THE COMPUTATIONAL APPROACH TO COMMUTING ORDINARY DIFFERENTIAL OPERATORS OF ORDERS SIX AND NINE

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Abstract

Entirely elementary methods are employed to determine explicit formulae for the coefficients of commuting ordinary differential operators of orders six and nine which correspond to an elliptic curve. These formulae come from solving the nonlinear ordinary differential equations which are equivalent to the commutativity condition. Most solutions turn out to be rational expressions in one or two arbitrary functions and their derivatives. The corresponding Burchnall-Chaundy curves are computed.

1. Introduction

The theory of commuting ordinary differential operators had its beginnings with the pioneering work of Burchnall and Chaundy [2, 3, 4] and Baker [1], and after some time in dormancy, has recently enjoyed a revival of interest with the application of commuting operators to completely integrable equations [9, 10, 11, 13, 14, 15, 17, 18, 22]. In their application to finding solutions of the Kadomtsev-Petviashvili (KP) equation for example, the basic idea is to search for solutions which lie in a special stationary manifold given by the condition of commutativity of two differential operators. It was in trying to make this idea exact that Krichever and Novikov and others were led to investigate the problem of finding explicit formulae for the coefficients of commuting operators whose orders were not coprime. The KP equation itself,

$$\tfrac{3}{4}U_{yy} = \left(U_t - \tfrac{3}{4}UU_x - \tfrac{1}{4}U_{xxx}\right)_x,$$

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for a function $U(x, y, t)$ where x and y are space coordinates and t is time, is of independent interest as a model for weakly dispersive waves in two space dimensions. The search for solutions of this equation is also the source of our motivation.

Burchnall and Chaundy showed that any two commuting ordinary differential operators M and N satisfy identically an algebraic relation $F(M, N) = 0$ (F algebraic). We call such an (M, N) a commuting pair and say loosely that the pair lies on, or corresponds to, the curve F. We do not review the theory of commuting operators as this is adequately done elsewhere [2, 3, 4, 12, 18, 19, 22], but give a brief commentary on progress in the search for their explicit formulae. As already mentioned, of most interest in applications is the determination of the coefficients of commuting pairs (M, N) for which $r := \gcd(\text{ord} M, \text{ord} N)$ is greater than one. In this case the dimension of the common eigenspace has the possibility of exceeding one, giving a rich geometry of a higher dimensional vector bundle over the Burchnall-Chaundy curve F. It is a result of Krichever [12] that the general solution to this problem is parametrized by $r - 1$ arbitrary functions and some arbitrary constants. Some authors [19] believe the presence of these arbitrary functions to be the main cause of the difficulty of finding explicit formulae for commuting operators.

The case of $r = 1$ was solved completely by Burchnall and Chaundy and will not be mentioned further. The case where $r = 2$ and F determines an elliptic curve has been solved in three different ways by Dehornoy [5], Krichever and Novikov [13] (with corrections by Grinevich [7]) and Grünbaum [8]. These papers give different but equivalent forms of the coefficients of the most general commuting pair of operators of orders four and six. Both Krichever and Novikov [13, 14] and Grünbaum [9] have successfully applied the forms of their solutions to find associated explicit solutions of the KP equation. These authors reduce the KP equation to the so-called Krichever-Novikov equation, a completely integrable nonlinear partial differential equation in one less space dimension.

In this paper, we employ the computational approach of Grünbaum [8], to study the problem of determining explicit formulae for the coefficients of commuting pairs (L_6, L_9), differential operators of orders six and nine, which lie on an elliptic curve. This might be thought of as the next most interesting case, $r = 3$. In fact this problem has already been solved using geometric methods by Mokhov [16, 17] and previously Dehornoy [5] had given a partial solution. Even earlier, Dixmier [6] had given the first example of an $r = 3$ commuting pair with polynomial coefficients. With the problem already solved, one might justifiably ask: why solve it again? The answer comes from the

desire to use the explicit solution in applications, and in particular to apply it
to the KP equation. From this point of view, it is desirable to give a more
elementary characterization of the solution, in particular, one which avoids the
appearence of elliptic functions. We offer here such an elementary approach.
As justification for this attitude, we cite the success of Grünbaum in [9] for
$r = 2$ and the partial answer in [15] for $r = 3$.

There are three main steps constituting the computational approach, which
can be summarized as follows;

(i) by imposing that F be elliptic, identify, and then compute, L_9 as the
differential operator part of an appropriate fractional power of L_6,

(ii) compute the commutator $[L_9, L_6]$ and form the commutativity equations,
which are the nonlinear ordinary differential equations in the coefficients of L_6,
obtained by equating the coefficients of each power of d/dx in the commutator
to zero, and

(iii) solve the commutativity equations derived in (ii).

The easy statement of our problem as the simple condition $[L_9, L_6] = 0$ is
maligned by the rather complicated nature of the commutativity equations arising
from step (ii). Although we do obtain explicit formulae for the coefficients of
commuting L_6 and L_9 which depend on two arbitrary functions, we pay the price
of being unable, in a computational way at least, to show that they are the most
general possible. This, in part, rests on our inability to completely perform step
(iii). Despite falling short on this score, the computational approach has none the
less a number of advantages. First, the appearance of the arbitrary functions and
constants in our solutions is very explicit. This makes the design of commuting
pairs with coefficients of a particular type, for example, polynomial, rational,
periodic, very easy. One has only to choose the arbitrary functions and constants
appropriately. Second, Grünbaum's computational approach is simplicity itself.
Nothing more complicated than the pseudo-differential calculus of I. Schur [20],
based on the Leibnitz rule, is needed to carry out steps (i) and (ii). Because of
this, this approach readily lends itself to implementation by computer. Finally,
the method is easy. Once the commutativity equations are computed, solutions
of them are relatively easy to find. This is particularly true in certain degenerate
cases where all that is required is to solve second order *linear* differential
equations.

In Section 2, we perform steps (i) and (ii) and set up the five nonlinear
ordinary differential equations for the coefficients of L_6 which are equivalent to
the requirement that $[L_9, L_6] = 0$. It is shown that these differential equations
are easily integrated once, and that all but one of five integration constants

appearing in this process must be taken to be zero if the Burchnall-Chaundy curve F is elliptic. Sections 3 and 4 are devoted to step (iii). Section 3 solves completely the case where L_6 has no second derivative term. This is one of the easy cases mentioned above. In Section 4, the commutativity equations are further manipulated by a transformation of the last coefficient of L_6. It is then possible to obtain solutions of the commutativity equations for the case where L_6 has no first derivative term. A solution which depends on two arbitrary functions is given by reducing the five nonlinear commutativity equations to a single nonlinear integro-differential equation for one coefficient. This integro-differential equation has appearing in its coefficients, the two arbitrary functions parametrizing the solution. Also in this section, solutions for two special cases together with some examples are given. Finally, the last section offers some comments as to the appropriateness of a major assumption used in step (iii).

