CO-COHEN-MACAULAY ARTINIAN MODULES OVER COMMUTATIVE RINGS

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0. Introduction. In [**7**], Z. Tang and H. Zakeri introduced the concept of co-Cohen–Macaulay Artinian module over a quasi-local commutative ring R (with identity): a non-zero Artinian R-module A is said to be a *co-Cohen–Macaulay* module if and only if codepth $A = \dim A$, where codepth A is the length of a maximal A-cosequence and $\dim A$ is the Krull dimension of A as defined by R. N. Roberts in [**2**]. Tang and Zakeri obtained several properties of co-Cohen–Macaulay Artinian R-modules, including a characterization of such modules by means of the modules of generalized fractions introduced by Zakeri and the present second author in [**6**]; this characterization is explained as follows.

Let $\mathfrak{m}$ denote the maximal ideal of R, and let A be a non-zero Artinian R-module of Krull dimension $d > 0$. Roberts [**2**, Theorem 6] proved that d is equal to the least integer i for which there exists a proper ideal $\mathfrak{q}$ of R generated by i elements such that $(0:_A\mathfrak{q})$ has finite length. A sequence $a_1, \ldots, a_d$ of d elements of $\mathfrak{m}$ is called a *system of parameters* (s.o.p.) for A if $(0:_A(a_1, \ldots, a_d))$ has finite length. For an integer j with $1 \leq j \leq d$ and $b_1, \ldots, b_j \in R$, we say that the sequence $b_1, \ldots, b_j$ is a *partial system of parameters* (p.s.o.p.) for A if there exist $b_{j+1}, \ldots, b_d \in \mathfrak{m}$ such that $b_1, \ldots, b_j, b_{j+1}, \ldots, b_d$ is a s.o.p. for A. Tang and Zakeri set, for each $i \in \mathbb{N}$ (we use $\mathbb{N}$ (respectively $\mathbb{N}_0$) to denote the set of positive (respectively non-negative) integers),

$$U_i := \{(r_1, \ldots, r_i) \in R^i : \text{there exists } j \text{ with } 0 \leq j \leq i \text{ such that } r_1, \ldots, r_j \text{ is a p.s.o.p. for } A \text{ and } r_{j+1} = \ldots = r_i = 1\}.$$

They showed [**7**, Theorem 3.10] that $\mathcal{U} = (U_n)_{n \in \mathbb{N}}$ is a chain of triangular sets on R in the sense of L. O'Carroll [**1**, p. 420] (so that the complex $C(\mathcal{U}, R)$ of modules of generalized fractions can be formed, as described in [**1**, p. 420]), and that A is co-Cohen–Macaulay if and only if the complex $\mathrm{Hom}_R(C(\mathcal{U}, R), A)$ is exact.

Tang and Zakeri did not extend the notion of co-Cohen–Macaulay module to Artinian modules over arbitrary commutative rings, and so we make here the following obvious definition.

0.1. Definition. Let A be a non-zero Artinian module over the commutative ring R (with identity). Observe that $A_\mathfrak{m}$ is an Artinian $R_\mathfrak{m}$-module for each maximal ideal $\mathfrak{m} \in \mathrm{Supp}(A)$. We say that A is a *co-Cohen–Macaulay* R-module precisely when $A_\mathfrak{m}$ is a co-Cohen–Macaulay $R_\mathfrak{m}$-module for each maximal ideal $\mathfrak{m} \in \mathrm{Supp}(A)$.

It should be noted that, in the situation of this definition, the support of the Artinian R-module A consists of maximal ideals; furthermore, it is an elementary exercise to show that $\mathrm{Supp}(A)$ is equal to the finite set of maximal ideals $\mathfrak{m}$ of R for which $\mathrm{Soc}(A)$, the socle of A, has a submodule isomorphic to $R/\mathfrak{m}$.

One of the aims of this paper is to provide a generalization of Tang's and Zakeri's generalized fraction characterization of co-Cohen–Macaulay Artinian modules that applies in non-quasi-local situations. This is dealt with in Section 2 of the paper.

The other aim of this paper, dealt with in Section 1, is to put on a more formal footing the sense in which the notion of co-Cohen–Macaulay Artinian module is "dual" to the well-known concept of Cohen–Macaulay Noetherian module. In order to describe our results in this area, we introduce now notation which will be used throughout the paper.

