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THE FIXED POINT PROPERTY IN DIRECT SUMS AND MODULUS $R(a, X)$

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Abstract

We show that the direct sum $(X_1 \oplus \cdots \oplus X_r)_\psi$ with a strictly monotone norm has the weak fixed point property for nonexpansive mappings whenever $M(X_i) > 1$ for each $i = 1, \ldots, r$. In particular, $(X_1 \oplus \cdots \oplus X_r)_\psi$ enjoys the fixed point property if Banach spaces X_i are uniformly nonsquare. This combined with the earlier results gives a definitive answer for $r = 2$: a direct sum $X_1 \oplus_\psi X_2$ of uniformly nonsquare spaces with any monotone norm has the fixed point property. Our results are extended to asymptotically nonexpansive mappings in the intermediate sense.

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1. Introduction

A Banach space X is said to have the fixed point property (FPP) if every nonexpansive mapping $T : C \to C$, that is,

$$\|Tx - Ty\| \leq \|x - y\|, \quad x, y \in C,$$

acting on a (nonempty) bounded closed and convex subset C of X has a fixed point. A Banach space X is said to have the weak fixed point property (WFPP) if we additionally assume that C is weakly compact. The fixed point theorem of Kirk [19] asserts that every Banach space with weak normal structure has the WFPP. Recall that a Banach space X has weak normal structure if $r(C) < \operatorname{diam} C$ for all weakly compact convex subsets C of X consisting of more than one point, where $r(C) = \inf_{x\in C} \sup_{y\in C} \|x - y\|$ is the Chebyshev radius of C. For more information regarding metric fixed point theory we refer the reader to [1, 14, 21].

The permanence properties of normal structure and other conditions which guarantee the FPP under the direct sum operation have been studied extensively since the 1968 theorem of Belluce *et al.* [3], which states that a direct sum of two Banach spaces with normal structure, endowed with the maximum norm, also has normal structure. Nowadays, there exist many results concerning permanence properties of conditions which imply normal structure (see [8] for a survey). However, the problem is more difficult if at least one of the spaces lacks weak normal structure (see [29] and references therein) and it is quite well understood only for nonexpansive mappings defined on rectangles $C_1 \times C_2$ (see [20, 22]).

It was recently proved in [29] that if a Banach space X has the WFPP and Y has the generalised Gossez–Lami Dozo property (which is a slightly stronger property than weak normal structure) or is uniformly convex in every direction, then the direct sum $X \oplus Y$ with a strictly monotone norm has the WFPP. The present paper is concerned with direct sums $(X_1 \oplus \cdots \oplus X_r)_\psi$ of Banach spaces with $M(X_i) > 1$. (See Section 2 for the definitions of a ψ-direct sum, $M(X)$ and $R(a, X)$.) Dhompongsa *et al.* [7] proved that $(X_1 \oplus \cdots \oplus X_r)_\psi$ has the WFPP whenever $M(X_i) > 1$ for each $i = 1, \ldots, r$ and $\psi \in \Psi_r$ is strictly convex. Kato and Tamura [17] proved that if $\psi \neq \psi_1$ then $R(X_1 \oplus_\psi X_2) < 2$ if and only if $R(X_1) < 2$ and $R(X_2) < 2$. Subsequently, Kato and Tamura [18] showed that $R(a, (X_1 \oplus \cdots \oplus X_r)_\infty) = \max_{1 \le i \le r} R(a, X_i)$ and, consequently, $(X_1 \oplus \cdots \oplus X_r)_\infty$ has the WFPP whenever $M(X_i) > 1$ for each i.

In this paper we show that $(X_1 \oplus \cdots \oplus X_r)_\psi$ has the WFPP if $M(X_i) > 1$ for each $i = 1, \ldots, r$ and the norm $\|\cdot\|_\psi$ is strictly monotone. In particular, $(X_1 \oplus \cdots \oplus X_r)_\psi$ has the FPP if Banach spaces X_i are uniformly nonsquare and $\|\cdot\|_\psi$ is strictly monotone. This, combined with the aforementioned results, gives a definitive answer for $r = 2$: a direct sum $X_1 \oplus_\psi X_2$ of uniformly nonsquare spaces with any monotone norm has the FPP. Theorems 3.8 and 3.9 extend our results to asymptotically nonexpansive mappings in the intermediate sense.

2. Preliminaries

The modulus $R(a, X)$ of a Banach space X was defined by Domínguez Benavides [9] as a generalisation of the coefficient $R(X)$ introduced by García Falset [11]. Recall that, for a given $a \ge 0$,

$$R(a, X) = \sup\{\liminf_{n\to\infty} \|x_n + x\|\},$$

where the supremum is taken over all $x \in X$ with $\|x\| \le a$ and all weakly null sequences in the unit ball B_X such that

$$D((x_n)) = \limsup_{n\to\infty} \limsup_{m\to\infty} \|x_n - x_m\| \le 1.$$

Define

$$M(X) = \sup\Big\{\frac{1 + a}{R(a, X)} : a \ge 0\Big\}.$$

Then the condition $M(X) > 1$ implies that X has the WFPP for nonexpansive mappings (see [9]). We shall need the following characterisation of $M(X)$ proved in [5, Lemma 4.3] (see also [12, Corollary 4.3]).