2. Commutativity equations

Let L_6 be a general sixth order ordinary differential operator with locally smooth coefficients $a_i(x)$. We write this operator with the *standard normalization* in which the leading coefficient is 1 and the second coefficient is 0:

$$L_6 = \left(D^3 + a_4 D + D a_4 + a_3\right)^2 + a_2 D^2 + D^2 a_2 + a_1 D + D a_1 + a_0, \quad (2.1)$$

where $D = d/dx$. We will need to work with a sixth root $L_6^{1/6}$, of L_6, in the ring of formal pseudo-differential expressions of the form $C = \sum_{-\infty}^{N} c_i D^i$. Given such an expression, define the differential operator part C_+ as the sum of those terms of C with nonnegative powers of D, and the formal integral operator part C_-, by $C_- = C - C_+$. According to the results of Schur [20], any differential operator L_9, of order 9, which commutes with L_6 must be a constant linear combination of the operators $(L_6^{j/6})_+$ where $0 \leq j \leq 9$. As mentioned in the introduction, any two commuting differential operators lie on some Burchnall-Chaundy curve. In this paper, we will consider only the case where the Burchnall-Chaundy curve is an elliptic curve of the form,

$$L_9^2 = L_6^3 + g_1 L_6 + g_0, \quad (2.2)$$

for some constants g_0 and g_1. One can show by a careful consideration of the powers of D which occur, that insisting on the curve (2.2) implies that only $j = 9$ appears in L_9. We therefore take from now on:

$$L_9 = (L_6^{3/2})_+. \quad (2.3)$$

With this L_9, another way of writing the curve (2.2) is as

$$L_9^2 - L_6^3 = -\left\{L_9(L_6^{3/2})_- + (L_6^{3/2})_- L_9\right\}_+, \tag{2.4}$$

which is often a more convenient form for the purposes of computation. Calculating the commutator we get,

$$[L_9, L_6] = \alpha(x)D^4 + \beta(x)D^3 + \gamma(x)D^2 + \delta(x)D + \epsilon(x), \tag{2.5}$$

where $\alpha(x)$, $\beta(x)$, $\gamma(x)$, $\delta(x)$ and $\epsilon(x)$ are all differential polynomials in the coefficients $a_i(x)$ of L_6. If we assign the k-th derivative of a_i, $a_i^{(k)}$, a weight of $6 - i + k$, then as is well known [22], the coefficient of D^j in (2.5) is homogeneous of weight $13 - j$. All together, these differential polynomials contain 858 terms and so it is not possible to give their expressions here. From (2.5) the condition for commutativity is, now,

$$\alpha(x) = \beta(x) = \gamma(x) = \delta(x) = \epsilon(x) = 0. \tag{2.6}$$

From the point of view of solving these nonlinear ordinary differential equations, it is much more useful to work not with (2.6) but linear combinations of them and their integrals and derivatives. It turns out that both α and $\beta - 2\alpha'$ are exact derivatives of differential polynomials which we denote by I_α and I_{β_1} respectively. Define the new differential polynomials,

$$
\begin{aligned}
\beta_1 &= \beta - 2\alpha', \\
\gamma_1 &= \gamma - 3\beta_1'/2 - 11\alpha''/6 - 2a_4\alpha/3 + 4a_4'I_\alpha/3, \\
\delta_1 &= \delta - \gamma_1' - 5\beta_1''/6 - 2a_4\beta_1/3 + 2a_4'I_{\beta_1}/3 \\
&\quad -5\alpha'''/6 - 2a_4\alpha'/3 + 2a_4'\alpha/3 + 2/3(a_3' + 2a_4'')I_\alpha, \\
\epsilon_1 &= \epsilon - \delta_1'/2 - \gamma_1''/3 - 2a_4\gamma_1/3 - \beta_1'''/6 - a_4'\beta_1'/3 + 1/3(a_3' + a_4'')I_{\beta_1} \\
&\quad -\alpha^{(iv)}/6 - a_4\alpha''/3 + (a_4'' + a_3' - a_2/2)\alpha/3 + (a_4''' + a_3'' + a_2'/2)I_\alpha/3.
\end{aligned}
\tag{2.7}
$$

Here the prime denotes differentiation with respect to x. These combinations have been chosen to simplify as much as possible the expessions for β_1, γ_1, δ_1 and ϵ_1. They also preserve the homogeneity and weight, and produce exact derivatives, that is, the differential polynomials α, β_1, γ_1, δ_1, ϵ_1 are the first derivatives of certain other differential polynomials denoted by I_α, I_{β_1}, I_{γ_1}, I_{δ_1}, I_{ϵ_1} respectively. In terms of these, the commutativity equations (2.6) are

equivalent to

$$
\begin{aligned}
I_\alpha + K_1 &= 0, \\
I_{\beta_1} + K_2 &= 0, \\
I_{\gamma_1} + 4a_4 K_1/3 + K_3 &= 0, \\
I_{\delta_1} + 2a_4 K_2/3 + 2a_3 K_1/3 + K_4 &= 0, \\
I_{\epsilon_1} - a_4'' K_1/9 - 4a_4^2 K_1/9 + a_2 K_1/6 + a_3 K_2/3 + K_5 &= 0,
\end{aligned} \tag{2.8}
$$

where the K_j are arbitrary integration constants. We will be concerned here only with a very special choice of these constants which is forced by our assumption that, when they commute, L_6 and L_9 lie on an elliptic curve.

LEMMA 1. *Assume that* $[L_9, L_6] = 0$ *where* L_9 *is given by* (2.3). *Then* L_6 *and* L_9 *lie on the curve* (2.2) *if and only if*

$$
K_1 = K_2 = K_4 = K_5 = 0 \qquad and \qquad K_3 = 3g_1. \qquad .
$$

This is proved by explicitly expressing the coefficients of $(L_6^{3/2})_-$, given that (2.8) holds, in terms of the K_j, and then forcing the form (2.2) on (2.4). We omit these somewhat complicated computations.

The constants K_j have an interesting interpretation if we introduce a formal integral operator $\Psi = 1 + \sum_1^\infty \psi_j(x) D^{-j}$ such that $\Psi^{-1} L_6 \Psi = D^6$. If coefficients $f_j(x)$ are defined by $\Psi^{-1} L_9 \Psi = D^9 - \sum_1^\infty f_j(x) D^{-j}$, then by conjugating the commutator $[L_9, L_6]$ with Ψ, it is easy to show that L_6 and L_9 commute if and only if all the f_j are constant. In this case the commutativity equations (2.8) are satisfied, and it can be shown that $f_j = -K_j/6$ for $j = 1, \ldots, 5$. All the other f_j are determined recursively in terms of the first five. The choice of constants given in Lemma 1 then implies that only powers of D^{-3} occur in $\Psi^{-1} L_9 \Psi$.