0.2. NOTATION. All rings considered in this paper will be commutative and will have identity elements. Throughout, let R denote such a ring, and let A be a non-zero Artinian R-module. Let the distinct maximal ideals in Supp(A) be $\mathfrak{m}_1, \ldots, \mathfrak{m}_s$, and set $\mathfrak{j} := \bigcap_{i=1}^{s} \mathfrak{m}_i$; each element of A is annihilated by a power of $\mathfrak{j}$, and so A has a natural structure as a module over

$$\hat{R}^{(\mathfrak{j})} := \varprojlim_{n} R/\mathfrak{j}^n,$$

the $\mathfrak{j}$-adic completion of R. Let $R' := \hat{R}^{(\mathfrak{j})}/(0:_{\hat{R}^{(\mathfrak{j})}} A)$: in [**4**], it was shown that R' is a complete semi-local commutative Noetherian ring. (Incidentally, when we refer to a commutative semi-local Noetherian ring as "complete", we shall mean that it is complete with respect to the topology defined by its Jacobson radical.) We shall let $\psi: R \to R'$ denote the natural ring homomorphism.

Also, R will only be assumed to be quasi-local when this is explicitly stated; in such circumstances, we shall take for granted, without further comment, the automatic simplifications that $s = 1$ and $\mathfrak{m}_1$ is the unique maximal ideal of R.

Set $E' := E_{R'}(R'/\mathrm{Jac}(R'))$ (where "Jac" is used to denote the Jacobson radical and "E" is used to denote injective envelope). We shall use D' to denote the additive, exact, R'-linear functor $\mathrm{Hom}_{R'}(\ \ , E')$ from $\mathscr{C}(R')$, the category of all R'-modules and R'-homomorphisms, to itself. We shall refer to D' as the *Matlis duality functor on* R': note that (by [**5**, (3.5)] for example) a version of Matlis duality is available for R'.

The completion of a quasi-local ring U with respect to the topology defined by its maximal ideal will be denoted by $\hat{U}$.

Now A has a natural structure as a module over R', in such a way that a subset of A is an R-submodule if and only if it is an R'-submodule. Thus A is a faithful Artinian R'-module, so that $D'(A)$ is a Noetherian R'-module. Our main results in Section 1 are that A is co-Cohen–Macaulay as an R-module if and only if it is co-Cohen–Macaulay as an R'-module, and that this is the case if and only if $D'(A)$ is a Cohen–Macaulay R'-module.

1. Use of Matlis duality. First in this section, we show that, when R is quasi-local, so that R' is a complete local ring, the extension $\psi(\mathfrak{m}_1)R'$ of the maximal ideal of R is primary for the maximal ideal of R'. We provide one preparatory lemma.

1.1. LEMMA. *Let R'' be a complete (Noetherian) local ring with maximal ideal $\mathfrak{m}''$. Suppose that G is a faithful R''-submodule of $E'' := E_{R''}(R''/\mathfrak{m}'')$. Then $G = E''$.*

Proof. The argument given in the proof of [**4**, Corollary 1.4] works in this situation.

1.2. Proposition. *Suppose that R is quasi-local, and let $\mathfrak{m}'$ denote the maximal ideal of the complete local ring R'. Then the extension $\psi(\mathfrak{m}_1)R'$ of the maximal ideal of R is $\mathfrak{m}'$-primary.*

Proof. We first treat the case in which $\operatorname{Soc}(A) \cong R/\mathfrak{m}_1$. Since a subset of A is an R-submodule if and only if it is an R'-submodule, $\operatorname{Soc}(A)$ is the same whether we consider A as an R-module or R'-module. Thus, as an R'-module, $\operatorname{Soc}(A) \cong R'/\mathfrak{m}'$.

Since A is an essential extension of its socle, it can be embedded as an R'-submodule of E'. Since A is a faithful R'-module, it follows from 1.1 that $A \cong E'$. Since

$$\operatorname{Soc}(A) = (0:_A \mathfrak{m}_1) = (0:_A \psi(\mathfrak{m}_1)R') = (0:_A \mathfrak{m}')$$

and $\psi(\mathfrak{m}_1)R' \subseteq \mathfrak{m}'$, it follows that $D'(\mathfrak{m}'/\psi(\mathfrak{m}_1)R') = 0$, so that $\psi(\mathfrak{m}_1)R' = \mathfrak{m}'$ by Matlis duality for R'. Thus, in this special case, we have proved that the extension of $\mathfrak{m}_1$ to R' under the natural homomorphism is equal to the maximal ideal of R'.