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LEMMA 2.1. *Let X be a Banach space. The following conditions are equivalent:*

(i) $M(X) > 1$;

(ii) *there exists* $a > 0$ *such that* $R(a, X) < 1 + a$;

(iii) *for every* $a > 0$, $R(a, X) < 1 + a$.

Let us now recall terminology concerning direct sums. A norm $\|\cdot\|$ on $\mathbb{R}^n$ is said to be monotone if

$$\|(x_1, \ldots, x_n)\| \le \|(y_1, \ldots, y_n)\|$$

whenever $|x_1| \le |y_1|, \ldots, |x_n| \le |y_n|$. A norm $\|\cdot\|$ is said to be strictly monotone if

$$\|(x_1, \ldots, x_n)\| < \|(y_1, \ldots, y_n)\|$$

whenever $|x_i| \le |y_i|$ for $i = 1, \ldots, n$ and $|x_{i_0}| < |y_{i_0}|$ for some i_0. It is easy to see that ℓ_p^n-norms, $1 \le p < \infty$, are strictly monotone. A norm $\|\cdot\|$ on $\mathbb{R}^n$ is said to be absolute if

$$\|(x_1, \ldots, x_n)\| = \|(|x_1|, \ldots, |x_n|)\|$$

for every $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. It is well known (see [2]) that a norm is absolute if and only if it is monotone.

We will assume that the norm is normalised, that is,

$$\|(1, 0, \ldots, 0)\| = \cdots = \|(0, \ldots, 0, 1)\| = 1.$$

Bonsall and Duncan [6] showed that the set of all absolute and normalised norms on $\mathbb{R}^2$ ($\mathbb{C}$) is in one-to-one correspondence with the set Ψ of all continuous convex functions on $[0, 1]$ satisfying $\psi(0) = \psi(1) = 1$ and $\max\{1 - t, t\} \le \psi(t) \le 1$ for $0 \le t \le 1$, where the correspondence is given by

$$\psi(t) = \|(1 - t, t)\|, \quad 0 \le t \le 1. \tag{2.1}$$

Conversely, for any $\psi \in \Psi$ define

$$\|(x_1, x_2)\|_\psi = (|x_1| + |x_2|)\psi\Big(\frac{|x_2|}{|x_1| + |x_2|}\Big)$$

for $(x_1, x_2) \ne (0, 0)$ and $\|(0, 0)\|_\psi = 0$. Then $\|\cdot\|_\psi$ is an absolute and normalised norm which satisfies (2.1) (see [6, 24]). For example, the ℓ_p^2-norms correspond to the functions

$$\psi_p(t) = \begin{cases} ((1 - t)^p + t^p)^{1/p} & \text{if } 1 \le p < \infty, \\ \max\{1 - t, t\} & \text{if } p = \infty. \end{cases}$$

It was proved in [27, Corollary 3] that a norm $\|\cdot\|_\psi$ in $\mathbb{R}^2$ is normalised and strictly monotone if and only if

$$\psi(t) > \psi_\infty(t)$$

for all $0 < t < 1$.

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Saito *et al.* [25] generalised the result of [6] to the n-dimensional case. Let

$$\triangle_n = \{(s_1, \ldots, s_{n-1}) \in \mathbb{R}^{n-1} : s_1 + \cdots + s_{n-1} \le 1, s_i \ge 0, i = 1, \ldots, n-1\}$$

and denote by Ψ_n the set of all continuous convex functions on $\triangle_n$ which satisfy the following conditions:

$$\psi(1, 0, \ldots, 0) = \psi(0, 1, 0, \ldots, 0) = \cdots = \psi(0, \ldots, 0, 1) = 1,$$
$$\psi(s_1, \ldots, s_{n-1}) \ge (s_1 + \cdots + s_{n-1})\psi\Big(\frac{s_1}{s_1 + \cdots + s_{n-1}}, \ldots, \frac{s_{n-1}}{s_1 + \cdots + s_{n-1}}\Big),$$
$$\psi(s_1, \ldots, s_{n-1}) \ge (1 - s_1)\psi\Big(0, \frac{s_2}{1 - s_1}, \ldots, \frac{s_{n-1}}{1 - s_1}\Big),$$
$$\vdots$$
$$\psi(s_1, \ldots, s_{n-1}) \ge (1 - s_{n-1})\psi\Big(\frac{s_1}{1 - s_{n-1}}, \ldots, \frac{s_{n-2}}{1 - s_{n-1}}, 0\Big).$$