From now on, we adhere to the choice of constants in Lemma 1 and so will be interested in obtaining all solutions of the nonlinear differential equations represented by:

$$
I_\alpha = I_{\beta_1} = 0, \qquad I_{\gamma_1} + K_3 = 0, \qquad I_{\delta_1} = I_{\epsilon_1} = 0. \tag{2.9}
$$

These equations are considerably shorter than those given by (2.6) but are still too large to put into print. To finish this section, we give the explicit expression

for L_9 as computed from (2.3),

$$
\begin{aligned}
L_9 \;=\; & \left(D^3 + a_4 D + D a_4 + a_3\right)^3 + 3 a_2 D^5 + 3\left(a_1 + 5 a_2'/2\right) D^4 \\
& + 3\left(2 a_2 a_4 + 17 a_2''/4 + 2 a_1' + a_0/2\right) D^3 \\
& + 3\left(3 a_2 a_4' + 3 a_2' a_4 + 2 a_1 a_4 + a_2 a_3 + 31 a_2'''/8 + 3 a_1'' + 3 a_0'/4\right) D^2 \\
& + 3\left(a_2 a_4'' - a_2' a_4'/2 + 3 a_2'' a_4/2 + 2(a_1 a_4)' + a_0 a_4 + (a_2 a_3)'\right. \\
& \qquad \left. + a_1 a_3 + 25 a_2^{(\mathrm{iv})}/16 + a_2^2/2 + 2 a_1''' + 9 a_0''/8\right) D \\
& + 3\left(-5 a_2' a_4''/4 - a_2'' a_4' + a_2''' a_4/4 - a_1' a_4' + (a_0 a_4)'/2 - a_2' a_3' + (a_1 a_3)'/2\right. \\
& \qquad \left. - a_2'' a_3/4 + a_0 a_3/2 + 7 a_2^{(5)}/32 + a_2 a_2'/2 + a_1 a_2 + 7 a_0'''/16\right).
\end{aligned}
$$

$$(2.10)$$

Again giving $a_i^{(k)}$ a weight of $6 - i + k$, the coefficient of D^j in L_9 is a homogeneous differential polynomial of weight $9 - j$.

3. The case $a_2 \equiv 0$

When a_2 is identically zero in (2.1), the commutativity equations (2.9) assume a particularly simple form. In this case, the problem of determining all commuting operators of orders six and nine reduces to the solution of second order linear differential equations, and we are therefore able to obtain the general solution. In this solution, a_1 is essentially the arbitrary function parametrizing the commuting operators.

Letting $a_2 = 0$ in (2.9) gives

$$
I_\alpha = -\tfrac{9}{2}(a_0' a_4)' - \tfrac{9}{2}(a_1 a_3' + 3 a_1' a_3)' + 9 a_1^2 - 63 a_0^{(\mathrm{iv})}/16 = 0, \tag{3.1}
$$

$$
\begin{aligned}
I_{\beta_1} = \; & 6\left(a_1 a_4^{(\mathrm{iv})} + 5 a_1' a_4''' + 9 a_1'' a_4'' + 8 a_1''' a_4' + 4 a_1^{(\mathrm{iv})} a_4\right) \\
& + 12\left(a_1 a_4 a_4'' + a_1 a_4'^2/2 + 2(a_1' a_4^2)'\right) \\
& - \tfrac{9}{4}\left(2 a_0' a_3' + 3 a_0'' a_3\right) + 6 a_1^{(6)} + 9 a_0 a_1 = 0, \tag{3.2}
\end{aligned}
$$

$$
\begin{aligned}
I_{\gamma_1} = \; & \tfrac{3}{8}\left(4 a_0' a_4''' + 14 a_0'' a_4'' + 21 a_0''' a_4' + 5 a_0^{(\mathrm{iv})} a_4\right) \\
& + 3\left(3 a_1 a_3 a_4'' + a_1 a_3' a_4' + 9 a_1' a_3 a_4' - a_1 a_3'' a_4 - 2 a_1' a_3' a_4 + 3 a_1'' a_3 a_4\right) \\
& + 3(a_0' a_4)' a_4 + \tfrac{3}{4}\left(a_1 a_3^{(\mathrm{iv})} + 6 a_1' a_3''' + 12 a_1'' a_3'' + 14 a_1''' a_3' + 15 a_1^{(\mathrm{iv})} a_3\right) \\
& + 33 a_0^{(6)}/32 - 9 a_1 a_1'' + 9 a_0^2/4 = -K_3, \tag{3.3}
\end{aligned}
$$

$$
I_{\delta_1} = -9/4\left(a_0'' a_1 - 2 a_0' a_1'\right) = 0, \tag{3.4}
$$

$$
\begin{aligned}
I_{\epsilon_1} = \; & 3\left(a_1^2 a_4'' + 2 a_1 a_1' a_4' + 2(a_1 a_1'' - a_1'^2) a_4\right) + 3\left(a_1 a_1^{(\mathrm{iv})} - a_1' a_1'''\right) + 9 a_0'^2/16 \\
& = 0. \tag{3.5}
\end{aligned}
$$

The general solution of these equations is the next result.

PROPOSITION 2. *Let $g(x)$ be an arbitrary nonconstant function and suppose that $a_2 \equiv 0$. Define the coefficients of L_6 by*

$$
\begin{aligned}
a_4 &= -\frac{g'''}{2g'} + \frac{3(g'')^2}{8(g')^2} + \frac{K_{51}g + K_{52} - 3K_{41}^2 g^2/32}{g'}, \\
a_3 &= \frac{g^2 - 3K_{41}g'g''/8 - K_{41}(K_{51}g^2/2 + K_{52}g - K_{41}^2 g^3/32) + K_{11}g + K_{12}}{(g')^{3/2}}, \\
a_2 &= 0, \\
a_1 &= \sqrt{g'}, \\
a_0 &= K_{41}g + K_{42},
\end{aligned}
$$

(3.6)

where $K_{11}, K_{12}, K_{41}, K_{42}, K_{51}, K_{52}$ are constants. Then $[L_9, L_6] = 0$ if and only if

$$
Q_1 := 9/4(K_{41}^2 K_{52} + 8K_{51}^2/3 + 4K_{42} - 2K_{11}K_{41}) = 0. \tag{3.7}
$$

Moreover, when (3.7) holds, the elliptic curve (2.2) is given by

$$
\begin{aligned}
g_1 &= -\left(K_{41}K_{51}K_{52} - 2K_{52} + K_{11}K_{51} + 3K_{42}^2/4 - 9K_{12}K_{41}^2/16\right), \\
g_0 &= K_{41}^2 K_{52}^2 - K_{41}K_{42}K_{51}K_{52}/2 + K_{42}K_{52} - K_{11}K_{41}K_{52} - K_{11}K_{42}K_{51}/2 \\
&\quad + 3K_{12}K_{41}K_{51}/2 - K_{42}^3/4 + 9K_{12}K_{41}^2 K_{42}/32 - K_{12} + K_{11}^2/4.
\end{aligned}
$$

(3.8)

The formulae (3.6) under the condition (3.7) give the most general commuting pair for which $a_2 \equiv 0$ and $a_1 \not\equiv 0$.

PROOF. The labelling K_{ij} generally refers to the jth integration constant arising in the solution of the ith commutativity equation in (3.1)–(3.5). Solving (3.4) gives a_0 in terms of a_1. Next solve (3.5) for a_4, and lastly, solve (3.1) for a_3. This solves three of the equations producing (3.6), and evaluating the left hand side of (3.2) with the coefficients (3.6) gives, $I_{\beta_1} = Q_1 a_1$, hence the algebraic condition (3.7). Substituting into (3.3) gives $I_{\gamma_1} = Q_1 a_0/2 - K_{42}Q_1/2 - R_1$, where R_1 is a constant. We therefore take $K_3 = R_1 = 3g_1$ to obtain a solution to all the commutativity equations. The coefficients in (3.6) can now be used together with (3.7) in (2.4) to compute g_0.

Special cases of the solution (3.6) have appeared in the literature before, and we point out the correspondences.