Now we revert to the general case. Certainly, $\psi(\mathfrak{m}_1)R' \subseteq \mathfrak{m}'$. By [**4**, Lemma 3.1], there exist $t \in \mathbb{N}$ and submodules $A_1, \ldots, A_t$ of A such that $0 = \bigcap_{j=1}^{t} A_j$ and, for each $j = 1, \ldots, t$, the Artinian R-module A/A_j satisfies $\operatorname{Soc}(A/A_j) \cong R/\mathfrak{m}_1$. Let $\hat{\mathfrak{m}}$ denote the maximal ideal of the quasi-local ring $\hat{R}$. Let $r' \in \hat{\mathfrak{m}}$. Let $\theta: R \to \hat{R}$ denote the natural homomorphism. By the special case covered in the first two paragraphs of this proof, for each $j = 1, \ldots, t$, there exists $s_j \in \theta(\mathfrak{m}_1)\hat{R}$ such that $r' - s_j \in (A_j :_{\hat{R}} A)$. Hence

$$\prod_{j=1}^{t} (r' - s_j) \in \bigcap_{j=1}^{t} (A_j :_{\hat{R}} A) = (0 :_{\hat{R}} A),$$

and the claim follows from this.

1.3. Corollary. *Suppose that R is quasi-local. Then $\dim_R A = \dim_{R'} A$ and $\operatorname{codepth}_R A = \operatorname{codepth}_{R'} A$. Hence A is co-Cohen–Macaulay as an R-module if and only if it is co-Cohen–Macaulay as an R'-module, and this is the case if and only if $D'(A)$ is a Cohen–Macaulay R'-module.*

Proof. That $\dim_R A = \dim_{R'} A$ is immediate from Roberts' theorem [**2**, Theorem 6] and the definition of Krull dimension.

Let $d := \operatorname{codepth}_R A$, and let $r_1, \ldots, r_d$ be an A-cosequence of elements of R. Let $B := (0:_A(r_1, \ldots, r_d))$. It is clear that $\psi(r_1), \ldots, \psi(r_d)$ is an A-cosequence of elements of R', and that $B := (0:_A(\psi(r_1), \ldots, \psi(r_d)))$. Now $\mathfrak{m}_1 \in \operatorname{Att}_R B$, since $r_1, \ldots, r_d$ is a maximal A-cosequence in $\mathfrak{m}_1$. Further, $\operatorname{Att}_R B = \psi^{-1}(\operatorname{Att}_{R'} B)$: to see this, one only has to use a minimal secondary representation of B as an R'-module to obtain a minimal secondary representation of B as an R-module. However, it follows from 1.2 that $\mathfrak{m}'$ is the one and only prime ideal of R' which contracts under ψ to $\mathfrak{m}_1$. Hence $\mathfrak{m}' \in \operatorname{Att}_{R'} B$, and $d = \operatorname{codepth}_{R'} A$.

It is now immediate that A is co-Cohen–Macaulay as an R-module if and only if it is co-Cohen–Macaulay as an R'-module. The equivalence of the second of these statements with the statement that $D'(A)$ is a Cohen–Macaulay R'-module is an elementary consequence of Matlis duality: one uses [**3**, (2.4)] to show that $\dim_{R'} A = \dim_{R'} D'(A)$ and $\operatorname{codepth}_{R'} A = \operatorname{depth}_{R'} D'(A)$.

1.4. THEOREM. (The notation is as in 0.2). *The following statements are equivalent*:
(i) *A is co-Cohen–Macaulay as an R-module*;
(ii) *A is co-Cohen–Macaulay as an R'-module*;
(iii) *D'(A) is a Cohen–Macaulay R'-module.*

Proof. The natural homomorphism $\alpha: A \to \bigoplus_{i=1}^{s} A_{\mathfrak{m}_i}$ which is such that, for each $a \in A$ and each $i = 1, \ldots, s$, the component of $\alpha(a)$ in $A_{\mathfrak{m}_i}$ is $a/1$, is an isomorphism: to see this, just note that each of Ker α and Coker α has no maximal ideal in its support.

