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DEGREE-FREE BOUNDS FOR DEPENDENCE RELATIONS

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Abstract

Let $\alpha_1, \ldots, \alpha_n$ be non-zero algebraic numbers and let $l_1(\alpha_1), \ldots, l_n(\alpha_n)$ denote arbitrary fixed values of the logarithms of $\alpha_1, \ldots, \alpha_n$, respectively. Given that $l_1(\alpha_1), \ldots, l_n(\alpha_n)$ are linearly dependent over $\mathbf{Q}$, the existence of a non-trivial dependence relation between these numbers with integer coefficients of low absolute values can be proved. Existing results of this kind give bounds for the absolute values of the coefficients which are expressions in the degree $D = [\mathbf{Q}(\alpha_1, \ldots, \alpha_n): \mathbf{Q}]$, the heights of $\alpha_1, \ldots, \alpha_n$ and the magnitudes of the logarithms involved.

In the present paper it is shown by means of Baker's method that one can suppress the dependence on D completely–at the price of the occurrence of more branches of logarithms in the bounds. An application of this feature is given.

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1. Introduction

Let $\alpha_1, \ldots, \alpha_n \in \mathbf{C}$ be non-zero algebraic numbers; let $l_1(\alpha_1), \ldots, l_n(\alpha_n)$ denote arbitrary fixed values of the logarithms of $\alpha_1, \ldots, \alpha_n$, respectively. Put $D = [\mathbf{Q}(\alpha_1, \ldots, \alpha_n): \mathbf{Q}]$ and let $H(\alpha_\nu)$ denote the classical height of α_ν, that is to say, the maximum of the absolute values of the coefficients in its minimal polynomial in $\mathbf{Z}[X]$. We consider the following problem. Let it be given that $l_1(\alpha_1), \ldots, l_n(\alpha_n)$ are linearly dependent over $\mathbf{Q}$, that is that there exist $q_1, \ldots, q_n \in \mathbf{Z}$, not all zero, satisfying

$$(1) \qquad q_1 l_1(\alpha_1) + \cdots + q_n l_n(\alpha_n) = 0.$$

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Prove that the q_ν may be chosen such that their absolute values are relatively small in terms of n, D, the $H(\alpha_\nu)$ and the $|l_\nu(\alpha_\nu)|$.

Results of this kind can be found in Baker (1975), Bijlsma (1978) p. 53 and Waldschmidt (1980). The related problem of multiplicative dependence was treated by Van der Poorten and Loxton (1977). They assumed the existence of $q_1, \ldots, q_n \in \mathbf{Z}$, not all zero, satisfying

$$\alpha_1^{q_1} \cdots \alpha_n^{q_n} = 1 \tag{2}$$

and showed that relatively small q_ν exist with the same properties.

In all quoted results, the dependence on D (if already given explicitly) is expressed as a factor greater than D^{2n}, when D is large, in the occurring bounds. We remark, that a refinement of the method of Bijlsma (1978) enables one to reach a factor about D^n. On the other hand, taking $n = 2$, $\alpha_1 = 2$, $\alpha_2 = 2^{1/D}$ and principal values of the logarithms shows that the dependence on D cannot be suppressed completely. The purpose of this paper is nevertheless to give bounds which have no explicit dependence on D.

As a basis for the technique that we shall use, we state the following lemma:

LEMMA 1. *For $\alpha_1, \ldots, \alpha_n \in \mathbf{C}$ and $x_1, \ldots, x_n \in \mathbf{Z}$, not all zero, satisfying $(x_1, \ldots, x_n) = 1$, the following statements are equivalent:*

i) *there exist values $l_1(\alpha_1), \ldots, l_n(\alpha_n)$ of the logarithms of $\alpha_1, \ldots, \alpha_n$ respectively such that*

$$x_1 l_1(\alpha_1) + \cdots + x_n l_n(\alpha_n) = 0;$$

ii) $\alpha_1^{x_1} \cdots \alpha_n^{x_n} = 1$.

PROOF. Clearly i) implies ii). Now assume that ii) holds; this implies that there is a rational integer k satisfying

$$x_1 \operatorname{Log} \alpha_1 + \cdots + x_n \operatorname{Log} \alpha_n = 2k\pi i.$$

As $(x_1, \ldots, x_n) = 1$, there exist $k_1, \ldots, k_n \in \mathbf{Z}$ satisfying $k_1 x_1 + \cdots + k_n x_n = -k$. If we define $l_\nu(\alpha_\nu) := \operatorname{Log} \alpha_\nu + 2k_\nu \pi i$ for $\nu = 1, \ldots, n$, we have

$$\sum_{\nu=1}^{n} x_\nu l_\nu(\alpha_\nu) = \sum_{\nu=1}^{n} x_\nu \operatorname{Log} \alpha_\nu + 2\pi i \sum_{\nu=1}^{n} k_\nu x_\nu = 2k\pi i - 2k\pi i = 0.$$

Multiplicative dependence, once established, is invariant under conjugation; it is a consequence of Lemma 1 that the same is true for linear dependence of logarithms. This is formulated in the following property:

Let $b_1, \ldots, b_n \in \mathbf{Z}$, not all zero, such that

$$b_1 l_1(\alpha_1) + \cdots + b_n l_n(\alpha_n) = 0.$$

Let σ be a $\mathbf{Q}$*-isomorphism of* $\mathbf{Q}(\alpha_1, \ldots, \alpha_n)$ *into* $\mathbf{C}$. *Then there also exist values* $l_1^*(\sigma(\alpha_1)), \ldots, l_n^*(\sigma(\alpha_n))$ *of the logarithms of* $\sigma(\alpha_1), \ldots, \sigma(\alpha_n)$ *respectively such that*

$$b_1 l_1^*(\sigma(\alpha_1)) + \cdots + b_n l_n^*(\sigma(\alpha_n)) = 0.$$

Indeed, put $x_\nu := b_\nu/(b_1, \ldots, b_n)$ for $\nu = 1, \ldots, n$. Then $\alpha_1, \ldots, \alpha_n$ and $x_1, \ldots, x_n$ possess property i) in Lemma 1, and thereby property ii); from this we deduce that $\sigma(\alpha_1), \ldots, \sigma(\alpha_n)$ and $x_1, \ldots, x_n$ possess property ii) and thereby property i). The statement follows upon multiplication by $(b_1, \ldots, b_n)$.