EXAMPLE. Let $u(x)$ be an arbitrary nonzero function and set $g'(x) = u^2(x)$ in (3.6). Taking $K_{11} = K_{41} = K_{42} = K_{51} = K_{52} = 0$ and $K_{12} = a$ then gives Mokhov's example (12) in [16], (see also [17, page 1307]). Further setting $u(x) = -1$ and $g(x) = x$ gives the famous example of Dixmier [6]. For both of these examples, the elliptic curve from (3.8) is, $L_9^2 = L_6^3 - a$.

If we take $a_1 = a_2 = 0$ in the commutativity equations (2.9), then the situation is very simple. From (3.5), we conclude that if a_1 and a_2 are identically zero, then $[L_9, L_6] = 0$ iff $a_0' = 0$. This solution is parametrized by the two arbitrary functions a_3 and a_4; however, we will consider this case trivial, since L_6 and L_9 are both constant coefficient polynomials in the operator $S = D^3 + a_4 D + D a_4 + a_3$. In fact, from (2.10), $L_9 = S^3 + 3/2 a_0 S$ and the curve (2.2) can be obtained by eliminating S.

4. The case $a_2 \not\equiv 0$

The main cause of the complexity of the commutativity equations is that in general, $a_2 \not\equiv 0$. Assuming this is so, (2.9) can be simplified somewhat by the introduction of two auxiliary functions which we now introduce.

Let $h(x)$ be any (fixed) nonconstant function such that

$$a_2 = h', \tag{4.1}$$

and define a new coefficient $u_0(x)$ by

$$a_0 = 8a_2 a_4/3 + 5a_2''/6 - 2h^2/3 + u_0. \tag{4.2}$$

The motivation behind this last definition comes from the case $a_1 \equiv 0$ where it is possible to solve the equation $I_{\beta_1} = 0$ explicitly and find precisely the relation (4.2) but with u_0 linear in h. As with (3.1), most of the terms in I_α are derivatives (see (4.8) below). We therefore denote by J_α the formal integral of I_α, that is, the expression for which $D J_\alpha = I_\alpha$. In terms of this new integral, the first equation in (2.9) is equivalent to

$$J_\alpha + K_{12} = 0, \tag{4.3}$$

where K_{12} is an integration constant. Define two new expressions by

$$\begin{aligned} I_{\gamma_3} &= I_{\gamma_1} + \alpha'/6 + 2h J_\alpha/3, \\ I_{\epsilon_2} &= I_{\epsilon_1} + a_2 I_\alpha/6. \end{aligned} \tag{4.4}$$

For the first of these, we find that

$$I_{\gamma_3} = 3(\chi a_3' + 3\chi' a_3) + \sigma + 2a_4 I_\alpha/3, \tag{4.5}$$

where

$$\chi = a_1 a_4' + 2a_1' a_4 + a_2 a_3' + 3a_2' a_3/2 + a_1''' - a_1 h, \tag{4.6}$$

and σ is a fairly complicated expression in which we use (4.2) to eliminate a_0:

$$\begin{aligned}
\sigma &= \tfrac{3}{8}u_0^{(vi)} + \tfrac{3}{8}\big(2u_0'a_4''' + 9u_0''a_4'' + 15u_0'''a_4' + 10u_0^{(iv)}a_4\big) + 6\big(a_4^2 u_0'' + a_4 a_4' u_0'\big) \\
&\quad + 6a_2 a_4 u_0 + \tfrac{21}{4}\big(a_2'' u_0 + a_2' u_0' - 4a_2 u_0''/7\big) + 9u_0^2/4 - 3ha_4 u_0' - 3h^2 u_0 \\
&\quad - 21hu_0'''/8 - h^4/3 + 6h\int\big\{a_1^2 + a_2 u_0 - 7a_2'^2/12 + 2a_2^2 a_4\big\} + 7a_2 a_2' h/2 \\
&\quad + a_2^2 a_4'' + 5a_2 a_2' a_4'/2 + 5a_2 a_2'' a_4 - 5a_2'^2 a_4/2 + 4a_2^3 a_4^2 - 6a_1^2 a_4 \\
&\quad - 9a_1 a_2 a_3 + \tfrac{1}{2}\big(a_2 a_2^{(iv)} - a_2' a_2'' + a_2'^2/2\big) + a_2^3 - 6a_1 a_1'' + 3a_1'^2.
\end{aligned} \tag{4.7}$$

Using (4.2) to eliminate a_0 from I_α, I_{β_1}, I_{δ_1} and I_{ϵ_2} as well, we find that $a_2^{-3/2}$ is an integrating factor for most of the terms of the equation $I_{\delta_1} = 0$. With the definitions (4.1), (4.2), (4.4), (4.5), (4.6), and (4.7), the commutativity equations (2.9) are equivalent to the following set of integro-differential equations:

$$\begin{aligned}
I_\alpha &= \tfrac{3}{4}\big(2a_2 a_4''' + 9a_2' a_4'' + 15a_2'' a_4' + 10a_2''' a_4\big)' + 12\big(a_2 a_4 a_4' + a_2' a_4^2\big)' \\
&\quad + 6(a_2 h a_4)' + 6a_2^2 a_4 - \tfrac{9}{2}\big(a_1 a_3' + 3a_1' a_3\big)' + 3a_2^{(6)}/4 \\
&\quad + 21a_2''' h/4 + 15a_2 a_2'' + 9a_2'^2/2 - 6a_2 h^2 + 9a_1^2 - 63u_0^{(iv)}/16 \\
&\quad - \tfrac{9}{2}(u_0' a_4)' + 9a_2 u_0 = 0, \tag{4.8} \\[4pt]
I_{\beta_1}/3 &= 2\big(\chi a_4' + 2\chi' a_4 + \chi''' + \chi h\big) - \tfrac{3}{2}\big(a_3' u_0' + 3a_3 u_0''/2\big) + 3a_1 u_0 \\
&= 0, \tag{4.9} \\[4pt]
I_{\gamma_3} - 2a_4 I_\alpha/3 &= 3\big(\chi a_3' + 3\chi' a_3\big) + \sigma + 2h K_{12}/3 + K_3 = 0, \tag{4.10} \\[4pt]
a_2^{1/2}\!\int a_2^{-3/2} I_{\delta_1}/3 &= \chi - \tfrac{3}{8}a_2^{1/2}\!\int a_2^{-3/2}(u_0'' a_1 - 2u_0' a_1') - K_{41}a_2^{1/2} = 0, \tag{4.11} \\[4pt]
I_{\epsilon_2}/3 &= a_1 \chi' - a_1' \chi + a_4 H_1(u_0) + H_1''(u_0)/8 + \tfrac{3}{16}H_2(u_0) - \tfrac{5}{16}H_3(u_0) \\
&= 0, \tag{4.12}
\end{aligned}$$

where the H_i in the last equation are differential operators acting on u_0,

$$\begin{aligned}
H_1(u_0) &= a_2 u_0'' - a_2' u_0', \\
H_2(u_0) &= (u_0')^2 + \tfrac{8}{3}\big(-a_2 h u_0' + a_2^2 u_0\big), \tag{4.13} \\
H_3(u_0) &= a_2' u_0''' - a_2'' u_0'' = \big(a_2' H_1'(u_0) - a_2'' H_1(u_0)\big)/a_2,
\end{aligned}$$

and K_3, K_{12} and K_{41} are constants. Our point of view in trying to solve the commutativity equations (4.8) through (4.12) will be that a_1 and a_2 will play the role of the arbitrary functions which parametrize the solution. By this assumption, we mean that we will be searching for expressions for a_3, a_4 and u_0 in terms of a_1 and a_2 (and their derivatives and integrals).