Let i be an integer between 1 and s. Since $A_{\mathfrak{m}_i}$ is an Artinian module over the quasi-local ring $R_{\mathfrak{m}_i}$, we can apply the results of [**4**] to deduce that $A_{\mathfrak{m}_i}$ has a natural structure as a (faithful Artinian) module over the complete (Noetherian) local ring

$$W_i := \widehat{R_{\mathfrak{m}_i}}/(0:_{\widehat{R_{\mathfrak{m}_i}}} A_{\mathfrak{m}_i});$$

furthermore, 1.3 shows that $A_{\mathfrak{m}_i}$ is co-Cohen–Macaulay as an $R_{\mathfrak{m}_i}$-module if and only if it is co-Cohen–Macaulay as a W_i-module. We can regard $\bigoplus_{i=1}^{s} A_{\mathfrak{m}_i}$ as a module over

$$W := \prod_{i=1}^{s} W_i = \prod_{i=1}^{s} \widehat{R_{\mathfrak{m}_i}}/(0:_{\widehat{R_{\mathfrak{m}_i}}} A_{\mathfrak{m}_i})$$

in a natural way, and it now follows that A is co-Cohen-Macaulay as an R-module if and only if $\bigoplus_{i=1}^{s} A_{\mathfrak{m}_i}$ is co-Cohen–Macaulay as a W-module.

However, the obvious ring isomorphisms

$$R \Big/ \Big(\bigcap_{i=1}^{s} \mathfrak{m}_i\Big)^n \xrightarrow{\cong} \prod_{i=1}^{s} R_{\mathfrak{m}_i}/(\mathfrak{m}_i R_{\mathfrak{m}_i})^n \qquad (n \in \mathbb{N})$$

lead to a ring isomorphism

$$\beta: \hat{R}^{(\mathrm{i})} \xrightarrow{\cong} \prod_{i=1}^{s} \widehat{R_{\mathfrak{m}_i}}$$

for which $\beta((0:_{\hat{R}^{(\mathrm{i})}} A)) = \prod_{i=1}^{s} (0:_{\widehat{R_{\mathfrak{m}_i}}} A_{\mathfrak{m}_i})$. There is therefore induced a ring isomorphism

$$\gamma: R' \xrightarrow{\cong} W = \prod_{i=1}^{s} \widehat{R_{\mathfrak{m}_i}}/0:_{\widehat{R_{\mathfrak{m}_i}}} A_{\mathfrak{m}_i}),$$

and it is straightforward to check that, when $\bigoplus_{i=1}^{s} A_{\mathfrak{m}_i}$ is regarded as an R'-module via γ, then α is actually an R'-isomorphism. The equivalence of statements (i) and (ii) now follows.

Let $N' := D'(A)$. The equivalence of statements (ii) and (iii) can now be proved by straightforward techniques of elementary commutative algebra: bearing in mind that the ring W, to which R' is isomorphic, is given as a direct product of s complete Noetherian local rings, one shows that, for each maximal ideal $\mathfrak{n}'$ of R', there are $R'_{\mathfrak{n}'}$-isomorphisms

$$A_{\mathfrak{n}'} \cong (D'(N))_{\mathfrak{n}'} \cong \mathrm{Hom}_{R'_{\mathfrak{n}'}}(N_{\mathfrak{n}'}, E_{R'_{\mathfrak{n}'}}(R'_{\mathfrak{n}'}/\mathfrak{n}'R'_{\mathfrak{n}'}));$$

and, since $R'_{\mathfrak{n}'}$ is a complete Noetherian local ring, it follows that $N_{\mathfrak{n}'}$ is Cohen–Macaulay over $R'_{\mathfrak{n}'}$ if and only if $A_{\mathfrak{n}'}$ is co-Cohen–Macaulay over $R'_{\mathfrak{n}'}$.

2. A characterization by generalized fractions.

2.1. Definition. (The notation is as in 0.2.) We say that the sequence $r_1, \ldots, r_n$ of elements of R is a *locally weak system of parameters for A* if, for each $i = 1, \ldots, s$, there exists $t_i \in \mathbb{N}_0$ with $0 \leq t_i \leq n$ such that the sequence of natural images $r_1/1, \ldots, r_{t_i}/1$ in $R_{\mathfrak{m}_i}$ is a p.s.o.p. for $A_{\mathfrak{m}_i}$ and $r_{t_i+1}/1, \ldots, r_n/1$ are all units in $R_{\mathfrak{m}_i}$.

We shall use "l.w.s.o.p." as an abbreviation for "locally weak system of parameters".

The following lemma will be quite helpful in situations where we need to construct locally weak systems of parameters.