Then the set of all absolute and normalised norms on $\mathbb{R}^n$ ($\mathbb{C}^n$) is in one-to-one correspondence with the set Ψ_n, where the correspondence is given by

$$\psi(s_1, \ldots, s_{n-1}) = \Big\|\Big(1 - \sum_{i=1}^{n-1} s_i, s_1, \ldots, s_{n-1}\Big)\Big\|, \quad (s_1, \ldots, s_{n-1}) \in \triangle_n, \tag{2.2}$$

and if $\psi \in \Psi_n$ then

$$\|(x_1, x_2, \ldots, x_n)\|_\psi = (|x_1| + |x_2| + \cdots + |x_n|) \times \psi\Big(\frac{|x_2|}{|x_1| + |x_2| + \cdots + |x_n|}, \ldots, \frac{|x_n|}{|x_1| + |x_2| + \cdots + |x_n|}\Big),$$

defined for $(x_1, x_2, \ldots, x_n) \ne (0, 0, \ldots, 0)$, $\|(0, 0, \ldots, 0)\|_\psi = 0$, is an absolute and normalised norm which satisfies (2.2) (see [25, Theorem 3.4]).

Let $X_1, \ldots, X_r$ be Banach spaces and $\psi \in \Psi_r$. Following [27], we shall write $(X_1 \oplus \cdots \oplus X_r)_\psi$ for the ψ-direct sum with the norm $\|(x_1, \ldots, x_r)\|_\psi = \|(\|x_1\|, \ldots, \|x_r\|)\|_\psi$, where $(x_1, \ldots, x_r) \in X_1 \times \cdots \times X_r$.

3. Fixed point theorems

Let $T : C \to C$ be a nonexpansive mapping, where C is a nonempty weakly compact convex subset of a Banach space X. By the Kuratowski–Zorn lemma, there exists a minimal (in the sense of inclusion) convex and weakly compact set $K \subset C$ which is invariant under T. Let (x_n) be an approximate fixed point sequence for T in K, that is, $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. The Goebel–Karlovitz lemma (see [13, 15]) asserts that

$$\lim_{n\to\infty} \|x_n - x\| = \operatorname{diam} K$$

for every $x \in K$.

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Suppose now that $X_1, X_2, \ldots, X_r$ are Banach spaces, $\psi \in \Psi_r$ and $T : K \to K$ is a nonexpansive mapping acting on a weakly compact convex and minimal invariant subset K of a direct sum $(X_1 \oplus \cdots \oplus X_r)_\psi$. Suppose further that $\operatorname{diam} K = 1$ and $(w_n) = ((x_n^{(1)}, \ldots, x_n^{(r)}))$ is an approximate fixed point sequence for T in K weakly converging to $(0, \ldots, 0) \in K$ such that $\lim_{n\to\infty} \|x_n^{(r)}\| = 0$. The following construction (for $r = 2$) was proposed in [29]. Fix an integer $k \geq 1$ and a sequence (ε_n) in $(0, 1)$. Since $\|Tw_n - w_n\|_\psi$ and $\|x_n^{(r)}\|$ converge to 0, we can choose w_{n_1} in such a way that $\|Tw_{n_1} - w_{n_1}\|_\psi < \varepsilon_1$ and $\|x_{n_1}^{(r)}\| < \varepsilon_1$. Let us put

$$D_1^1 = \{w_{n_1}\}, D_2^1 = \operatorname{conv}(\{w_{n_1}, Tw_{n_1}\}), \ldots, D_k^1 = \operatorname{conv}(D_{k-1}^1 \cup T(D_{k-1}^1))$$

where $\operatorname{conv} A$ denotes the convex hull of A. We obtain a family $D_1^1 \subset D_2^1 \subset \cdots \subset D_k^1$ of relatively compact convex subsets of K. It follows from the Goebel–Karlovitz lemma and the relative compactness of D_k^1 that there exists $n_2 > n_1$ such that $\|Tw_{n_2} - w_{n_2}\|_\psi < \varepsilon_2$, $\|x_{n_2}^{(r)}\| < \varepsilon_2$ and $\|w_{n_2} - z\|_\psi > 1 - \varepsilon_2$ for all $z \in D_k^1$. Put

$$D_1^2 = \operatorname{conv}(\{w_{n_1}, w_{n_2}\}), \ldots, D_k^2 = \operatorname{conv}(D_{k-1}^2 \cup T(D_{k-1}^2)).$$

Again, we can find $n_3 > n_2$ such that $\|Tw_{n_3} - w_{n_3}\|_\psi < \varepsilon_3$, $\|x_{n_3}^{(r)}\| < \varepsilon_3$ and $\|w_{n_3} - z\|_\psi > 1 - \varepsilon_3$ for all $z \in D_k^2$. Put

$$D_1^3 = \operatorname{conv}(\{w_{n_1}, w_{n_2}, w_{n_3}\}), \ldots, D_k^3 = \operatorname{conv}(D_{k-1}^3 \cup T(D_{k-1}^3)).$$

Continuing in this fashion, we obtain by induction a subsequence (w_{n_j}) of (w_n) and a family $\{D_j^i\}_{1\leq j\leq k, i\geq 1}$ of subsets of K such that:

(i) $\|Tw_{n_i} - w_{n_i}\|_\psi < \varepsilon_i$;
(ii) $\|x_{n_i}^{(r)}\| < \varepsilon_i$;
(iii) $\|w_{n_i} - z\|_\psi > 1 - \varepsilon_i$ for all $z \in D_k^{i-1}$ (where $D_k^0 = \emptyset$);
(iv) $D_1^i = \operatorname{conv}(\{w_{n_1}, w_{n_2}, \ldots, w_{n_i}\})$;
(v) $D_{j+1}^i = \operatorname{conv}(D_j^i \cup T(D_j^i))$,

for every $i \geq 1$ and $j = 1, \ldots, k - 1$ (see [29, Lemma 3.1]).