An interesting feature of the new commutativity equations is the special form of (4.12). If $H_1(u_0)$ were nonzero and known in terms of a_1 and a_2, which is equivalent to knowing u_0, then (4.11) gives χ which can be put into (4.12) to give a_4 directly. Then the definition (4.6) can be used to solve a linear first order differential equation to give a_3. In this paper, we will only be concerned with the situation in which this procedure is *not* possible. We therefore make the following assumption for the remainder of the paper.

ASSUMPTION 4.14. *Assume that* $H_1(u_0) \equiv 0$.

Granted this assumption, the definition of H_1 implies that $u_0 = K_{51}h + K_{52}$ for some constants K_{51} and K_{52}. Substituting this u_0 into (4.11) and assuming a_1 is not proportional to $a_2^{1/2}$ gives

$$\chi = -3K_{51}a_1/4 + K_{41}a_2^{1/2}. \tag{4.15}$$

Now putting this χ into (4.12), using (4.13) and cancelling a factor a_2 (which is nonzero), gives that

$$\tfrac{3}{16}\big(K_{51}^2 + 8K_{52}/3\big)a_2 - K_{41}\big(a_1/a_2^{1/2}\big)' = 0. \tag{4.16}$$

The case $a_1 \equiv 0$ When a_1 is identically zero, we gain some simplification of the commutativity equations (4.8) through (4.12). If in addition, the integration constant K_{41} is nonzero, it is easy, in this case, to obtain the most general commuting pair which depend on a single arbitrary function.

PROPOSITION 3. *Let* $h(x)$ *be an arbitrary nonconstant function, suppose that* $a_1 \equiv 0$ *and assume that* (4.14) *holds. Define the coefficients of* L_6 *by*

$$
\begin{aligned}
a_4 &= -\frac{h'''}{2h'} + \frac{3h''^2}{8h'^2} + \frac{K_{11} + \tfrac{3}{4}K_{51}h - \tfrac{1}{2}h^2}{h'}, \\
a_3 &= (K_{41}h + K_{42})/a_2^{3/2}, \\
a_2 &= h', \\
a_1 &= 0, \\
a_0 &= -h'''/2 + h''^2/h' - 2h^2 + 3K_{51}h + K_{52} + 8K_{11}/3,
\end{aligned}
\tag{4.17}
$$

where K_{11}, K_{41}, K_{51} and K_{52} are constants. Then $[L_9, L_6] = 0$ if and only if

$$Q_1 = 9(K_{52} + 3K_{51}^2/8) = 0. \qquad (4.18)$$

If $K_{41} \neq 0$, then (4.17) under the condition (4.18) gives the most general commuting pair for which $a_1 = 0$ and (4.14) holds. When (4.18) is satisfied, the elliptic curve (2.2) is given by

$$
\begin{aligned}
g_1 &= -\big(K_{41}^2 + 3K_{52}^2/4 + 4K_{11}K_{52} + 9K_{11}K_{51}^2/4 + 4K_{11}^2/3\big), \\
g_0 &= K_{42}^2 + 3K_{51}K_{41}K_{42}/2 - K_{52}K_{41}^2/2 - 4K_{11}K_{41}^2/3 \qquad (4.19) \\
&\quad - K_{52}^3/4 + K_{11}K_{52}^2 - 4K_{11}^2K_{52}/3 + 16K_{11}^3/27.
\end{aligned}
$$

PROOF. If $a_1 = 0$, then from either (4.11) or (4.16), $\chi = K_{41}a_2^{1/2}$, and solving this using (4.6) gives the expression for a_3. If $K_{41} \neq 0$, then substituting for χ and a_3 in (4.9) and integrating once, we obtain

$$(a_2a_4)' + a_2^{1/2}\big(a_2^{1/2}\big)''' + a_2h - \tfrac{3}{4}\big\{\tfrac{3}{2}(u_0'h/a_2)' + \tfrac{3}{2}(K_{42}/K_{41})(u_0'/a_2)' - u_0'/2\big\} = 0.$$

When $H_1(u_0) = 0$ and $a_1 = 0$, (4.16) gives the condition (4.18), and so $u_0 = K_{51}h - 3K_{51}^2/8$. Putting this into the last relation produces the formula for a_4. At this point (4.9), (4.11) and (4.12) have been solved. The solution is then checked in (4.8) to reveal that $I_\alpha = Q_1a_2$, while if instead of (4.10) we use the equivalent equation $I_{\gamma_1} + K_3 = 0$, we find $I_{\gamma_1} = Q_1a_0/2 - (K_{52}+8K_{11}/3)Q_1/2 - R_1$ where R_1 is a polynomial given by $R_1 = 3g_1$. Thus taking $K_3 = R_1 = 3g_1$ gives a solution of all the commutativity equations provided $Q_1 = 0$. Using (4.17) and (4.18) in the right-hand side of (2.4) gives the expression for g_0.

This proof uses in an essential way that $K_{41} \neq 0$, to obtain the formula for a_4. If $K_{41} = 0$, then (4.9) actually *implies* (4.14). A statement about the generality of (4.17) in this case however, cannot be made, but some insight can be gained from the next case we treat where a_1 and a_2 are entirely arbitrary.

When (4.18) is satisfied, we get for a_0 (cf. (4.2));

$$a_0 = 8a_2a_4/3 + 5a_2''/6 - \tfrac{2}{3}\big(h - \tfrac{3}{4}K_{51}\big)^2, \qquad (4.20)$$

so that K_{51} amounts to no more than a translation of h. In view of (4.1), K_{51} may thus seem like an irrelevant constant; however, we carry it along as it has a use in later examples.

The case of a_1 and a_2 arbitrary   In this part we operate under the assumption that a_1 and a_2 *must* be the two independent arbitrary functions parametrizing the commuting operators L_6 and L_9, and look for corresponding solutions of the commutativity equations. We therefore make a loose definition of independence of arbitrary functions. We will say that a_1 and h (or a_1 and a_2 given (4.1)) are *independent* arbitrary functions if the requirement that (4.16) hold for all x implies $K_{41} = 0$.

If a_1 and a_2 are independent arbitrary functions, then (4.16) gives $K_{41} = 0$ and $K_{52} = -3K_{51}^2/8$. We note here that there are two interesting cases when K_{41} is not forced to be zero, namely $a_1 = a_2^{1/2}$ and $a_1 = a_2^{1/2}h$. These two cases will be considered in the next part. We now get from (4.11), $\chi = -3K_{51}a_1/4$. Substituting this χ and $u_0 = K_{51}h - 3K_{51}^2/8$ into (4.9) produces a copy of (4.11) (with $K_{41} = 0$). Thus (4.9) and (4.11) are the same equation and so there are only three equations to solve. Since (4.11) is linear and first order in both a_4 and a_3, it can be solved for a_3 and thus this variable can be eliminated. This is the idea behind the next result.