2.2. Lemma. *Let $\mathfrak{a}$ be an ideal of R, and let $r_1, \ldots, r_{n-1}$* (where $n \in \mathbb{N}$) *be a l.w.s.o.p. for A composed of elements of $\mathfrak{a}$ such that, for all $j \in \{1, \ldots, s\}$ with $\mathfrak{a} \subseteq \mathfrak{m}_j$, we have*

$$\dim(0:_{A_{\mathfrak{m}_j}} \mathfrak{a}R_{\mathfrak{m}_j}) < \dim(0:_{A_{\mathfrak{m}_j}} (r_1, \ldots, r_{n-1})R_{\mathfrak{m}_j}).$$

Then there exists $r_n \in \mathfrak{a}$ such that $r_1, \ldots, r_{n-1}, r_n$ is a l.w.s.o.p. for A.

Proof. Let $J := \{j \in \{1, \ldots, s\} : \mathfrak{a} \subseteq \mathfrak{m}_j\}$ and $K = \{1, \ldots, s\} \setminus J$. Let $j \in J$, and let

$$(0:_{A_{\mathfrak{m}_j}} (r_1, \ldots, r_{n-1})R_{\mathfrak{m}_j}) = \sum_{i=1}^{h_j} S_{j,i}$$

be a minimal secondary representation for $(0:_{A_{\mathfrak{m}_j}} (r_1, \ldots, r_{n-1})R_{\mathfrak{m}_j})$, with $S_{j,i}$ a $\mathfrak{p}_{j,i}R_{\mathfrak{m}_j}$-secondary $R_{\mathfrak{m}_j}$-submodule (and $\mathfrak{p}_{j,i} \in \mathrm{Spec}(R)$ and $\mathfrak{p}_{j,i} \subseteq \mathfrak{m}_j$) for all $i = 1, \ldots, h_j$. Suppose also that the $S_{j,i}$ are numbered so that

$$\dim S_{j,i} \begin{cases} = \dim_{R_{\mathfrak{m}_j}} A_{\mathfrak{m}_j} - (n-1) & \text{for } i = 1, \ldots, g_j, \\ < \dim_{R_{\mathfrak{m}_j}} A_{\mathfrak{m}_j} - (n-1) & \text{for } i = g_j + 1. \ldots, h_j. \end{cases}$$

(See [**7**, 2.7].) We claim that

$$\mathfrak{a} \not\subseteq \left(\bigcup_{j \in J} (\mathfrak{p}_{j,1} \cup \ldots \cup \mathfrak{p}_{j,g_j}) \right) \cup \left(\bigcup_{k \in K} \mathfrak{m}_k \right).$$

If this were not the case, then, since $\mathfrak{a} \not\subseteq \mathfrak{m}_k$ for all $k \in K$, we would have $\mathfrak{a} \subseteq \mathfrak{p}_{j,i}$ for some $j \in J$ and $i \in \mathbb{N}$ with $1 \leq i \leq g_j$. Since there exists $m \in \mathbb{N}$ such that $(\mathfrak{p}_{j,i}R_{\mathfrak{m}_j})^m S_{j,i} = 0$ (by [**3**, (2.5)]), this would mean that

$$S_{j,i} \subseteq (0:_{A_{\mathfrak{m}_j}} (\mathfrak{a}R_{\mathfrak{m}_j})^m),$$

and hence, on use of [7, 2.5 and 2.7], that

$$\begin{aligned}\dim(0:_{A_{\mathfrak{m}_j}}(r_1,\ldots,r_{n-1})R_{\mathfrak{m}_j}) &= \dim_{R_{\mathfrak{m}_j}} A_{\mathfrak{m}_j} - (n-1)\\ &= \dim S_{j,i}\\ &\leqslant \dim(0:_{A_{\mathfrak{m}_j}}(\mathfrak{a}R_{\mathfrak{m}_j})^m)\\ &= \dim(0:_{A_{\mathfrak{m}_j}}\mathfrak{a}R_{\mathfrak{m}_j}),\end{aligned}$$

a contradiction to the hypotheses. Hence, there exists

$$r_n \in \mathfrak{a}\bigg\backslash\bigg(\bigg(\bigcup_{j\in J}(\mathfrak{p}_{j,1}\cup\ldots\cup\mathfrak{p}_{j,g_j})\bigg)\cup\bigg(\bigcup_{k\in K}\mathfrak{m}_k\bigg)\bigg).$$