Furthermore (see [29, Lemma 3.2]), for every $i \geq 1$, $j = 1, \ldots, k$ and $u \in D_j^{i+1}$, there exists $z \in D_j^i$ such that

$$\|z - u\|_\psi + \|u - w_{n_{i+1}}\|_\psi \leq \|z - w_{n_{i+1}}\|_\psi + 3(j - 1)\varepsilon_{i+1}.$$

We need the following simple observation regarding the set D_k^1.

Lemma 3.1. *If $u = (y^{(1)}, \ldots, y^{(r)}) \in D_k^1$ then $\|y^{(r)}\| < k\varepsilon_1$.*

Proof. We first show by induction that for every $j \in \{2, \ldots, k\}$ and for every $v \in D_j^1$,

$$\|v - w_{n_1}\|_\psi < (j - 1)\varepsilon_1. \tag{3.1}$$

For $j = 2$, $D_2^1 = \operatorname{conv}(\{w_{n_1}, Tw_{n_1}\})$ and inequality (3.1) follows from the fact that $\|Tw_{n_1} - w_{n_1}\|_\psi < \varepsilon_1$.

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Now fix $j \in \{2, \ldots, k-1\}$ and assume that for every $u \in D_j^1$, $\|u - w_{n_1}\|_\psi < (j-1)\varepsilon_1$. Since

$$D_{j+1}^1 = \operatorname{conv}(D_j^1 \cup T(D_j^1)),$$

it is enough to show that $\|v - w_{n_1}\|_\psi < j\varepsilon_1$ for every $v \in T(D_j^1)$. Let $v = Tu$ for some $u \in D_j^1$. Then

$$\|v - w_{n_1}\|_\psi \le \|Tu - Tw_{n_1}\|_\psi + \|Tw_{n_1} - w_{n_1}\|_\psi \le \|u - w_{n_1}\|_\psi + \varepsilon_1 < j\varepsilon_1,$$

and the proof of (3.1) is complete.

Let $w_{n_1} = (x_{n_1}^{(1)}, \ldots, x_{n_1}^{(r)})$, $u = (y^{(1)}, \ldots, y^{(r)}) \in D_k^1$. It follows from (3.1) that

$$\|y^{(r)} - x_{n_1}^{(r)}\| \le \|u - w_{n_1}\|_\psi < (k-1)\varepsilon_1$$

and consequently, since $\|x_{n_1}^{(r)}\| < \varepsilon_1$, $\|y^{(r)}\| < k\varepsilon_1$. □

Notice that if the norm $\|\cdot\|_\psi$ on $\mathbb{R}^r$ is strictly monotone then, for every $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that if $(t^{(1)}, \ldots, t^{(r-1)}, t'), (t^{(1)}, \ldots, t^{(r-1)}, t'')$ belong to the unit ball of $(\mathbb{R}^r, \|\cdot\|_\psi)$ and

$$\|(t^{(1)}, \ldots, t^{(r-1)}, t')\|_\psi < \|(t^{(1)}, \ldots, t^{(r-1)}, t'')\|_\psi + \delta(\varepsilon),$$

then $|t'| < |t''| + \varepsilon$. Indeed, otherwise, there exist $\varepsilon_0 > 0$ and two sequences $((t_n^{(1)}, \ldots, t_n^{(r-1)}, t_n'))$, $((t_n^{(1)}, \ldots, t_n^{(r-1)}, t_n''))$ in the unit ball such that

$$\|(t_n^{(1)}, \ldots, t_n^{(r-1)}, t_n')\|_\psi < \|(t_n^{(1)}, \ldots, t_n^{(r-1)}, t_n'')\|_\psi + \frac{1}{n}$$

and $|t_n'| \ge |t_n''| + \varepsilon_0$. Passing to convergent subsequences and taking limits,

$$\|(g^{(1)}, \ldots, g^{(r-1)}, g')\|_\psi \le \|(g^{(1)}, \ldots, g^{(r-1)}, g'')\|_\psi$$

and $|g'| \ge |g''| + \varepsilon_0$, which contradicts the strict monotonicity of $\|\cdot\|_\psi$.

Thus, using Lemma 3.1, we can follow the arguments in [29, Lemma 3.3] and obtain the following result.