PROPOSITION 4. *Let $h(x)$ and $g'(x)$ be independent arbitrary functions with h' not identically zero and assume that* (4.14) *holds. Define the coefficients of L_6 by*

$$
\begin{aligned}
a_4 &= -\frac{h'''}{2h'} + \frac{3h''^2}{8h'^2} - \frac{3g'^2}{2h'^2} + \frac{K_{11} + \frac{3}{4}K_{51}h - \frac{1}{2}h^2}{h'} + v_4, \\
a_3 &= -\frac{g'''}{h'} + \frac{g''h''}{2h'^2} + \frac{g'}{2h'}\left(\frac{h'''}{h'} - \frac{h''^2}{h'^2}\right) + 2\frac{g'^3}{h'^3} + \frac{g'(h^2 - \frac{3}{2}K_{51}h - 2K_{11})}{h'^2} \\
&\quad - \frac{2g'}{h'}v_4 + K_{42}(h')^{-3/2} + (h')^{-3/2}\int g'(h')^{-1/2}(h'v_4)', \\
a_2 &= h', \\
a_1 &= g', \\
a_0 &= 8h'a_4/3 + 5h'''/6 - \frac{2}{3}(h - \frac{3}{4}K_{51})^2,
\end{aligned}
$$

(4.21)

where K_{11}, K_{42} and K_{51} are constants. Then $[L_9, L_6] = 0$ if and only if v_4 solves the equation

$$
\begin{aligned}
0 &= \left(a_2^{1/2}(a_2v_4)'\right)' + 4a_2^{3/2}v_4^2 - 4(h - \frac{3}{2}K_{51})a_2^{1/2}hv_4 + 8K_{11}a_2^{1/2}v_4 \\
&\quad - 9a_1a_2^{-1}\int a_1a_2^{-1/2}(a_2v_4)' + 4(h - \frac{3}{4}K_{51})a_2^{-1/2}\int a_2^2v_4 - 9a_1a_2^{-1}K_{42} \\
&\quad + (3K_{51}K_{11} + \frac{2}{3}K_{12})a_2^{-1/2}h + (\frac{81}{256}K_{51}^4 - \frac{1}{2}K_{51}K_{12} + 4K_{11}^2 + K_3)a_2^{-1/2},
\end{aligned}
$$

(4.22)

with $a_1 = g'$ and $a_2 = h'$.

PROOF. The definition of a_3 and a_4 in terms of v_4 can be derived by an iterative interplay between (4.8) and (4.11). We assume, as in the case $a_1 \equiv 0$, that the $3a_2^{(6)}/4$ term in (4.8) is cancelled by the appropriate terms in a_4; hence $a_4 = -a_2''/(2a_2) + 3a_2'^2/(8a_2^2) + \cdots$. Substituting this into (4.11) gives the first few terms of a_3 and again reverting to (4.8), a_4 must have the $-3a_1^2/(2a_2^2)$ term to cancel the first term in a_3. From (4.11), a_3 hence gains a $2a_1^3/a_2^3$. Continuing in this way leads to the first two expressions in (4.21) where v_4 represents unknown terms in a_4.

The independence of a_1 and h implies from (4.16) that $K_{41} = 0$. With $K_{41} = 0$, $u_0 = K_{51}h - 3K_{51}^2/8$ and with a_3 and a_4 as given in (4.21), the equations (4.9), (4.11) and (4.12) are solved still with v_4 arbitrary, leaving only (4.8) and (4.10) to determine v_4. We concentrate on (4.10). Putting χ and u_0 into (4.10) gives a result which can be simplified by subtracting $K_{51}/2$ times (4.3) from it. Doing this and evaluating the resulting equation with the above expressions for a_3 and a_4 gives the integro-differential equation (4.22) for v_4. If L_6 and L_9 are to commute, then apart from v_4 solving (4.22), any solution of this equation must give a solution of (4.8). Solving (4.22) implicitly for v_4 and substituting into (4.8) we get, $I_\alpha = 0$ and in fact $J_\alpha = -K_{12}$ so the solution is consistent with (4.3).

The general solution of (4.22) seems difficult to find. A transformation of the dependent variable via $u_4 = -2a_2 v_4/3$ and the independent variable via $t = \int^x a_2^{-1/2}$ suggests that elliptic functions are involved, but this is difficult to make explicit. It also poses a considerable computational problem to compute the elliptic curve (2.2) given only that v_4 solves (4.22). We do not attempt this here; however, given that there would normally be two new integration constants entering in the solution of (4.22), it is reasonable to expect the curve to be nonsingular.

EXAMPLE. Certain obvious solutions of (4.22) are apparent. If C is an arbitrary constant, then $v_4 = C/a_2$ is a solution provided we choose $K_{12} = -9/2(K_{11} + C)K_{51}$, $K_{42} = 0$ and $K_3 = -(4(K_{11}+C)^2 - K_{51}K_{12}/2 + 81K_{51}^4/256)$. Hence if C is taken to be nonzero, it amounts only to a translation of K_{11}. Thus we lose no generality by taking $C = 0$, i.e. $v_4 = 0$, (and $K_{42} = 0$) in (4.21). The elliptic curve for L_6 and L_9 is then given by

$$L_9^2 = (L_6 + 2b)(L_6 - b)^2, \tag{4.23}$$

where $b = (9K_{51}^2 + 32K_{11})/48$. A translation of h indeed shows that when $v_4 = 0$ the formulae (4.21) depend only on the single constant b. That (4.23)

is singular in this case is somewhat disappointing; however, this example was used with some success in an application to the Kadomtsev-Petviashvili equation in [15]. The outcome of this work has a strong connection with the results of Painlevé analysis as done by Weiss [21].

Two special cases   Here we consider separately each of the two special cases noted above. If a_1 is a linear combination of $a_2^{1/2}$ and $a_2^{1/2}h$ then (4.16) does not imply that $K_{41} = 0$. If $a_1 = a_2^{1/2}$, then K_{41} remains arbitrary, while if $a_1 = a_2^{1/2}h$, then K_{41} is bound within an algebraic relation with K_{51} and K_{52}.

If $a_1 = a_2^{1/2}$, then the integral term in (4.11) vanishes under (4.14) to give $\chi = K_{41}a_2^{1/2}$. For all but one exceptional value of K_{41}, the most general commuting differential operators can easily be found.