We now show that $r_1,\ldots,r_{n-1},r_n$ is a l.w.s.o.p. for A. For each $k\in K$, the element $r_n/1$ in $R_{\mathfrak{m}_k}$ is a unit. Let $j\in J$. Then it follows from [7, 2.14] that $r_n/1$ in $R_{\mathfrak{m}_k}$ is a p.s.o.p. for $(0:_{A_{\mathfrak{m}_j}}(r_1,\ldots,r_{n-1})R_{\mathfrak{m}_j})$, so that (by [7, 2.7] again) $r_1/1,\ldots,r_{n-1}/1, r_n/1$ is a p.s.o.p. for A_{m_j}. Hence $r_1,\ldots,r_{n-1}, r_n$ is a l.w.s.o.p. for A.

The concept of poor A-cosequence, which features in the next theorem, is explained in [7, p. 2175]. This theorem can be viewed as an extension, to the non-quasi-local situation, of [7, Proposition 2.15].

2.3. Theorem. (The notation is as in 0.2.) *The (non-zero, Artinian) R-module A is co-Cohen–Macaulay if and only if every locally weak system of parameters for A is a poor A-cosequence.*

Proof. ($\Rightarrow$) Assume that A is co-Cohen–Macaulay, and let $r_1,\ldots,r_n$ be a l.w.s.o.p. for A. It is enough to show that, for each $i=1,\ldots,s$, the sequence of elements $r_1/1,\ldots,r_n/1$ in $R_{\mathfrak{m}_i}$ is a poor $A_{\mathfrak{m}_i}$-cosequence. However, by definition, there exists $t_i\in\mathbb{N}_0$ with $0\leqslant t_i\leqslant n$ such that $r_1/1,\ldots,r_{t_i}/1$ is a p.s.o.p. for $A_{\mathfrak{m}_i}$ and $r_{t_i+1}/1,\ldots,r_n/1$ are all units in $R_{\mathfrak{m}_i}$. Since $A_{\mathfrak{m}_i}$ is a co-Cohen–Macaulay $R_{\mathfrak{m}_i}$-module, it is immediate from [7, 2.15] that $r_1/1,\ldots,r_n/1$ is a poor $A_{\mathfrak{m}_i}$-cosequence.

($\Leftarrow$) Assume that every l.w.s.o.p. for A is a poor A-cosequence. To show that A is co-Cohen–Macaulay, we show that, for each $i=1,\ldots,s$, the $R_{\mathfrak{m}_i}$-module $A_{\mathfrak{m}_i}$ is co-Cohen–Macaulay.

To achieve this, let $d:=\dim_{R_{\mathfrak{m}_i}} A_{\mathfrak{m}_i}$. We show that it is possible to construct a l.w.s.o.p. $r_1,\ldots,r_d$ for A such that the sequence of natural images $r_1/1,\ldots,r_d/1$ in $R_{\mathfrak{m}_i}$ is a s.o.p. for $A_{\mathfrak{m}_i}$; the assumption that every l.w.s.o.p. for A is a poor A-cosequence will then show that $\operatorname{codepth}_{R_{\mathfrak{m}_i}} A_{\mathfrak{m}_i} = d = \dim_{R_{\mathfrak{m}_i}} A_{\mathfrak{m}_i}$, so that the proof will be complete. So suppose, inductively, that $j\in\mathbb{N}$ with $j\leqslant d$ and we have constructed a l.w.s.o.p. $r_1,\ldots,r_{j-1}$ for A with $r_1,\ldots,r_{j-1}\in\mathfrak{m}_i$. Then, on use of [7, 2.7], we see that

$$\dim(0:_{A_{\mathfrak{m}_i}}\mathfrak{m}_iR_{\mathfrak{m}_i}) = 0 < d-(j-1) = \dim(0:_{A_{\mathfrak{m}_i}}(r_1,\ldots,r_{j-1})R_{\mathfrak{m}_i}).$$

Hence, by 2.2, there exists $r_j\in\mathfrak{m}_i$ such that $r_1,\ldots,r_{j-1}, r_j$ is a l.w.s.o.p. for A. A l.w.s.o.p. for A with the desired properties can therefore be constructed by induction.

Our main aim in this section is to establish a generalization, available for non-quasi-local situations, of [7, Theorem 3.10], which characterizes co-Cohen–Macaulay modules (among non-zero Artinian modules) over a quasi-local ring in terms of modules of generalized fractions. For this work, we need some appropriate triangular sets, and locally weak systems of parameters provide these for us.

