10

Spinning charges

The Lorentz model includes by necessity the inner rotation of charges and, beyond the translational degrees of freedom, one has to determine its effective dynamics. This will lead to a derivation of the Bargmann–Michel–Telegdi (BMT) equation from a microscopic basis including an expression for the gyromagnetic ratio. We will also discuss the Abraham model with spin, a little-explored territory, since it is more easily controlled mathematically and it teaches us how the BMT equation is modified when Lorentz invariance is no longer available.

10.1 Effective spin dynamics of the Lorentz model

Let us recall the equations of motion for an extended charge, where for the moment the interaction with the self-field is ignored,

$$\dot{\mathbf{p}} = \mathbf{f}, \quad \dot{\mathbf{s}} + \boldsymbol{\Omega}_{\mathrm{FW}} \cdot \mathbf{s} = \mathbf{t}\,. \tag{10.1}$$

Here the external force $\boldsymbol{f}$, respectively the external torque $\boldsymbol{t}$, are defined through (2.92), respectively (2.95). Equation (10.1) must be supplemented by

$$\mathbf{p} = m_{\mathrm{g}}\mathbf{u}, \quad \mathbf{s} = I_{\mathrm{b}}\mathbf{w}, \tag{10.2}$$

which define the bare gyrational mass m_{g} and the bare moment of inertia I_{b}. Both depend on $|\mathbf{w}|$.

We assume now that the external field tensor is slowly varying, by replacing $\mathbf{F}(\mathbf{q})$ by the scaled field tensor $\varepsilon\mathbf{F}(\varepsilon\mathbf{q})$ in (2.92), (2.95). Note that this prescription automatically includes slow variation in time. $\mathbf{f}$ and $\mathbf{t}$ simplify in the limit of small ε and, on the macroscopic scale, (10.1) becomes

$$\dot{\mathbf{p}} = e\mathbf{F} \cdot \mathbf{u}, \quad \dot{\mathbf{s}} + \boldsymbol{\Omega}_{\mathrm{FW}} \cdot \mathbf{s} = \mu(\mathbf{F} \cdot \mathbf{w})^{\perp} \tag{10.3}$$

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with the magnetic moment

$$\mu = \frac{1}{3} e \int \mathrm{d}^3 x \varphi(\boldsymbol{x}) \boldsymbol{x}^2 \tag{10.4}$$

and $\mathbf{a}^{\perp} = (\mathbf{g} + \mathbf{u} \otimes \mathbf{u}) \cdot \mathbf{a}$. Since $|\mathbf{w}|$ is conserved, the translational motion is autonomous, whereas the spin follows the local fields as they are encountered.

As a next step we have to include the coupling to the self-field. In principle the scheme of chapter 7 has to be repeated, but we prefer to take the static short-cut. The energy–momentum relation for the Lorentz model was computed in chapter 4. Thus we stipulate that the bare gyrational mass m_{g} is renormalized to $m_{\mathrm{g}} + m_{\mathrm{f}}$ and the bare moment of inertia to $I_{\mathrm{b}} + I_{\mathrm{f}}$; see (4.43), (4.45), respectively (4.49), (4.51). This means that instead of (10.2) we have

$$\mathbf{p} = (m_{\mathrm{g}} + m_{\mathrm{f}})\mathbf{u}, \quad \mathbf{s} = (I_{\mathrm{b}} + I_{\mathrm{f}})\mathbf{w}\,. \tag{10.5}$$

Equation (10.3) together with (10.5) is the effective dynamics in the adiabatic limit on the Hamiltonian level neglecting radiation damping.

We want to compare our spin dynamics with the BMT equation which reads

$$\dot{\mathbf{w}} + \boldsymbol{\Omega}_{\mathrm{FW}} \cdot \mathbf{w} = \frac{g}{2}\frac{e}{m}(\mathbf{F} \cdot \mathbf{w})^{\perp}\,, \tag{10.6}$$

where m is the experimental mass and g the gyromagnetic ratio, which like the charge is an intrinsic property of the particle. Using the fact that $\boldsymbol{\Omega}_{\mathrm{FW}}$ is determined by Newton's translational equations of motion one arrives at the perhaps more familiar three-vector form for the angular velocity,

$$\begin{aligned} \dot{\boldsymbol{\omega}} = \frac{e}{mc}\boldsymbol{\omega} \times \Big[\Big(\frac{g}{2} - 1 + \frac{1}{\gamma}\Big)\boldsymbol{B}_{\mathrm{ex}} - \Big(\frac{g}{2} - 1\Big)\frac{\gamma}{1+\gamma} c^{-2}(\boldsymbol{v} \cdot \boldsymbol{B}_{\mathrm{ex}})\boldsymbol{v} \\ - \Big(\frac{g}{2} - \frac{\gamma}{1+\gamma}\Big) c^{-1} \boldsymbol{v} \times \boldsymbol{E}_{\mathrm{ex}} \Big]. \end{aligned} \tag{10.7}$$