Lemma 3.2. *Let $X_1, X_2, \ldots, X_r$ be Banach spaces, $\psi \in \Psi_r$, and $T : K \to K$ a nonexpansive mapping acting on a weakly compact convex and minimal invariant subset K of a direct sum $(X_1 \oplus \cdots \oplus X_r)_\psi$ with a strictly monotone norm $\|\cdot\|_\psi$. Suppose that* $\operatorname{diam} K = 1$ *and $(w_n) = ((x_n^{(1)}, \ldots, x_n^{(r)}))$ is an approximate fixed point sequence for T in K weakly converging to $(0, \ldots, 0) \in K$ such that $\lim_{n\to\infty} \|x_n^{(r)}\| = 0$. Then for every positive integer k, there exist a sequence (ε_n) in $(0, 1)$, a subsequence (w_{n_j}) of (w_n) and a family $\{D_j^i\}_{1\le j\le k, i\ge 1}$ of subsets of K (depending on k) such that the above conditions* (i)–(v) *are satisfied and $\|y^{(r)}\| < 1/k$ for every $u = (y^{(1)}, \ldots, y^{(r)}) \in \bigcup_{i=1}^\infty D_k^i$.*

We show that the appropriate approximate fixed point sequence (w_n) exists if $M(X_i) > 1$, $i = 1, \ldots, r$.

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Lemma 3.3. *Let* $X_1, \ldots, X_r$ *be Banach spaces with* $M(X_i) > 1$ *for each* $i = 1, \ldots, r$, $\psi \in \Psi_r$, *and* $T : K \to K$ *a nonexpansive mapping acting on a weakly compact convex and minimal invariant subset* K *of a direct sum* $(X_1 \oplus \cdots \oplus X_r)_\psi$ *with a strictly monotone norm* $\|\cdot\|_\psi$. *Suppose that* $\operatorname{diam} K = 1$ *and* $(w_n) = ((x_n^{(1)}, \ldots, x_n^{(r)}))$ *is an approximate fixed point sequence for* T *in* K *weakly converging to* $(0, \ldots, 0) \in K$. *Then* $\liminf_{n\to\infty} \|x_n^{(i_0)}\| = 0$ *for some* $i_0 \in \{1, \ldots, r\}$.

Proof. Let $(w_n) = ((x_n^{(1)}, \ldots, x_n^{(r)}))$ be an approximate fixed point sequence for T in K weakly converging to $(0, \ldots, 0) \in K$ and suppose, contrary to our claim, that for each $i = 1, \ldots, r$, $\liminf_{n\to\infty} \|x_n^{(i)}\| > 0$. We can assume, passing to a subsequence, that

$$d_1 = \min\{\lim_{n\to\infty} \|x_n^{(1)}\|, \ldots, \lim_{n\to\infty} \|x_n^{(r)}\|\} > 0.$$

We will show that for every $\varepsilon > 0$ there exists $\delta_1(\varepsilon) > 0$ such that if $v \in K$ and $\|Tv - v\|_\psi < \delta_1(\varepsilon)$ then $\|v\|_\psi > 1 - \varepsilon$. Indeed, otherwise, arguing as in [10, Lemma 2], there exist $\varepsilon_0 > 0$ and a sequence $(v_n) \subset K$ such that $\|Tv_n - v_n\|_\psi < 1/n$ and $\|v_n\|_\psi \le 1 - \varepsilon_0$ for every $n \in \mathbb{N}$. Thus (v_n) is an approximate fixed point sequence in K, but $\limsup_{n\to\infty} \|v_n\|_\psi \le 1 - \varepsilon_0$, which contradicts the Goebel–Karlovitz lemma since $(0, \ldots, 0) \in K$.

Furthermore, since $(\mathbb{R}^r, \|\cdot\|_\psi)$ is finite-dimensional and the norm $\|\cdot\|_\psi$ is strictly monotone, for every $\eta > 0$, there exists $\delta_2(\eta) > 0$ such that if $(t^{(1)}, \ldots, t^{(r)}), (s^{(1)}, \ldots, s^{(r)})$ belong to the unit ball, $|t^{(i)}| \le |s^{(i)}|$ for $i = 1, \ldots, r$ and $|t^{(i_0)}| < |s^{(i_0)}| - \eta$ for some i_0, then

$$\|(t^{(1)}, \ldots, t^{(r)})\|_\psi < \|(s^{(1)}, \ldots, s^{(r)})\|_\psi - \delta_2(\eta).$$

Let

$$A = \inf_{d_1/6r \le a \le 1} \min_{1\le i\le r} (1 + a - R(a, X_i))$$

and take $0 < \eta < Ad_1/3, \varepsilon < \min\{\delta_2(\eta), d_1/6\}$ (notice that $A > 0$ by Lemma 2.1). Choose $t = 1 - (d_1/3), \gamma < \min\{1, \delta_1(\varepsilon)\}$ and define the contraction $S_n : K \to K$ by

$$S_n x = (1 - \gamma)Tx + \gamma t w_n.$$

By the contractive mapping principle, for any $n \in \mathbb{N}$, there exists a unique fixed point $z_n = (y_n^{(1)}, \ldots, y_n^{(r)})$ of S_n. We can assume, passing to a subsequence, that (z_n) converges weakly to $z = (y^{(1)}, \ldots, y^{(r)})$ and the limits $\lim_{n\to\infty} \|y_n^{(i)}\|$, $\lim_{n,m\to\infty, n\ne m} \|y_n^{(i)} - y_m^{(i)}\|$ exist for each $i = 1, \ldots, r$ (see, for example, [26]).

