \in \mathbf{C} \mid |s - \tau| \leqslant r \} \subset \{ s \in \mathbf{C} \mid 1/2 < \mathrm{Re}(s) < 1 \}.$$

Let N *be an arbitrary positive real number. The set of all series*

$$\sum_{p > \mathrm{N}} \frac{x_p}{p^s} \quad \text{with} \quad (x_p) \in \widehat{\mathbf{S}}^1,$$

which converge in H(D)*, is dense in* H(D).

We will deduce the proposition from the density criterion of Theorem A.5.1 in Appendix A, applied to the space H(D) and the sequence (f_p) with $f_p(s) = p^{-s}$ for p prime. Since $\| f_p \|_\infty = p^{-\sigma_1}$, where $\sigma_1 = \tau - r > 1/2$, the condition

$$\sum_p \| f_p \|_\infty^2 < +\infty$$

holds. Furthermore, Proposition 3.2.9 certainly shows that there exist *some* $(x_p) \in \widehat{\mathbf{S}}^1$ such that the series $\sum_p x_p f_p$ converges in H(D). Hence the conclusion of Theorem A.5.1 is what we seek, and we only need to check the following lemma to establish the last hypothesis required to apply it:

Lemma 3.3.2 *Let $\mu \in \mathrm{C}(\bar{\mathrm{D}})'$ be a continuous linear functional. Let*

$$g(z) = \mu(s \mapsto e^{sz})$$

be its Laplace transform. If

$$\sum_p |g(\log p)| < +\infty, \tag{3.13}$$

then we have $g = 0$.

Indeed, the point is that $\mu(f_p) = \mu(s \mapsto p^{-s}) = g(\log p)$, so that the assumption (3.13) concerning g is precisely (A.3).

This is a statement that has some arithmetic content, as we will see, and indeed the proof involves the Prime Number Theorem.

Proof Let

$$\varrho = \limsup_{r \to +\infty} \frac{\log |g(r)|}{r},$$

which is finite by Lemma A.5.2 (1). By Lemma A.5.2 (3), it suffices to prove that $\varrho \leqslant 1/2$ to conclude that $g = 0$. To do this, we will use Theorem A.5.3, that provides access to the value of ϱ by "sampling" g along certain sequences of real numbers tending to infinity.

The idea is that (3.13) implies that $|g(\log p)|$ cannot be often of size at least $1/p = e^{-\log p}$, since the series $\sum p^{-1}$ diverges. Since the sequence $\log p$ increase slowly, this makes it possible to find real numbers $r_k \to +\infty$ growing linearly and such that $|g(r_k)| \leqslant e^{-r_k}$, and from this and Theorem A.5.3 we will get a contradiction.

To be precise, we first note that for $y \in \mathbf{R}$, we have

$$|g(iy)| \leqslant \|\mu\| \, \|s \mapsto e^{iys}\|_\infty \leqslant \|\mu\| e^{r|y|}$$

(since the maximum of the absolute value of the imaginary part of $s \in \bar{\mathrm{D}}$ is r), and therefore

$$\limsup_{\substack{y \in \mathbf{R} \\ |y| \to +\infty}} \frac{\log |g(iy)|}{|y|} \leqslant r.$$

We put $\alpha = r \leqslant 1/4$. Then the first condition of Theorem A.5.3 holds for the function g. We also take $\beta = 1$ so that $\alpha\beta < \pi$.