Here $\boldsymbol{v}$, $\boldsymbol{E}_{\mathrm{ex}}$, $\boldsymbol{B}_{\mathrm{ex}}$ are to be evaluated along the given orbit. To compare (10.6) with (10.3) one uses (10.6) and notes that, since $|\boldsymbol{w}|$ is a constant of motion,

$$\dot{\mathbf{w}} + \boldsymbol{\Omega}_{\mathrm{FW}} \cdot \mathbf{w} = \frac{\mu}{I_{\mathrm{b}} + I_{\mathrm{f}}}(\mathbf{F} \cdot \mathbf{w})^{\perp}\,. \tag{10.8}$$

Therefore the gyromagnetic ratio of the Lorentz model is given by

$$g = \frac{2\mu}{e}\frac{m_{\mathrm{g}} + m_{\mathrm{f}}}{I_{\mathrm{b}} + I_{\mathrm{f}}}. \tag{10.9}$$

The magnetic moment μ depends on the charge distribution, all other terms in (10.9) on the mass distribution. Through their variation any value of g can be realized, unless the charge and mass form factors are equal to each other, as assumed already. In the case of a uniformly charged sphere [ball] of radius R the integrals in (10.9) can be evaluated with the result (the first term refers to a sphere and [...] to a ball)

$$\mu = \frac{1}{3}eR^2 , \quad \Big[= \frac{1}{5}eR^2 \Big] , \tag{10.10}$$

$$m_{\mathrm{g}} = m_{\mathrm{b}} \frac{1}{\omega R} \mathrm{arctanh} \omega R , \quad \Big[= m_{\mathrm{b}} \frac{3}{2(\omega R)^3} \big(\omega R - (1 - (\omega R)^2) \mathrm{arctanh} \omega R \big) \Big] , \tag{10.11}$$

$$m_{\mathrm{f}} = \frac{1}{2} \frac{e^2}{4\pi R} \Big(1 + \frac{2}{9} (\omega R)^2 \Big) , \quad \Big[= \frac{1}{2} \frac{e^2}{4\pi R} \Big(\frac{6}{5} + \frac{4}{35} (\omega R)^2 \Big) \Big] , \tag{10.12}$$

$$I_{\mathrm{b}} = m_{\mathrm{b}} \frac{1}{2\omega^2} \Big(-1 + \frac{1 + (\omega R)^2}{\omega R} \mathrm{arctanh} \omega R \Big) ,$$

$$\Big[= m_{\mathrm{b}} \frac{1}{2\omega^2} \frac{3}{4(\omega R)^3} \big(3\omega R - (\omega R)^3 + (-3 + 2(\omega R)^2 + (\omega R)^4) \mathrm{arctanh} \omega R \big) \Big] , \tag{10.13}$$

$$I_{\mathrm{f}} = \frac{2}{9} \frac{e^2}{4\pi R} , \quad \Big[= \frac{4}{35} \frac{e^2}{4\pi R} \Big] . \tag{10.14}$$

In the limit $e \to 0$, g_{sphere} decreases from 1 to $2/3$ and g_{ball} from 1 to $2/5$ as ωR increases from 0 to 1. In the opposite limit $m_{\mathrm{b}} \to 0$, one obtains

$$g_{\mathrm{sphere}} = \frac{3}{2} + \frac{1}{3} (\omega R)^2 , \quad g_{\mathrm{ball}} = \frac{21}{10} + \frac{1}{5} (\omega R)^2 . \tag{10.15}$$

10.2 The Abraham model with spin

Abraham models the charge as a nonrelativistic rigid body with mass distribution $m_{\mathrm{b}}\varphi$ and charge distribution $e\varphi$, which for notational simplicity we take to be proportional to each other. A complete mechanical description must specify both the center of mass, $\boldsymbol{q}(t)$, and the angular velocity, $\boldsymbol{\omega}(t) \in \mathbb{R}^3$, relative to the center. The spinning charge generates the current

$$\boldsymbol{j}(\boldsymbol{x}, t) = \big(\boldsymbol{v}(t) + \boldsymbol{\omega}(t) \times (\boldsymbol{x} - \boldsymbol{q}(t)) \big) e\varphi(\boldsymbol{x} - \boldsymbol{q}(t)) , \tag{10.16}$$

which satisfies charge conservation, since φ is radial. Therefore the Maxwell equations have a modified source term and read

$$\begin{aligned} &\partial_t \boldsymbol{B}(\boldsymbol{x}, t) = -\nabla \times \boldsymbol{E}(\boldsymbol{x}, t) , \\ &\partial_t \boldsymbol{E}(\boldsymbol{x}, t) = \nabla \times \boldsymbol{B}(\boldsymbol{x}, t) - \big(\boldsymbol{v}(t) + \boldsymbol{\omega}(t) \times (\boldsymbol{x} - \boldsymbol{q}(t)) \big) e\varphi(\boldsymbol{x} - \boldsymbol{q}(t)) , \\ &\nabla \cdot \boldsymbol{E}(\boldsymbol{x}, t) = e\varphi(\boldsymbol{x} - \boldsymbol{q}(t)), \quad \nabla \cdot \boldsymbol{B}(\boldsymbol{x}, t) = 0. \end{aligned} \tag{10.17}$$