It is not difficult to see that

$$\|Tz_n - z_n\|_\psi \le \gamma < \delta_1(\varepsilon)$$

and hence

$$\|z_n\|_\psi > 1 - \varepsilon, \tag{3.2}$$

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for every $n \in \mathbb{N}$. Furthermore, for every $n, m \in \mathbb{N}$,

$$\|z_n - z_m\|_\psi \le t$$

and, from the weak lower semicontinuity of the norm,

$$\|z_n - z\|_\psi \le \liminf_{m\to\infty} \|z_n - z_m\|_\psi \le t.$$

Thus

$$\|z\|_\psi \ge \|z_n\|_\psi - \|z_n - z\|_\psi > 1 - \varepsilon - t > \frac{d_1}{6}$$

and, consequently, there exists $i_0 \in \{1, \ldots, r\}$ such that

$$\|y^{(i_0)}\| > \frac{d_1}{6r},$$

since otherwise

$$\|z\|_\psi \le \|y^{(1)}\| + \|y^{(2)}\| + \cdots + \|y^{(r)}\| \le \frac{d_1}{6}.$$

Furthermore,

$$\|z_n - w_n\|_\psi \le (1-\gamma)\|Tz_n - Tw_n\|_\psi + (1-\gamma)\|Tw_n - w_n\|_\psi + \gamma(1-t)\|w_n\|_\psi,$$

which gives

$$\|z_n - w_n\|_\psi \le 1 - t + \frac{(1-\gamma)}{\gamma}\|Tw_n - w_n\|_\psi.$$

It follows from the weak lower semicontinuity of the norm that, for each $i = 1, \ldots, r$,

$$\|y^{(i)}\| \le \liminf_{n\to\infty} \|y_n^{(i)} - x_n^{(i)}\| \le \liminf_{n\to\infty} \|z_n - w_n\|_\psi \le 1 - t = \frac{d_1}{3}$$

and, from the triangle inequality,

$$\|y_n^{(i)} - y^{(i)}\| \ge \|x_n^{(i)}\| - \|x_n^{(i)} - y_n^{(i)}\| - \|y^{(i)}\|.$$

Hence

$$\lim_{n\to\infty} \|y_n^{(i)} - y^{(i)}\| \ge \frac{d_1}{3}, \tag{3.3}$$

for each $i = 1, \ldots, r$. Write

$$d_2 = \lim_{n,m\to\infty, n\neq m} \|y_n^{(i_0)} - y_m^{(i_0)}\|$$

and notice that

$$d_2 = \limsup_{n\to\infty} \limsup_{m\to\infty} \|y_n^{(i_0)} - y_m^{(i_0)}\| \ge \lim_{n\to\infty} \|y_n^{(i_0)} - y^{(i_0)}\| \ge \frac{d_1}{3}$$

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by (3.3). It follows that

$$\begin{aligned}\lim_{n\to\infty}\|y_n^{(i_0)}\| &= \lim_{n\to\infty}\left\|\frac{y_n^{(i_0)}-y^{(i_0)}}{d_2}+\frac{y^{(i_0)}}{d_2}\right\|d_2\\ &\le R\left(\frac{\|y^{(i_0)}\|}{d_2},X_{i_0}\right)d_2 \le \left(1+\frac{\|y^{(i_0)}\|}{d_2}-A\right)d_2\\ &\le d_2+\|y^{(i_0)}\|-Ad_2 < d_2+\|y^{(i_0)}\|-\eta,\end{aligned}$$

since $d_1/6r \le \|y^{(i_0)}\|/d_2 \le 1$. Therefore,

$$\begin{aligned}\lim_{n\to\infty}\|z_n\|_\psi &< \|(\lim_{n\to\infty}\|y_n^{(1)}\|,\dots,d_2+\|y^{(i_0)}\|,\dots,\lim_{n\to\infty}\|y_n^{(r)}\|)\|_\psi-\delta_2(\eta)\\ &\le \|(\lim_{n\to\infty}\|y_n^{(1)}-y^{(1)}\|,\dots,\lim_{n\to\infty}\|y_n^{(r)}-y^{(r)}\|)\|_\psi\\ &\quad+\ \|(\|y^{(1)}\|,\dots,\|y^{(r)}\|)\|_\psi-\varepsilon\\ &\le \lim_{n,m\to\infty,n\neq m}\|z_n-z_m\|_\psi+\liminf_{n\to\infty}\|z_n-w_n\|_\psi-\varepsilon\\ &\le t+1-t-\varepsilon\le 1-\varepsilon,\end{aligned}$$

which contradicts (3.2). □

We can now formulate our main result. The proof combines the arguments of [29, Theorem 3.4] and Lemma 3.3.