For any $k \geqslant 0$, let I_k be the set of primes p such that $e^k \leqslant p < e^{k+1}$. By the Mertens Formula (C.4), or the Prime Number Theorem, we have

$$\sum_{p \in \mathrm{I}_k} \frac{1}{p} \sim \frac{1}{k}$$

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as $k \to +\infty$. Let further A be the set of those $k \geqslant 0$ for which the inequality

$$|g(\log p)| \geqslant \frac{1}{p}$$

holds for *all* primes $p \in \mathrm{I}_k$, and let B be its complement among the nonnegative integers. We then note that

$$\sum_{k\in \mathrm{A}} \frac{1}{k} \ll \sum_{k\in \mathrm{A}} \sum_{p\in \mathrm{I}_k} \frac{1}{p} \ll \sum_{k\in \mathrm{A}} \sum_{p\in \mathrm{I}_k} |g(\log p)| < +\infty.$$

This shows that B is infinite. For $k \in \mathrm{B}$, let p_k be a prime in I_k such that $|g(\log p_k)| < p_k^{-1}$. Let $r_k = \log p_k$. We then have

$$\limsup_{k\to+\infty} \frac{\log|g(r_k)|}{r_k} \leqslant -1.$$

Since $p_k \in \mathrm{I}_k$, we have

$$r_k = \log p_k \sim k.$$

Furthermore, if we order B in increasing order, the fact that

$$\sum_{k\notin \mathrm{B}} \frac{1}{k} < +\infty$$

implies that the kth element n_k of B satisfies $n_k \sim k$.

Now we consider the sequence formed from the r_{2k}, arranged in increasing order. We have $r_{2k}/k \to 2$ from the above. Moreover, since $r_k \in \mathrm{I}_k$, we have

$$r_{2k+2} - r_{2k} \geqslant 1,$$

by construction, hence $|r_{2k} - r_{2l}| \gg |k - l|$. Since $|g(r_{2k})| \leqslant e^{-r_{2k}}$ for all $k \in \mathrm{B}$, we can apply Theorem A.5.3 to this increasing sequence and we get

$$\varrho = \limsup_{k\to+\infty} \frac{\log|g(r_{2k})|}{r_{2k}} \leqslant -1 < 1/2,$$

as desired. □

There remains a last lemma to prove, that allows us to go from the support of the series $\mathrm{P}(s)$ of independent random variables to that of the full series $\tilde{\mathrm{P}}(s)$.

Lemma 3.3.3 *The support of* $\tilde{\mathrm{P}}(s)$ *is* $\mathrm{H}(\mathrm{D})$.

Proof We can write

$$\tilde{\mathrm{P}} = -\sum_{p} \log(1 - \mathrm{X}_p p^{-s}),$$

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where the random variables $(\log(1-X_p p^{-s}))_p$ are independent, and the series converges almost surely in H(D). Therefore it is enough by Proposition B.10.8 to prove that the set of convergent series

$$-\sum_p \log(1 - x_p p^{-s}) \quad \text{and} \quad (x_p) \in \widehat{\mathbf{S}}^1$$

is dense in H(D).

Fix $f \in \mathrm{H(D)}$ and $\varepsilon > 0$ be fixed. For $\mathrm{N} \geqslant 1$ and any $(x_p) \in \widehat{\mathbf{S}}^1$, let

$$h_{\mathrm{N}}(s) = \sum_{p>\mathrm{N}} \sum_{k \geqslant 2} \frac{x_p^k}{kp^{ks}}.$$

This series converges absolutely for any s such that $\mathrm{Re}(s) > 1/2$ and $(x_p) \in \widehat{\mathbf{S}}^1$, and we have

$$\|h_{\mathrm{N}}\|_\infty \leqslant \sum_{p>\mathrm{N}} \sum_{k \geqslant 2} \frac{1}{kp^{k/2}} \to 0$$

as $\mathrm{N} \to +\infty$, uniformly with respect to $(x_p) \in \widehat{\mathbf{S}}^1$. Fix N such that $\|h_{\mathrm{N}}\|_\infty < \frac{\varepsilon}{2}$ for any $(x_p) \in \widehat{\mathbf{S}}^1$.