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The momentum of the center of mass is $m_\mathrm{b}\boldsymbol{v}(t)$ and the angular momentum relative to $\boldsymbol{q}(t)$ is

$$\boldsymbol{s} = I_\mathrm{b}\boldsymbol{\omega} \quad \text{with} \quad I_\mathrm{b} = \frac{2}{3} m_\mathrm{b} \int \mathrm{d}^3 x \varphi(\boldsymbol{x})\boldsymbol{x}^2 \,. \tag{10.18}$$

Therefore Newton's equations of motion for the translational degrees of freedom become

$$\frac{\mathrm{d}}{\mathrm{d}t} m_\mathrm{b}\boldsymbol{v}(t) = \int \mathrm{d}^3 x e\varphi(\boldsymbol{x} - \boldsymbol{q}(t))\big[\boldsymbol{E}(\boldsymbol{x},t) + \big(\boldsymbol{v}(t) + \boldsymbol{\omega}(t) \times (\boldsymbol{x} - \boldsymbol{q}(t))\big) \times \boldsymbol{B}(\boldsymbol{x},t)\big] \tag{10.19}$$

and for the rotational degrees of freedom

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} I_\mathrm{b}\,\boldsymbol{\omega}(t) = \int \mathrm{d}^3 x e\varphi(\boldsymbol{x} - \boldsymbol{q}(t))(\boldsymbol{x} - \boldsymbol{q}(t)) \\ \times \big[\boldsymbol{E}(\boldsymbol{x},t) + \big(\boldsymbol{v}(t) + \boldsymbol{\omega}(t) \times (\boldsymbol{x} - \boldsymbol{q}(t))\big) \times \boldsymbol{B}(\boldsymbol{x},t)\big] .\end{aligned} \tag{10.20}$$

If in addition there are external forces acting on the charge, then $\boldsymbol{E}$ and $\boldsymbol{B}$ in (10.19), (10.20) would have to be replaced by $\boldsymbol{E} + \boldsymbol{E}_\mathrm{ex}$ and $\boldsymbol{B} + \boldsymbol{B}_\mathrm{ex}$, respectively.

The Abraham model of section 2.4 is obtained by formally setting $\boldsymbol{\omega}(t) = 0$. Note that this is not consistent with Newton's torque equation (10.20), since $\dot{\boldsymbol{\omega}}(t) \neq 0$, in general, even for $\boldsymbol{\omega}(t) = 0$.

The Abraham model with spin conserves the energy

$$\mathcal{E} = \frac{1}{2} m_\mathrm{b}\boldsymbol{v}^2 + \frac{1}{2} I_\mathrm{b}\boldsymbol{\omega}^2 + \frac{1}{2}\int \mathrm{d}^3 x (\boldsymbol{E}^2 + \boldsymbol{B}^2)\,, \tag{10.21}$$

the linear momentum

$$\mathcal{P} = m_\mathrm{b}\boldsymbol{v} + \int \mathrm{d}^3 x \boldsymbol{E} \times \boldsymbol{B}\,, \tag{10.22}$$

and in addition the total angular momentum

$$\mathcal{J} = \boldsymbol{q} \times m_\mathrm{b}\boldsymbol{v} + I_\mathrm{b}\boldsymbol{\omega} + \int \mathrm{d}^3 x \boldsymbol{x} \times (\boldsymbol{E} \times \boldsymbol{B})\,. \tag{10.23}$$

Of course, also the spinless Abraham model is invariant under rotations and there must exist a correspondingly conserved quantity, only it does not have the standard form of a total angular momentum, which from a somewhat different perspective indicates that inner rotations must be included.

In the by now established tradition, we assume that the external forces are slowly varying and want to derive in this adiabatic limit an effective equation of motion for the particle including its spin. As a first step of this program we have

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to determine the charge solitons. We set

$$\boldsymbol{q}(t) = \boldsymbol{v}t, \quad \boldsymbol{\omega}(t) = \boldsymbol{\omega}, \quad \boldsymbol{E}(\boldsymbol{x}, t) = \boldsymbol{E}(\boldsymbol{x} - \boldsymbol{v}t), \quad \boldsymbol{B}(\boldsymbol{x}, t) = \boldsymbol{B}(\boldsymbol{x} - \boldsymbol{v}t) \tag{10.24}$$

and have to determine the solutions of

$$\begin{aligned} &-\boldsymbol{v} \cdot \nabla \boldsymbol{B} = -\nabla \times \boldsymbol{E}, \quad -\boldsymbol{v} \cdot \nabla \boldsymbol{E} = \nabla \times \boldsymbol{B} - (\boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{x}) e\varphi\,, \\ &\nabla \cdot \boldsymbol{E} = e\varphi, \quad \nabla \cdot \boldsymbol{B} = 0\,, \end{aligned} \tag{10.25}$$

$$0 = \int \mathrm{d}^3 x e\varphi(\boldsymbol{x}) \big[\boldsymbol{E}(\boldsymbol{x}) + (\boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{x}) \times \boldsymbol{B}(\boldsymbol{x})\big]\,, \tag{10.26}$$

$$0 = \int \mathrm{d}^3 x e\varphi(\boldsymbol{x}) \boldsymbol{x} \times \big[\boldsymbol{E}(\boldsymbol{x}) + (\boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{x}) \times \boldsymbol{B}(\boldsymbol{x})\big]\,, \tag{10.27}$$