2.4. THEOREM. (The notation is as in 0.2.) *For each* $n \in \mathbb{N}$, *let*

$$U_n := \{(r_1, \dots, r_n) \in R^i : r_1, \dots, r_n \text{ is a l.w.s.o.p. for } A\}.$$

Then $\mathcal{U} = (U_n)_{n \in \mathbb{N}}$ *is a chain of triangular sets on R in the sense of L. O'Carroll* [**1**, p. 420], *and (the non-zero, Artinian R-module) A is co-Cohen–Macaulay if and only if the complex* $\mathrm{Hom}_R(C(\mathcal{U}, R), A)$ *is exact.*

Proof. Let $n \in \mathbb{N}$. It is obvious that $(1, \dots, 1) \in R^n$ is an element of U_n, and it is an easy consequence of [**7**, 2.5 and 2.7] that, if $(u_1, \dots, u_n) \in U_n$, then $(u_1^{h_1}, \dots, u_n^{h_n}) \in U_n$ for all $h_1, \dots, h_n \in \mathbb{N}$.

Suppose that $(u_1, \dots, u_n)$, $(v_1, \dots, v_n) \in U_n$; suppose, inductively, that $i \in \mathbb{N}$ with $i \leq n$ and we have constructed a l.w.s.o.p. $w_1, \dots, w_{i-1}$ for A such that

$$w_j \in \left(\sum_{\gamma=1}^{j} Ru_\gamma\right) \cap \left(\sum_{\gamma=1}^{j} Rv_\gamma\right) \qquad \text{for all } j = 1, \dots, i-1.$$

We propose to use 2.2. Let k be an integer between 1 and s such that $\left(\sum_{\gamma=1}^{i} Ru_\gamma\right) \cap \left(\sum_{\gamma=1}^{i} Rv_\gamma\right) \subseteq \mathfrak{m}_k$. Then either $\left(\sum_{\gamma=1}^{i} Ru_\gamma\right) \subseteq \mathfrak{m}_k$ or $\left(\sum_{\gamma=1}^{i} Rv_\gamma\right) \subseteq \mathfrak{m}_k$, and so, on use of [**7**, 2.4 and 2.7], we see that

$$\begin{aligned} \dim(0 :_{A_{\mathfrak{m}_k}} ((u_1, \dots, u_i) \cap (v_1, \dots, v_i)) R_{\mathfrak{m}_k}) &= \dim_{R_{\mathfrak{m}_k}} A_{\mathfrak{m}_k} - i \\ &< \dim_{R_{\mathfrak{m}_k}} A_{\mathfrak{m}_k} - (i-1) \\ &= \dim(0 :_{A_{\mathfrak{m}_k}} (w_1, \dots, w_{i-1}) R_{\mathfrak{m}_k}). \end{aligned}$$

Hence, by 2.2, there exists

$$w_i \in \left(\sum_{\gamma=1}^{i} Ru_\gamma\right) \cap \left(\sum_{\gamma=1}^{i} Rv_\gamma\right)$$

such that $w_1, \dots, w_{i-1}, w_i$ is a l.w.s.o.p. for A. We can now complete our inductive construction. We conclude that U_n is a triangular subset of R^n.

It is now easy to see that $\mathcal{U} = (U_n)_{n \in \mathbb{N}}$ is a chain of triangular sets on R. By 2.3, the (non-zero, Artinian) R-module A is co-Cohen-Macaulay if and only if, for all $n \in \mathbb{N}$, each element of U_n is a poor A-cosequence; by [**7**, 3.3], this is the case if and only if the complex $\mathrm{Hom}_R(C(\mathcal{U}, R), A)$ is exact.

2.5. REMARK. Let the situation be as in Theorem 2.4, but assume, in addition, that R is quasi-local. For each $n \in \mathbb{N}$, the triangular subset of R^n consisting of all locally weak systems of parameters for A has the triangular set

$$\{(r_1, \dots, r_n) \in R^n : \text{there exists } j \text{ with } 0 \leq j \leq n \text{ such that } r_1, \dots, r_j \text{ is a p.s.o.p. for } A \text{ and } r_{j+1} = \dots = r_n = 1\}$$

as a cofinal subset (with respect to the quasi-order considered in [**6**, p. 39]), and so Theorem 2.4 can indeed be considered as an extension of [**7**, Theorem 3.10].

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