Now let $x_p = 1$ for $p \leqslant \mathrm{N}$ and define $f_0 \in \mathrm{H(D)}$ by

$$f_0(s) = f(s) + \sum_{p \leqslant \mathrm{N}} \log(1 - x_p p^{-s}).$$

For any choice of $(x_p)_{p>\mathrm{N}}$ such that the series

$$\sum_p \frac{x_p}{p^s}$$

defines an element of H(D), we can then write

$$f(s) + \sum_p \log(1 - x_p p^{-s}) = g_{\mathrm{N}}(s) + f_0(s) + h_{\mathrm{N}}(s),$$

for $s \in \mathrm{D}$, where

$$g_{\mathrm{N}}(s) = \sum_{p>\mathrm{N}} \frac{x_p}{p^s}.$$

By Proposition 3.3.1, there exists $(x_p)_{p>\mathrm{N}}$ such that the series g_{N} converges in H(D) and $\|g_{\mathrm{N}} + f_0\|_\infty < \frac{\varepsilon}{2}$. We then have

$$\left\| f + \sum_p \log(1 - x_p p^{-s}) \right\|_\infty < \varepsilon.$$

□

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Exercise 3.3.4 This exercise uses Voronin's Theorem to deduce that the Riemann zeta function is not the solution to any algebraic differential equation.

(1) For $(a_0, \ldots, a_m) \in \mathbf{C}^{m+1}$ such that $a_0 \neq 0$, prove that there exists $(b_0, \ldots, b_m) \in \mathbf{C}^{m+1}$ such that we have

$$\exp\left(\sum_{k=0}^{m} b_k s^k\right) = \sum_{k=0}^{m} \frac{a_k}{k!} s^k + \mathrm{O}(s^{m+1})$$

for $s \in \mathbf{C}$.

Now fix a real number σ with $\frac{1}{2} < \sigma < 1$, and let g be a holomorphic function on $\mathbf{C}$ which does not vanish.

(2) For any $\varepsilon > 0$, prove that there exists a real number t and $r > 0$ such that

$$\sup_{|s| \leqslant r} |\zeta(s + \sigma + it) - g(s)| < \varepsilon \frac{r^k}{k!}.$$

(3) Let $n \geqslant 1$ be an integer. Prove that there exists $t \in \mathbf{R}$ such that for all integers k with $0 \leqslant k \leqslant n-1$, we have

$$|\zeta^{(k)}(\sigma + it) - g^{(k)}(0)| < \varepsilon.$$

(4) Let $n \geqslant 1$ be an integer. Prove that the image in $\mathbf{C}^n$ of the map

$$\begin{cases} \mathbf{R} \longrightarrow \mathbf{C}^n, \\ t \mapsto (\zeta(\sigma + it), \ldots, \zeta^{(n-1)}(\sigma + it)) \end{cases}$$

is dense in $\mathbf{C}^n$.

(5) Using (4), prove that if $n \geqslant 1$ and $\mathrm{N} \geqslant 1$ are integers, and $\mathrm{F}_0, \ldots, \mathrm{F}_\mathrm{N}$ are continuous functions $\mathbf{C}^n \to \mathbf{C}$, not all identically zero, then the function

$$\sum_{k=0}^{\mathrm{N}} s^k \mathrm{F}_k(\zeta(s), \zeta'(s), \ldots, \zeta^{(n-1)}(s))$$

is not identically zero. In particular, the Riemann zeta function satisfies no algebraic differential equation.

3.4 Generalizations

If we look back at the proof of Bagchi's Theorem, and at the proof of Voronin's Theorem, we can see precisely which arithmetic ingredients appeared. They are the following:

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- the crucial link between the arithmetic objects and the probabilistic model is provided by Proposition 3.2.5, which depends on the unique factorization of integers into primes; this is an illustration of the asymptotic independence of prime numbers, similarly to Proposition 1.3.7;
- the proof of Bagchi's Theorem then relies on the mean-value property (3.11) of the Riemann zeta function; this estimate has arithmetic meaning;
- the Prime Number Theorem, which appears in the proof of Voronin's Theorem, in order to control the distribution of primes in (roughly) dyadic intervals.