for which we turn to Fourier space. The inhomogeneous Maxwell equations (10.25) are then solved by

$$\widehat{\boldsymbol{E}} = \widehat{\boldsymbol{E}}_1 + \widehat{\boldsymbol{E}}_2, \quad \widehat{\boldsymbol{B}} = \widehat{\boldsymbol{B}}_1 + \widehat{\boldsymbol{B}}_2 \tag{10.28}$$

with

$$\widehat{\boldsymbol{E}}_1(\boldsymbol{k}) = -\mathrm{i}\big[\boldsymbol{k}^2 - (\boldsymbol{k} \cdot \boldsymbol{v})^2\big]^{-1} (\boldsymbol{k} - (\boldsymbol{k} \cdot \boldsymbol{v})\boldsymbol{v}) e\widehat{\varphi}(\boldsymbol{k})\,, \tag{10.29}$$

$$\widehat{\boldsymbol{E}}_2(\boldsymbol{k}) = -\big[\boldsymbol{k}^2 - (\boldsymbol{k} \cdot \boldsymbol{v})^2\big]^{-1} (\boldsymbol{\omega} \times \boldsymbol{k})(\boldsymbol{v} \cdot \nabla_{\boldsymbol{k}}) e\widehat{\varphi}(\boldsymbol{k})\,, \tag{10.30}$$

and

$$\widehat{\boldsymbol{B}}_1(\boldsymbol{k}) = \mathrm{i}\big[\boldsymbol{k}^2 - (\boldsymbol{k} \cdot \boldsymbol{v})^2\big]^{-1} (\boldsymbol{k} \times \boldsymbol{v}) e\widehat{\varphi}(\boldsymbol{k})\,, \tag{10.31}$$

$$\widehat{\boldsymbol{B}}_2(\boldsymbol{k}) = -\big[\boldsymbol{k}^2 - (\boldsymbol{k} \cdot \boldsymbol{v})^2\big]^{-1} (\boldsymbol{k} \times (\boldsymbol{\omega} \times \nabla_{\boldsymbol{k}})) e\widehat{\varphi}(\boldsymbol{k})\,. \tag{10.32}$$

Note that $\widehat{\boldsymbol{E}}_1$, $\widehat{\boldsymbol{B}}_1$ are odd, and $\widehat{\boldsymbol{E}}_2$, $\widehat{\boldsymbol{B}}_2$ are even in $\boldsymbol{k}$.

Since the integral over an odd term vanishes, a zero Lorentz force results in the condition

$$\begin{aligned} &-\int \mathrm{d}^3 k \widehat{\varphi}^* \big[\boldsymbol{k}^2 - (\boldsymbol{v} \cdot \boldsymbol{k})^2\big]^{-1} (\boldsymbol{\omega} \times \boldsymbol{k})(\boldsymbol{v} \cdot \nabla_{\boldsymbol{k}}) \widehat{\varphi} \\ &-\int \mathrm{d}^3 k \widehat{\varphi}^* \big[\boldsymbol{k}^2 - (\boldsymbol{v} \cdot \boldsymbol{k})^2\big]^{-1} \boldsymbol{v} \times (\boldsymbol{k} \times (\boldsymbol{\omega} \times \nabla_{\boldsymbol{k}})) \widehat{\varphi} \\ &+\int \mathrm{d}^3 k \big[\boldsymbol{k}^2 - (\boldsymbol{v} \cdot \boldsymbol{k})^2\big]^{-1} ((\boldsymbol{\omega} \times \nabla_{\boldsymbol{k}}) \widehat{\varphi}^*) \times (\boldsymbol{k} \times \boldsymbol{v}) \widehat{\varphi} \\ &\quad = -\int \mathrm{d}^3 k \widehat{\varphi}^* \big[\boldsymbol{k}^2 - (\boldsymbol{v} \cdot \boldsymbol{k})^2\big]^{-1} |\boldsymbol{k}|^{-1} \widehat{\varphi}_r' \\ &\qquad \times \big((\boldsymbol{\omega} \times \boldsymbol{k})(\boldsymbol{v} \cdot \boldsymbol{k}) + \boldsymbol{v} \times (\boldsymbol{k} \times (\boldsymbol{\omega} \times \boldsymbol{k})) - ((\boldsymbol{\omega} \times \boldsymbol{k}) \cdot \boldsymbol{v})\boldsymbol{k}\big) = 0 \end{aligned} \tag{10.33}$$

for every $\boldsymbol{v}$ and $\boldsymbol{\omega}$, using the fact that $\widehat{\varphi}$ is radial.