PROPOSITION 5. *Let $h(x)$ be an arbitrary nonconstant function and assume (4.14) holds. Define the coefficients of L_6 by*

$$
\begin{aligned}
a_4 &= -\frac{h'''}{2h'} + \frac{3h''^2}{8h'^2} + \frac{K_{11} + \frac{3}{4}K_{51}h - \frac{1}{2}h^2}{h'}, \\
a_3 &= \frac{h^2 + (K_{41} - \frac{3}{4}K_{51})h + K_{42}}{(h')^{3/2}}, \\
a_2 &= h', \\
a_1 &= \sqrt{h'}, \\
a_0 &= 8h'a_4/3 + 5h'''/6 - \frac{2}{3}\left(h - \frac{3}{4}K_{51}\right)^2,
\end{aligned}
\tag{4.24}
$$

where K_{11}, K_{41} and K_{51} are arbitrary constants. Then $[L_9, L_6] = 0$ and the elliptic curve is given by

$$
\begin{aligned}
g_1 &= -\left(\tfrac{27}{256}K_{51}^4 + \tfrac{3}{4}K_{51}^2 K_{11} - \tfrac{3}{4}K_{51}K_{41} - 3K_{42} + K_{41}^2 + \tfrac{4}{3}K_{11}^2 - 2K_{11}\right), \\
g_0 &= \tfrac{27}{2048}K_{51}^6 + \tfrac{9}{64}K_{11}K_{51}^4 - \tfrac{9}{64}K_{41}K_{51}^3 + \tfrac{9}{16}K_{42}K_{51}^2 + \tfrac{3}{16}K_{41}^2 K_{51}^2 + \tfrac{1}{2}K_{11}^2 K_{51}^2 \\
&\quad + \tfrac{3}{4}K_{11}K_{51}^2 + \tfrac{9}{64}K_{51}^2 + \tfrac{3}{2}K_{41}K_{42}K_{51} - \tfrac{1}{2}K_{11}K_{41}K_{51} + K_{42}^2 + 2K_{11}K_{42} \\
&\quad - K_{42} - \tfrac{3}{8}K_{41}K_{51} - \tfrac{4}{3}K_{11}K_{41}^2 + \tfrac{1}{4}K_{41}^3 + \tfrac{16}{27}K_{11}^3 + \tfrac{8}{3}K_{11}^2.
\end{aligned}
\tag{4.25}
$$

If $K_{41} + 3K_{51}/4 \neq 0$ then (4.24) give the most general commuting pair for which $a_1 = a_2^{1/2}$ and (4.14) holds.

PROOF. Putting $a_1 = a_2^{1/2}$ in (4.11) gives $\chi = K_{41}a_2^{1/2}$. Using now the definition (4.6), we get

$$
(a_2 a_4)' + (a_2^{3/2}a_3)' + a_2^{1/2}(a_2^{1/2})''' - a_2 h = K_{41}a_2.
$$

From (4.16), $K_{52} = -3K_{51}^2/8$ and hence the formula for a_0. Substituting for χ in (4.9), eliminating via (4.11) the terms which contain a_4, and cancelling a factor of $K_{41} + 3K_{51}/4$ yields

$$-\left(a_2^{3/2}a_3\right)' + 2a_2h + (K_{41} - \tfrac{3}{4}K_{51})a_2 = 0.$$

This gives a_3 and then the previous equation gives the formula for a_4. Equation (4.8) is automatically satisfied and so is (4.10) provided K_3 is taken to be three times the expression for g_1 given in (4.25). Again g_0 can be computed from (2.4).

If $a_1 = a_2^{1/2}h$, then provided K_{41} is nonzero, we can also obtain the most general commuting operators under the assumption (4.14).

PROPOSITION 6. *Let $h(x)$ be an arbitrary nonconstant function and assume (4.14) holds. Define the coefficients of L_6 by*

$$
\begin{aligned}
a_4 &= -\frac{h'''}{2h'} + \frac{3h''^2}{8h'^2} + \frac{K_{11} + \tfrac{3}{4}K_{51}h - 2h^2}{h'}, \\
a_3 &= -\frac{3h''}{2h'^{1/2}} + \frac{K_{42} + (K_{41} - 2K_{11})h - \tfrac{3}{2}K_{51}h^2 + 3h^3}{h'^{3/2}}, \\
a_2 &= h', \\
a_1 &= h\sqrt{h'}, \\
a_0 &= 8h'a_4/3 + 5h'''/6 - 2h^2/3 + K_{51}h + K_{52},
\end{aligned}
\tag{4.26}
$$

where K_{11}, K_{41}, K_{42}, K_{51} and K_{52} are constants. Then $[L_9, L_6] = 0$ if and only if

$$K_{52} + \tfrac{3}{8}K_{51}^2 - 2K_{41} = 0. \tag{4.27}$$

If $K_{41} \neq 0$ then (4.26) under the condition (4.27) give the most general commuting pair for which $a_1 = a_2^{1/2}h$ and (4.14) holds. When (4.27) is satisfied, the elliptic curve is given by

$$
\begin{aligned}
g_1 &= -\left(\tfrac{3}{4}K_{52}^2 - 2K_{11}K_{52} - \tfrac{9}{4}K_{42}K_{51} + K_{41}^2 + 6K_{11}K_{41} + \tfrac{4}{3}K_{11}^2\right), \\
g_0 &= -\tfrac{1}{4}K_{52}^3 + K_{11}K_{52}^2 - \tfrac{1}{2}K_{41}^2K_{52} - 3K_{11}K_{41}K_{52} - \tfrac{4}{3}K_{11}^2K_{52} - \tfrac{27}{64}K_{42}K_{51}^3 \\
&\quad + \tfrac{15}{4}K_{41}K_{42}K_{51} - \tfrac{3}{2}K_{11}K_{42}K_{51} + \tfrac{13}{4}K_{42}^2 - \tfrac{4}{3}K_{11}K_{41}^2 + 4K_{11}^2K_{41} + \tfrac{16}{27}K_{11}^3.
\end{aligned}
\tag{4.28}
$$

PROOF. When $a_1 = a_2^{1/2}h$, (4.27) follows from (4.16) since again by (4.14), $u_0 = K_{51}h + K_{52}$. Equation (4.11) gives $\chi = -\frac{3}{4}K_{51}a_1 + K_{41}a_2^{1/2}$. Putting this χ into (4.10), using (4.27) and cancelling a factor of K_{41}, we find

$$(a_2a_4)' + a_2^{1/2}(a_2^{1/2})''' + 4a_2h - \tfrac{3}{4}K_{51}a_2 = 0,$$

which gives the expression in (4.26) for a_4. Setting (4.6) equal to the explicit χ in this case and using the formula for a_4, then gives a_3. The resulting solution is then found to solve (4.8) provided (4.27) holds, and again (4.10) gives the curve parameter $g_1 = K_3/3$. g_0 once more comes from (2.4).

Each of the results in Propositions 3, 5 and 6 contain statements as to the generality of the solutions. This generality is determined by a condition on the integration constant K_{41}. An obvious question is to ask what happens in the cases when the condition guaranteeing generality fails. We conclude by considering this question and give some examples.

The three special circumstances of interest are:

(i) $a_1 \equiv 0$ and $K_{41} = 0$,

(ii) $a_1 = a_2^{1/2}$ and $K_{41} = -3K_{51}/4$,

(iii) $a_1 = a_2^{1/2}h$ and $K_{41} = 0$.

All of these are in fact covered by the result of Proposition 4. When $K_{41} = 0$, which was assumed in the derivation of (4.22), (i) and (iii) are covered by inserting the special forms of a_1 into (4.21) and (4.22). Hence any solution of (4.22) in these cases leads to commuting operators via the formulae (4.21), and all commuting operators are obtained by solving (4.22). When $a_1 = a_2^{1/2}$, we get $\chi = K_{41}a_2^{1/2}$. Now $K_{41} = -3K_{51}/4$ means that $\chi = -3K_{51}a_2^{1/2}/4 = -3K_{51}a_1/4$, which is exactly the χ used in the derivation of (4.22) from (4.21). Hence for (ii), all commuting pairs are given by solving (4.22) with a_1 replaced by $a_2^{1/2}$ and the coefficients of L_6 are given by the corresponding reduction of (4.21).