Note that some arithmetic features remain in the Random Dirichlet Series that arises as the limit in Bagchi's Theorem, in contrast with the Erdős–Kac Theorem, where the limit is the universal Gaussian distribution. This means, in particular, that going beyond Bagchi's Theorem to applications (as in Voronin's Theorem) still naturally involves arithmetic problems, many of which are very interesting in their interaction with probability theory (see below for a few references).

From this analysis, it shouldn't be very surprising that Bagchi's Theorem can be generalized to many other situations. The most interesting concerns perhaps the limiting behavior, in $\mathrm{H}(\mathrm{D})$, of families of L-functions of the type

$$\mathrm{L}(f,s) = \sum_{n\geqslant 1} \lambda_f(n) n^{-s},$$

where f runs over some sequence of arithmetic objects with associated L-functions, ordered in a sequence of probability spaces (which need not be continuous like Ω_{T}). We refer to [59, Ch. 5] for a survey and discussion of L-functions, and to [69] for a discussion of families of L-functions. There are some rather elementary special cases, such as the vertical translates $\mathrm{L}(\chi, s+it)$ of a fixed Dirichlet L-function $\mathrm{L}(\chi,s)$, since almost all properties of the Riemann zeta function extend quite easily to this case. Another interesting case is the finite set Ω_q of nontrivial Dirichlet characters modulo a prime number q, with the uniform probability measure. Then one can look at the distribution of the restrictions to D of the Dirichlet L-functions $\mathrm{L}(s,\chi)$ for $\chi \in \Omega_q$, and indeed one can check that Bagchi's Theorem extends to this situation.

A second example, which is treated in [72] is, still for a prime $q \geqslant 2$, the set Ω_q of holomorphic cuspidal modular forms of weight 2 and level q, either with the uniform probability measure, or with that provided by the Petersson formula ([69, 31, ex. 8]). An analogue of Bagchi's Theorem holds, but the limiting random Dirichlet series is not the same as in Theorem 3.2.1: with the Petersson average, it is

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$$\prod_p (1 - \mathrm{X}_p p^{-s} + p^{-2s})^{-1}, \tag{3.14}$$

where (X_p) is a sequence of independent random variables, which are all distributed according to the Sato–Tate measure (the same that appears in Example B.6.1 (3)). This different limit is simply due to the form that “local spectral equidistribution” (in the sense of [69]) takes for this family (see [69, 38]). Indeed, the local spectral equidistribution property plays the role of Proposition 3.2.5. The analogue of (3.11) follows from a stronger mean-square formula, using the Cauchy–Schwarz inequality: there exists a constant $\mathrm{A} > 0$ such that, for any $\sigma_0 > 1/2$ and all $s \in \mathbf{C}$ with $\mathrm{Re}(s) \geqslant \sigma_0$, we have

$$\sum_{f \in \Omega_q} \omega_f |\mathrm{L}(f, s)|^2 \ll (1 + |s|)^{\mathrm{A}} \tag{3.15}$$

for $q \geqslant 2$, where ω_f is the Petersson-averaging weight (see [76, Prop. 5], which proves an even more difficult result where $\mathrm{Re}(s)$ can be as small as $\frac{1}{2} + c(\log q)^{-1}$).

However, extending Bagchi’s Theorem to many other families of L-functions (e.g., vertical translates of an L-function of higher rank) requires restrictions, in the current state of knowledge. The reason is that the analogue of the mean-value estimates (3.11) or (3.15) is usually only known when $\mathrm{Re}(s) \geqslant \sigma_0 > 1/2$, for some σ_0 such that $\sigma_0 < 1$. Then the only domains D for which one can prove a version of Bagchi’s Theorem are those contained in $\mathrm{Re}(s) > \sigma_0$.

[Further references: Titchmarsh [117], especially Chapter 11, discusses the older work of Bohr and Jessen, which has some interesting geometric aspects that are not apparent in modern treatments. Bagchi’s Thesis [4] contains some generalizations as well as more information concerning the limit theorem and Voronin’s Theorem.**]**

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