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The Lorentz torque requires more work. Using again the fact that the integral over an odd term vanishes, we have

$$\begin{aligned}
&\mathrm{i}\int \mathrm{d}^3k e\widehat{\varphi}^*\big(\nabla_{\boldsymbol{k}}\times\widehat{\boldsymbol{E}}_1+\nabla_{\boldsymbol{k}}\times(\boldsymbol{v}\times\widehat{\boldsymbol{B}}_1)+\nabla_{\boldsymbol{k}}\times((\boldsymbol{\omega}\times i\nabla_{\boldsymbol{k}})\times\widehat{\boldsymbol{B}}_2)\big) \quad (10.34)\\
&\quad=-e^2\int \mathrm{d}^3k|\boldsymbol{k}|^{-1}\widehat{\varphi}_r^{*\prime}\big[\boldsymbol{k}^2-(\boldsymbol{k}\cdot\boldsymbol{v})^2\big]^{-1}\widehat{\varphi}\big(\boldsymbol{k}\times(\boldsymbol{k}-(\boldsymbol{k}\cdot\boldsymbol{v})\boldsymbol{v})\\
&\qquad-\boldsymbol{k}\times(\boldsymbol{v}\times(\boldsymbol{k}\times\boldsymbol{v}))\big)+e\int \mathrm{d}^3k|\boldsymbol{k}|^{-1}\widehat{\varphi}_r^{*\prime}\boldsymbol{k}\times\big((\boldsymbol{\omega}\times\nabla_{\boldsymbol{k}})\times\widehat{\boldsymbol{B}}_2\big)\\
&\quad=e\int \mathrm{d}^3k|\boldsymbol{k}|^{-1}\widehat{\varphi}_r^{*\prime}\boldsymbol{k}\times(\nabla_{\boldsymbol{k}}(\boldsymbol{\omega}\cdot\widehat{\boldsymbol{B}}_2)-\boldsymbol{\omega}\nabla_{\boldsymbol{k}}\cdot\widehat{\boldsymbol{B}}_2)\\
&\quad=-e\int \mathrm{d}^3k|\boldsymbol{k}|^{-1}\widehat{\varphi}_r^{*\prime}\times(\boldsymbol{k}\times\boldsymbol{\omega})\nabla_{\boldsymbol{k}}\cdot\widehat{\boldsymbol{B}}_2\,.
\end{aligned}$$

For the divergence of $\widehat{\boldsymbol{B}}_2$ we obtain

$$\nabla_{\boldsymbol{k}}\cdot\widehat{\boldsymbol{B}}_2=2\big[\boldsymbol{k}^2-(\boldsymbol{k}\cdot\boldsymbol{v})^2\big]^{-2}\boldsymbol{k}^2(\boldsymbol{\omega}\cdot\nabla_{\boldsymbol{k}}-(\boldsymbol{v}\cdot\boldsymbol{\omega})(\boldsymbol{v}\cdot\nabla_{\boldsymbol{k}}))e\widehat{\varphi} \quad (10.35)$$

and therefore zero Lorentz torque results in the condition

$$\int \mathrm{d}^3k|\nabla_{\boldsymbol{k}}\widehat{\varphi}|^2 2\big[\boldsymbol{k}^2-(\boldsymbol{k}\cdot\boldsymbol{v})^2\big]^{-2}(\boldsymbol{k}\times\boldsymbol{\omega})(\boldsymbol{\omega}\cdot\boldsymbol{k}-(\boldsymbol{v}\cdot\boldsymbol{\omega})(\boldsymbol{v}\cdot\boldsymbol{k}))=0\,. \quad (10.36)$$

Taking into account that $\widehat{\varphi}$ is radial, the torque vanishes only if either $\boldsymbol{\omega}\parallel\boldsymbol{v}$ or $\boldsymbol{\omega}\perp\boldsymbol{v}$. If $\boldsymbol{v}=0$, the torque always vanishes. For $\boldsymbol{\omega}$ oblique to $\boldsymbol{v}$ Eqs. (10.17)–(10.20) have no soliton-like solution.

Physically the charge distribution is rigid, but the electromagnetic fields are Lorentz contracted along $\boldsymbol{v}$. This mismatch yields a nonvanishing torque unless $\boldsymbol{\omega}\parallel\boldsymbol{v}$, respectively $\boldsymbol{\omega}\perp\boldsymbol{v}$. Clearly, the mismatch is an artifact of the semirelativistic Abraham model. As discussed in the previous section, for a relativistic extended charge distribution there is a charged soliton for every $\boldsymbol{v}$ and $\boldsymbol{\omega}$. Because in the Abraham model some charge solitons are "missing", an analysis of the adiabatic limit is hampered at an early stage and we do not really know what happens. Through radiation damping the spin could be forced to remain parallel to $\boldsymbol{v}(t)$. There could be an effective dynamics separately for the parallel and perpendicular components of $\boldsymbol{\omega}(t)$. Only one particular case lends itself to a more detailed analysis. We simply make sure that $\boldsymbol{q}(t)=0$ for all t, e.g. by taking $\boldsymbol{E}_{\mathrm{ex}}=0$, $\boldsymbol{B}_{\mathrm{ex}}=\varepsilon\boldsymbol{B}$ with $\boldsymbol{B}$ a spatially constant, possibly time-dependent vector, and suitable initial conditions for the Maxwell field. Then the Abraham model without external forces has a stationary solution for every $\boldsymbol{\omega}$ and the adiabatic limit is meaningful and of interest. We take up this problem in the following section.

In the quantized version of the Abraham model, the Pauli–Fierz Hamiltonian to be discussed in chapter 13, the spin couples differently and the Lorentz torque is not the quantization of the right-hand side of (10.20). The Pauli–Fierz model has a two-fold degenerate ground state for every fixed total momentum (smaller than some critical value p_c). Associated to this subspace there is an adiabatic evolution which admits an arbitrary spin orientation. Thus through quantization one regains some features of the relativistic model.