For the particular choice of $h = x$, it is easy to obtain interesting solutions of (4.22) in the above three special cases.

EXAMPLES. For each of the examples below, we fix $h = x$ and so $a_2 = 1$. For this choice, (4.22) reduces to

$$v_4'' + 4v_4^2 - 4(x - \tfrac{3}{2}K_{51})xv_4 + 8K_{11}v_4 - 9a_1\int a_1v_4' + 4(x - \tfrac{3}{4}K_{51})\int v_4$$

$$-9a_1K_{42} + (3K_{51}K_{11} + \tfrac{2}{3}K_{12})x + \tfrac{81}{256}K_{51}^4 - \tfrac{1}{2}K_{12}K_{51} + 4K_{11}^2 + K_3 = 0.$$

$$(4.29)$$

We give only the formula for v_4 which solves the corresponding case of (4.29) and the elliptic curve for L_6 and L_9. The formulae for a_4, a_3 and a_0 may be generated by substituting for a_1, a_2, h and v_4 in (4.21). Our examples all have a similar form.

(i) When $a_1 = 0$ and $K_{41} = 0$, the function

$$v_4 = -\tfrac{3}{2}(x - \tfrac{3}{4}K_{51})^{-2} + \kappa(x - \tfrac{3}{4}K_{51})^2 - \tfrac{9}{32}K_{51}^2 - K_{11}$$

is a solution of (4.29) provided κ takes one of the values 0 or $2/3$, and the constants satisfy

$$K_{12} = \tfrac{81}{64}(2\kappa - 1)(4\kappa - 1)K_{51}^3, \qquad K_3 = -12 + 10\kappa.$$

For this case, the differential operators lie on the curve

$$L_9^2 = L_6^3 + \left(\tfrac{10}{3}\kappa - 4\right)L_6 + K_{42}^2.$$

(ii) When $a_1 = a_2^{1/2} = 1$ and $K_{41} = -3K_{51}/4$, the function

$$v_4 = -\tfrac{3}{2}(x - \tfrac{3}{4}K_{51})^{-2} + \kappa(x - \tfrac{3}{4}K_{51})^2 - \tfrac{9}{32}K_{51}^2 - K_{11} + \tfrac{9}{8}$$

is a solution of (4.29) provided κ again takes one of the values 0 or $2/3$ and the constants satisfy

$$K_{12} = \tfrac{81}{64}(2\kappa - 1)(4\kappa - 1)K_{51}^3 - 81K_{51}/16,$$
$$K_3 = 9K_{42} - 9K_{11} + 10\kappa - 111/16 - 81K_{51}^2/32.$$

The differential operators lie on the elliptic curve

$$L_9^2 = L_6^3 + \left(3K_{42} - 3K_{11} + \tfrac{10}{3}\kappa - \tfrac{37}{16} - \tfrac{27}{32}K_{51}^2\right)L_6$$
$$+ \left(K_{42} - K_{11} - \tfrac{9}{32}K_{51}^2\right)^2 - 5\kappa/2 + 165/64.$$

It is clear from both the curve and the formulae for a_4 and a_3 which come from (4.21), that one of K_{11} or K_{42} may be assumed to be zero since they appear only in the combination $K_{42} - K_{11}$.

(iii) When $a_1 = a_2^{1/2}h = x$ and $K_{41} = 0$, the function

$$v_4 = -\tfrac{3}{2}(x - \tfrac{3}{13}K_{51})^{-2} + \kappa(x - \tfrac{3}{13}K_{51})^2 - \tfrac{9}{104}K_{51}^2 - K_{11},$$

is a solution of (4.29) provided κ takes one of the values 0 or 13/6 and the constants satisfy

$$K_{12} = 27K_{42}/2 + 81(8\kappa^2/2197 - 3\kappa/3338 + 1/208)K_{51}^3,$$
$$K_3 = 10\kappa - 39 + 27K_{42}K_{51}/4 + 81(48\kappa^2/28561 - 35\kappa/8788 + 81/43264)K_{51}^4.$$

The elliptic curve here is given by

$$L_9^2 = L_6^3 + \tfrac{1}{3}K_3\,L_6 + g_0,$$

where K_3 is as given, and

$$g_0 = \begin{cases} \tfrac{13}{4}K_{42}^2 + K_{51}^2\left(\tfrac{3}{4} - 291K_{42}K_{51}/832 + 32499K_{51}^4/4499456\right), & \text{if } \kappa = \tfrac{13}{6}, \\ \tfrac{13}{4}K_{42}^2 + K_{51}^2\left(\tfrac{27}{16} - 243K_{42}K_{51}/832 + 19683K_{51}^4/4499456\right), & \text{if } \kappa = 0. \end{cases}$$

It is obvious that these three examples *cannot* be obtained from (4.17), (4.24) or (4.26) respectively, by simply making the appropriate replacements for a_1, a_2, h and K_{41} in those formulae. These examples can also be generalized slightly by taking $h = Kx$ where K is a constant. If one then allows the constants to depend on extra variables y and t, the expressions for a_4 can be made into quadratically growing single pole rational solutions of the KP equation [15].

5. Final remarks

All of the solutions given in the last section were obtained under the assumption (4.14), i.e. they all have $H_1(u_0) = 0$. As remarked earlier, if $H_1(u_0) \neq 0$ then the possibility exists of solving (4.12) directly for a_4. To do this in a way which keeps a_1 and a_2 the arbitrary functions in the solution, we need to know $H_1(u_0)$, or equivalently u_0, in terms of a_1 and a_2. Since such an expression for u_0 is not obvious from the commutativity equations themselves, one alternative is to look for a u_0 which when put into (4.12) will produce an a_4 of a similar form to the one in (4.21). Then the expression for a_3 can be obtained from (4.11) and the solution checked in the remaining equations. For example, setting $H_1(u_0) = a_2^{3/2}$ allows us to solve for u_0 and hence χ from (4.11), and the expression for a_4 from (4.12), has the same first two terms as in (4.21). Unfortunately the resulting expressions for a_3 and a_4 do *not* solve the commutativity equations. Of course, there may be other choices of $H_1(u_0)$ for which

this process does work. Support for this general idea comes from the similar looking nature of all the solutions above. They all have the form

$$a_4 = -\frac{f''}{2f'} + \frac{3f''^2}{8f'^2} + \cdots,$$

for some arbitrary function f. If solutions for which $H_1(u_0) \neq 0$ exist, it is reasonable to expect them also to have this form. Also, this similarity of the solutions might be explained by the use of Darboux transformations.

We have not been able to prove the existence or otherwise of solutions for which $H_1(u_0) \neq 0$. For solutions which do satisfy $H_1(u_0) = 0$, the situation is no better. We have made next to no progress toward finding the general solution of (4.22). As hinted earlier, it may indeed not be possible to achieve one of the primary aims of the computational approach: the avoidance of elliptic functions. In this regard, it would be of great interest to see how the explicit formulae above compare with those of Mokhov which are written entirely in terms of the Weierstrass $\wp$-function. This seems a computationally intensive task which is beyond the current resources of the author.

It is fair to say that the computational approach here has been somewhat less successful than in the case $r = 2$ [8].

Acknowledgements

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