Theorem 3.4. *Let $X_1,\dots,X_r$ be Banach spaces with $M(X_i)>1$ for each $i=1,\dots,r$, $\psi\in\Psi_r$. Then the direct sum $(X_1\oplus\cdots\oplus X_r)_\psi$ with a strictly monotone norm $\|\cdot\|_\psi$ has the WFPP.*

Proof. Assume that $(X_1\oplus\cdots\oplus X_r)_\psi$ does not have the WFPP. Then there exist a weakly compact convex subset C of $(X_1\oplus\cdots\oplus X_r)_\psi$ and a nonexpansive mapping $T: C\to C$ without a fixed point. By the Kuratowski–Zorn lemma, there exists a convex and weakly compact set $K\subset C$ which is minimal invariant under T and which is not a singleton. Let $(w_n)=((x_n^{(1)},\dots,x_n^{(r)}))$ be an approximate fixed point sequence for T in K. Without loss of generality we can assume that diam $K=1$, (w_n) converges weakly to $(0,\dots,0)\in K$ and the limits $\lim_{n,m\to\infty,n\neq m}\|w_n-w_m\|_\psi$, $\lim_{n\to\infty}\|x_n^{(i)}\|$, $i=1,\dots,r$, exist. Applying Lemma 3.3 gives $\lim_{n\to\infty}\|x_n^{(i_0)}\|=0$ for some $i_0\in\{1,\dots,r\}$, and by rearrangement of Banach spaces we can assume that $i_0=r$. Now we follow the arguments in [29]. Lemma 3.2 shows that, for every positive integer k, there exist a sequence (ε_n) in $(0,1)$, a subsequence (w_{n_j}) of (w_n) and a family $\{D_j^i\}_{1\le j\le k,i\ge 1}$ of subsets of K (depending on k) such that conditions (i)–(v) are satisfied and $\|y^{(r)}\|<1/k$ for every $u=(y^{(1)},\dots,y^{(r)})\in\bigcup_{i=1}^\infty D_k^i$.

Let $C_0=\{(0,\dots,0)\}$ and $C_j=\mathrm{conv}(C_{j-1}\cup T(C_{j-1}))$ for $j\ge 1$. It is not difficult to see that $\mathrm{cl}(\bigcup_{j=1}^\infty C_j)$ is a closed convex subset of K which is invariant for T (and hence equals K). Fix $k\ge 1$ and notice that $(0,\dots,0)\in\mathrm{cl}(\bigcup_{i=1}^\infty D_1^i)$, because the

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sequence $(w_{n_j})_{j\geq 1}$ converges weakly to $(0, \ldots, 0)$. Furthermore, for $j < k$,

$$T\Big(\mathrm{cl}\Big(\bigcup_{i=1}^{\infty} D_j^i\Big)\Big) = \mathrm{cl}\Big(\bigcup_{i=1}^{\infty} T(D_j^i)\Big) \subset \mathrm{cl}\Big(\bigcup_{i=1}^{\infty} D_{j+1}^i\Big)$$

and hence, by induction on j,

$$C_j \subset \mathrm{cl}\Big(\bigcup_{i=1}^{\infty} D_{j+1}^i\Big) \subset \mathrm{cl}\Big(\bigcup_{i=1}^{\infty} D_k^i\Big), \quad j < k.$$

It follows that if $(y^{(1)}, \ldots, y^{(r)}) \in C_j$ and $j < k$, then $(y^{(1)}, \ldots, y^{(r)}) \in \mathrm{cl}(\bigcup_{i=1}^{\infty} D_k^i)$ and, consequently, $\|y^{(r)}\| \leq 1/k$. Letting $k \to \infty$, we have $y^{(r)} = 0$ for every $(y^{(1)}, \ldots, y^{(r)}) \in \mathrm{cl}(\bigcup_{j=1}^{\infty} C_j) = K$. Therefore, K is a subset of $(X_1 \oplus \cdots \oplus X_{r-1} \oplus \{0\})_\psi$ which is isometric to $(X_1 \oplus \cdots \oplus X_{r-1})_{\psi'}$ with the strictly monotone norm $\|(x_1, \ldots, x_{r-1})\|_{\psi'} = \|(x_1, \ldots, x_{r-1}, 0)\|_\psi$. Repeating the above procedure $r - 1$ times, we deduce that K is isometric to some X_{j_0}, $j_0 \in \{1, \ldots, r\}$. Since $M(X_{j_0}) > 1$, T has a fixed point in K, which contradicts our assumption. □

García Falset *et al.* [12] introduced the modulus $RW(a, X)$, which plays an important role in fixed point theory for nonexpansive mappings. For a given $a \geq 0$,

$$RW(a, X) = \sup \min\{\liminf_n \|x_n + x\|, \liminf_n \|x_n - x\|\},$$

where the supremum is taken over all $x \in X$ with $\|x\| \leq a$ and all weakly null sequences in the unit ball B_X. Let

$$MW(X) = \sup\Big\{\frac{1 + a}{RW(a, X)} : a \geq 0\Big\}.$$

It was proved in [12, Theorem 3.3] that $M(X) \geq MW(X)$ whenever B_{X^*} is w^*-sequentially compact. Since a minimal invariant set K is always separable, we obtain the following corollary.