10.3 Adiabatic limit and the gyromagnetic ratio

We consider a spinning charge sitting forever at the origin and hence choose $\boldsymbol{E}_{\mathrm{ex}} = 0$, $\boldsymbol{B}_{\mathrm{ex}} = \varepsilon \boldsymbol{B}_0$ with a constant $\boldsymbol{B}_0$, the initial $\boldsymbol{E}$ field odd, and the initial $\boldsymbol{B}$ field even in $\boldsymbol{x}$. Then the equations of motion simplify. We recall them for completeness,

$$\partial_t \boldsymbol{B}(\boldsymbol{x}, t) = -\nabla \times \boldsymbol{E}(\boldsymbol{x}, t), \quad \partial_t \boldsymbol{E}(\boldsymbol{x}, t) = \nabla \times \boldsymbol{B}(\boldsymbol{x}, t) - (\boldsymbol{\omega}(t) \times \boldsymbol{x}) e\varphi(\boldsymbol{x}), \tag{10.37}$$

$$\nabla \cdot \boldsymbol{E}(\boldsymbol{x}, t) = e\varphi(\boldsymbol{x}), \quad \nabla \cdot \boldsymbol{B}(\boldsymbol{x}, t) = 0, \tag{10.38}$$

together with Newton's rotational equations of motion

$$I_{\mathrm{b}} \frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{\omega} = e \int \mathrm{d}^3 x \varphi(\boldsymbol{x}) \boldsymbol{x} \times \big(\boldsymbol{E}(\boldsymbol{x}, t) + (\boldsymbol{\omega}(t) \times \boldsymbol{x}) \times (\varepsilon \boldsymbol{B}_0 + \boldsymbol{B}(\boldsymbol{x}, t))\big). \tag{10.39}$$

To obtain the effective dynamics let us first argue statically. The angular momentum, $\boldsymbol{s}$, of the charge soliton is the sum $\boldsymbol{s} = \boldsymbol{s}_{\mathrm{b}} + \boldsymbol{s}_{\mathrm{f}}$ with $\boldsymbol{s}_{\mathrm{b}} = I_{\mathrm{b}} \boldsymbol{\omega}$ and

$$\boldsymbol{s}_{\mathrm{f}} = \int \mathrm{d}^3 x \boldsymbol{x} \times (\boldsymbol{E} \times \boldsymbol{B}) \tag{10.40}$$

for $\boldsymbol{E}, \boldsymbol{B}$ the charge soliton field at $\boldsymbol{v} = 0$ and $\boldsymbol{\omega}$. Inserting from (10.28)–(10.32) we obtain

$$\boldsymbol{s}_{\mathrm{f}} = I_{\mathrm{f}} \boldsymbol{\omega} \quad \text{with} \quad I_{\mathrm{f}} = \frac{2}{3} e^2 \int \mathrm{d}^3 k |\nabla_{\boldsymbol{k}} \widehat{\varphi}|^2 |\boldsymbol{k}|^{-2}. \tag{10.41}$$

Therefore

$$\boldsymbol{s} = (I_{\mathrm{b}} + I_{\mathrm{f}}) \boldsymbol{\omega}. \tag{10.42}$$

The external torque is $\boldsymbol{\mu} \times \boldsymbol{B}_{\mathrm{ex}}$ with the magnetic moment

$$\boldsymbol{\mu} = \mu \boldsymbol{\omega}, \quad \mu = \frac{1}{3} e \int \mathrm{d}^3 x \varphi(\boldsymbol{x}) \boldsymbol{x}^2, \tag{10.43}$$

and thus the spin precession reads

$$\frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{s} = \boldsymbol{\mu} \times \boldsymbol{B}_{\mathrm{ex}}, \quad (I_{\mathrm{b}} + I_{\mathrm{f}}) \frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{\omega} = \mu \boldsymbol{\omega} \times \boldsymbol{B}_{\mathrm{ex}}. \tag{10.44}$$

The conventional definition of the gyromagnetic ratio g is through

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\omega} = g\frac{e}{2m}\boldsymbol{\omega} \times \boldsymbol{B}_{\mathrm{ex}} \,, \tag{10.45}$$

where m is the mass of the particle; compare with the BMT equation (10.7) for small velocities. Equating (10.44) and (10.45) we deduce the effective g-factor of the Abraham model as

$$g = \frac{(\mu/e)2m}{I_{\mathrm{b}} + I_{\mathrm{f}}} = \frac{1 + \frac{2}{3}(e^2/m_{\mathrm{b}}) \int \mathrm{d}^3k |\widehat{\varphi}|^2 |\boldsymbol{k}|^{-2}}{1 + (e^2/m_{\mathrm{b}}) \int \mathrm{d}^3k |\nabla_{\boldsymbol{k}}\widehat{\varphi}|^2 |\boldsymbol{k}|^{-2} / \int \mathrm{d}^3k |\nabla_{\boldsymbol{k}}\widehat{\varphi}|^2} \,. \tag{10.46}$$

For $e \to 0$ we obtain $g = 1$, as it has to be. In the opposite limit, $m_{\mathrm{b}} \to 0$, only the second summands survive. We did not discover any simple bounds, but for a uniformly charged sphere and ball the integrals have already been computed at the end of section 10.1. One obtains with $R = R_{\varphi}$ the radius of the sphere, respectively ball,

$$g_{\mathrm{sphere}} = \frac{1 + (e^2/4\pi R m_{\mathrm{b}})(2/3)}{1 + (e^2/4\pi R m_{\mathrm{b}})(1/3)} \,, \quad g_{\mathrm{ball}} = \frac{1 + (e^2/4\pi R m_{\mathrm{b}})(4/5)}{1 + (e^2/4\pi R m_{\mathrm{b}})(2/7)} \,. \tag{10.47}$$

Thus $g_{\mathrm{sphere}} \to 2$, respectively $g_{\mathrm{ball}} \to 14/5$, for $Rm_{\mathrm{b}} \to 0$. For $g = 2$ the spin and orbital precession are exactly in phase, whereas for $g = 1$ the spin turns once during two cyclotron revolutions.

To provide dynamical support we follow the scheme of chapter 7. One integrates (10.37), (10.38) and inserts in the Lorentz torque taking into account that the initial fields decay quickly. Then

$$I_{\mathrm{b}}\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\omega}(t) = \varepsilon\mu\boldsymbol{\omega}(t) \times \boldsymbol{B}_0 + \boldsymbol{N}_{\mathrm{self}}(t) \,, \tag{10.48}$$

where, after some rearrangement, the retarded torque simplifies to

$$\boldsymbol{N}_{\mathrm{self}}(t) = \int_0^t \mathrm{d}s \frac{2}{3} e^2 \int \mathrm{d}^3k |\nabla_{\boldsymbol{k}}\widehat{\varphi}|^2 \times \Big(-(\cos|\boldsymbol{k}|(t-s))\boldsymbol{\omega}(s) + \frac{1}{|\boldsymbol{k}|}(\sin|\boldsymbol{k}|(t-s))\boldsymbol{\omega}(t) \times \boldsymbol{\omega}(s) \Big) \,. \tag{10.49}$$

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Let us denote the solution to (10.48) by $\boldsymbol{\omega}^\varepsilon(t) = \boldsymbol{\omega}(\varepsilon t)$. We insert this ansatz in (10.49) and Taylor-expand. Then

$$\begin{aligned} \boldsymbol{N}^\varepsilon_{\mathrm{self}}(\varepsilon^{-1}t) &= \int_0^{\varepsilon^{-1}t} \mathrm{d}s \frac{2}{3} e^2 \int \mathrm{d}^3k |\nabla_{\boldsymbol{k}} \widehat{\varphi}|^2 \Big(-(\cos |\boldsymbol{k}|(\varepsilon^{-1}t - s)) \boldsymbol{\omega}(\varepsilon s) \\ &\quad + \frac{1}{|\boldsymbol{k}|} (\sin |\boldsymbol{k}|(\varepsilon^{-1}t - s)) \boldsymbol{\omega}(\varepsilon t) \times \boldsymbol{\omega}(\varepsilon s) \Big) \\ &\cong \int_0^{\varepsilon^{-1}t} \mathrm{d}s \frac{2}{3} e^2 \int \mathrm{d}^3k |\nabla_{\boldsymbol{k}} \widehat{\varphi}|^2 \Big(-(\cos |\boldsymbol{k}| s) \big(\boldsymbol{\omega}(t) - \varepsilon s \dot{\boldsymbol{\omega}}(t) \\ &\quad + \frac{1}{2} \varepsilon^2 s^2 \ddot{\boldsymbol{\omega}}(t) \big) + \frac{1}{|\boldsymbol{k}|} (\sin |\boldsymbol{k}| s) \boldsymbol{\omega}(t) \big(\boldsymbol{\omega}(t) \\ &\quad - \varepsilon s \dot{\boldsymbol{\omega}}(t) + \frac{1}{2} \varepsilon^2 s^2 \ddot{\boldsymbol{\omega}}(t) \big) \Big) . \end{aligned} \tag{10.50}$$

Let

$$I_p = \int_0^\infty \mathrm{d}t\, t^p \int \mathrm{d}^3k |\nabla_{\boldsymbol{k}} \widehat{\varphi}|^2 \frac{1}{|\boldsymbol{k}|} \sin |\boldsymbol{k}| t , \quad J_p = \int_0^\infty \mathrm{d}t\, t^p \int \mathrm{d}^3k |\nabla_{\boldsymbol{k}} \widehat{\varphi}|^2 \cos |\boldsymbol{k}| t . \tag{10.51}$$