Corollary 3.5. *Let $X_1, \ldots, X_r$ be Banach spaces with $MW(X_i) > 1$ for each $i = 1, \ldots, r$, $\psi \in \Psi_r$. Then the direct sum $(X_1 \oplus \cdots \oplus X_r)_\psi$ with a strictly monotone norm $\|\cdot\|_\psi$ has the WFPP.*

Recall that a Banach space X is uniformly nonsquare if

$$J(X) = \sup_{x,y \in S_X} \min\{\|x + y\|, \|x - y\|\} < 2.$$

Corollary 4.4 in [12] shows that if X is uniformly nonsquare, then $MW(X) > 1$. Since uniformly nonsquare spaces are reflexive, the FPP and WFPP coincide and thus we obtain the following corollary.

Corollary 3.6. *Let $X_1, \ldots, X_r$ be uniformly nonsquare Banach spaces, $\psi \in \Psi_r$. Then the direct sum $(X_1 \oplus \cdots \oplus X_r)_\psi$ with a strictly monotone norm $\|\cdot\|_\psi$ has the FPP.*

In the case $r = 2$ we have a stronger, definitive result.

THEOREM 3.7. *Let X_1, X_2 be uniformly nonsquare Banach spaces, $\psi \in \Psi_2$. Then the direct sum $X_1 \oplus_\psi X_2$ has the FPP.*

PROOF. Kato *et al.* [16, Theorem 1] proved that $X_1 \oplus_\psi X_2$ is uniformly nonsquare if and only if X_1, X_2 are uniformly nonsquare and $\psi \neq \psi_1, \psi_\infty$ (see also [4]). Kato and Tamura [17, Theorem 3.6] showed that $R(X_1 \oplus_\infty X_2) < 2$ and hence $X_1 \oplus_\infty X_2$ has the FPP. The case $X_1 \oplus_1 X_2$ is covered by Corollary 3.6, since the norm $\|\cdot\|_{\psi_1}$ is strictly monotone. □

We conclude with two fixed point theorems for asymptotically nonexpansive mappings. Recall that a mapping $T : C \to C$ is said to be asymptotically nonexpansive in the intermediate sense if T is continuous and

$$\limsup_{n\to\infty} \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\|) \leq 0.$$

A Banach space X is said to have the super fixed point property for nonexpansive mappings (asymptotically nonexpansive mappings) if every Banach space Y which is finitely representable in X has the FPP for nonexpansive mappings (asymptotically nonexpansive mappings).

Theorem 2.4 in [30] shows that X has the super fixed point property for nonexpansive mappings if and only if X has the super fixed point property for asymptotically nonexpansive mappings in the intermediate sense. Since the direct sum $(X_1 \oplus \cdots \oplus X_r)_\psi$ of uniformly nonsquare spaces is stable under passing to the Banach space ultrapowers, it follows from the properties of ultrapowers and Corollary 3.6 (Theorem 3.7) that $(X_1 \oplus \cdots \oplus X_r)_\psi$ with a strictly monotone norm ($X_1 \oplus_\psi X_2$ with any monotone norm) has the super fixed property for nonexpansive mappings and, consequently, for asymptotically nonexpansive mappings in the intermediate sense. Thus we obtain the following theorem.

THEOREM 3.8. *Let $X_1, \ldots, X_r$ be uniformly nonsquare Banach spaces, $\psi \in \Psi_r$. Then the direct sum $(X_1 \oplus \cdots \oplus X_r)_\psi$ with a strictly monotone norm $\|\cdot\|_\psi$ has the (super) fixed point property for asymptotically nonexpansive mappings in the intermediate sense. The assumption about the strict monotonicity of the norm can be dropped if $r = 2$.*

Recall that [28, Theorem 2.3] shows that the direct sum $(X_1 \oplus \cdots \oplus X_r)_\psi$ of uniformly noncreasy spaces with a strictly monotone norm has the FPP. Since uniformly noncreasy spaces are stable under passing to the Banach space ultrapowers (see [23]), the conclusion of Theorem 3.8 is also valid in this case.

THEOREM 3.9. *Let $X_1, \ldots, X_r$ be uniformly noncreasy Banach spaces, $\psi \in \Psi_r$. Then the direct sum $(X_1 \oplus \cdots \oplus X_r)_\psi$ with a strictly monotone norm $\|\cdot\|_\psi$ has the super fixed point property for asymptotically nonexpansive mappings in the intermediate sense.*

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