Then, using the fact that $\widehat{\varphi}$ is radial,

$$\begin{aligned} I_0 &= \int \mathrm{d}^3k |\nabla_{\boldsymbol{k}} \widehat{\varphi}|^2 |\boldsymbol{k}|^{-2} , \quad I_1 = 0 , \\ I_2 &= \frac{1}{4\pi} \int \mathrm{d}^3x \int \mathrm{d}^3x' \varphi(\boldsymbol{x}) \varphi(\boldsymbol{x}') \boldsymbol{x} \cdot \boldsymbol{x}' |\boldsymbol{x} - \boldsymbol{x}'| = -\frac{1}{2\pi} \int \mathrm{d}^3k |\nabla_{\boldsymbol{k}} \widehat{\varphi}|^2 |\boldsymbol{k}|^{-4} , \end{aligned} \tag{10.52}$$

and

$$J_0 = 0, \quad J_p = -p I_{p-1}, \quad p = 1, 2, \ldots . \tag{10.53}$$

Therefore to order ε^2

$$\boldsymbol{N}^\varepsilon_{\mathrm{self}}(t) = -\varepsilon \frac{2}{3} e^2 I_0 \dot{\boldsymbol{\omega}}(t) + \varepsilon^2 \frac{1}{3} e^2 I_2 \boldsymbol{\omega}(t) \times \ddot{\boldsymbol{\omega}}(t) , \tag{10.54}$$

and inserted in (10.48)

$$I_{\mathrm{b}} \varepsilon \dot{\boldsymbol{\omega}}(t) = \varepsilon \mu \boldsymbol{\omega}(t) \times \boldsymbol{B}_0 - \varepsilon I_{\mathrm{f}} \dot{\boldsymbol{\omega}}(t) + \varepsilon^2 \frac{1}{3} e^2 I_2 \boldsymbol{\omega}(t) \times \ddot{\boldsymbol{\omega}}(t) , \tag{10.55}$$

where $I_{\mathrm{f}} = 2e^2 I_0/3$ in agreement with the static result (10.41).

Beyond the renormalization of I_{b} we have also obtained the radiation reaction $\boldsymbol{\omega}(t) \times \ddot{\boldsymbol{\omega}}(t)$. As for the translational degrees of freedom only the solution on the center manifold is of physical relevance. To compute the effective dynamics we

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regard (10.44) as the unperturbed dynamics and reinsert in (10.55). To be somewhat more general let us take $\boldsymbol{B}_0$ to be time dependent and varying on the slow time scale. One obtains

$$(I_b + I_f)\dot{\boldsymbol{\omega}} = \mu\boldsymbol{\omega} \times \boldsymbol{B}_0 + \varepsilon e^2\big(\mu I_2/3(I_b + I_f)\big)\big(\dot{\boldsymbol{\omega}}(\boldsymbol{\omega} \cdot \boldsymbol{B}_0) + (\boldsymbol{\omega} \times (\dot{\boldsymbol{B}}_0 \times \boldsymbol{\omega}))\big) . \tag{10.56}$$

Since $\boldsymbol{\omega}^2$ is conserved under (10.56), the radiation reaction only modifies the frequency of gyration to order ε. A second-order term like $\ddot{\boldsymbol{\omega}}$ would lead to friction in the effective equation. As can be seen from (10.53), its prefactor J_2 vanishes and radiation damping appears only at order ε^4 through $I_4\dddot{\boldsymbol{\omega}}$.

Notes and references

Section 10.1

BMT is an acronym for Bargmann, Michel and Telegdi (1959). The BMT equation is explained in Jackson (1999). Bailey and Picasso (1970) is an informative article on how the BMT equation is used in the analysis of the high-precision measurements of the electron and muon g-factor. The BMT equation with $g = 2$ is the semiclassical limit of the Dirac equation (Rubinow and Keller 1963; Bolte and Keppeler 1999; Spohn 2000b; Panati *et al.* 2002a). Appel and Kiessling (2001) compute the effective parameters for a charge distribution concentrated on a sphere.

Just as for translational degrees of freedom, one way to guess the effective spin dynamics is to impose Lorentz invariance. In addition, one could require that the equations of motion come from a Lagrangian action. In full generality, including an electric dipole moment, this program is carried out by Bhabha (1939), Bhabha and Corben (1941) with earlier work by Frenkel (1926). Alternative approaches are compared in Corben (1961) and Nyborg (1962). Concise summaries are Barut (1964), who discusses also how the BMT equation fits into the general scheme, Teitelbom *et al.* (1980), and Rohrlich (1990). A more microscopic approach would be to carry out the adiabatic limit for the Lorentz model of section 2.5. In Nodvik's version of the model such an expansion is pushed to the order where translational and rotational degrees of freedom couple (Nodvik 1964).

The Lorentz model simplifies if initial data are assumed such that the particle moves at constant velocity. Then translational and rotational degrees of freedom decouple. Appel and Kiessling (2002) study the existence of solutions and their long-time limit. In the adiabatic limit, compare with section 10.3, the angular momentum responds to an external torque through the effective gyromagnetic ratio of (10.9).

Section 10.2

The nonrelativistic model of a rotating charge is introduced by Abraham (1903) and studied by Herglotz (1903), Schwarzschild (1903), and Thomas (1927). Schwarzschild (1903) notes that a stationary solution exists only if $\boldsymbol{\omega}$ is either parallel or orthogonal to $\boldsymbol{v}$. Kiessling (1999) remarks that the standard form of the total angular momentum is conserved only if the inner rotation of the charged particle is included.

Section 10.3

Grandy and Aghazadeh (1982) compute the gyromagnetic ratio to order e^2. The validity of the equations of motion (10.56) is proved in Imaikin *et al.* (2004).