Thus, if there exists a linear dependence relation between logarithms of $\alpha_1, \ldots, \alpha_n$, we have in fact D such relations, all with the same coefficients, between logarithms of conjugates of $\alpha_1, \ldots, \alpha_n$. Since each of these relations can be considered as a vanishing linear form, we can apply Baker's method simultaneously to all these forms, with one auxiliary function for each form. As in Bijlsma (1978) we shall construct the desired low dependence relation from the frequencies, used in the auxiliary functions. The final bound will be independent of D, but will contain the magnitudes of all logarithms of all conjugates involved.

It should be noted, that the above procedure cannot be used in order to obtain similar theorems for multiplicative dependence instead of linear dependence of logarithms. For, if for example $n = 3$, $\alpha_1 = 2$, $\alpha_2 = 3$, $\alpha_3 = -1/6$, the numbers $\alpha_1, \alpha_2, \alpha_3$ are multiplicatively dependent because $\alpha_1^2\alpha_2^2\alpha_3^2 = 1$; but no product with relatively prime exponents can equal 1, and thus, by Lemma 1, no linear combination of logarithms of $\alpha_1, \alpha_2, \alpha_3$ can be zero.

The sequel of this paper consists of three sections. In the next section we state our main theorem and we formulate or give reference to the lemmas we need. We remark, that Lemma 3 is of some independent interest. In section 3 we prove the theorem; the final section contains an application.

2. The main theorem

We shall use the absolute logarithmic height $h(\alpha)$ of an algebraic number of degree d, defined by

$$h(\alpha) := d^{-1} \log M(\alpha)$$

where

$$M(\alpha) := a \prod_{\delta=1}^{d} \max(|\sigma_\delta(\alpha)|, 1);$$

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here, a is the leading coefficient of the minimal polynomial of α and $\sigma_1(\alpha), \ldots, \sigma_d(\alpha)$ denote the conjugates of α. For a detailed description of the properties of the absolute logarithmic height we refer to Waldschmidt (1980); here we only recall that

$$(3) \qquad h(\alpha) \leqslant d^{-1}\{\log H(\alpha) + \log d\} \leqslant \log H(\alpha) + 1.$$

Our theorem can be formulated as follows:

THEOREM. *Let* $\alpha_1, \ldots, \alpha_n \in \mathbf{C}$ *be non-zero algebraic numbers. Put* $\mathbf{K} := \mathbf{Q}(\alpha_1, \ldots, \alpha_n)$, $D := [\mathbf{K} : \mathbf{Q}]$. *Let* $\sigma_1, \ldots, \sigma_D$ *denote the* $\mathbf{Q}$*-isomorphisms of* $\mathbf{K}$ *into* $\mathbf{C}$, *where* σ_1 *is the identity; let* $l_{\mu,\nu}(\sigma_\mu(\alpha_\nu))$ *be an arbitrary fixed value of the logarithm of* $\sigma_\mu(\alpha_\nu)$ *for* $\mu = 1, \ldots, D$ *and* $\nu = 1, \ldots, n$, *with* $l_{1,\nu}(\sigma_1(\alpha_\nu)) \neq 0$, $\nu = 1, \ldots, n$.

Suppose

$$A_\nu \geqslant \max\Big(h(\alpha_\nu), \max_{\mu=1,\ldots,D} |l_{\mu,\nu}(\sigma_\mu(\alpha_\nu))|, 1\Big), \qquad \nu = 1, \ldots, n.$$

Define

$$\Omega := \prod_{\nu=1}^{n} A_\nu;$$
$$\Omega_\nu := \Omega / A_\nu, \qquad \nu = 1, \ldots, n;$$
$$\Omega_0 := \max(\Omega, e).$$

Let E *be a number satisfying*

$$(4) \qquad e \leqslant E \leqslant \min_{\nu=1,\ldots,n} \min\left(\exp(A_\nu), \frac{eA_\nu}{\max_{\mu=1,\ldots,D}|l_{\mu,\nu}(\sigma_\mu(\alpha_\nu))|}\right).$$

If there exist $b_1, \ldots, b_n \in \mathbf{Z}$, *not all zero, such that*

$$b_1 l_{\mu,1}(\sigma_\mu(\alpha_1)) + \cdots + b_n l_{\mu,n}(\sigma_\mu(\alpha_n)) = 0, \qquad \mu = 1, \ldots, D,$$

then there exist $q_1, \ldots, q_n \in \mathbf{Z}$, *not all zero, such that*

$$q_1 l_{1,1}(\alpha_1) + \cdots + q_n l_{1,n}(\alpha_n) = 0$$

and

$$(5) \qquad |q_\nu| \leqslant C\Omega_\nu \log^n \Omega_0 \cdot \log(\Omega_0 E) \log^{-n} E, \qquad \nu = 1, \ldots, n,$$

where C *is an effectively computable number that depends only on* n.

REMARK 1. When $A \geqslant \max(h(\alpha_\nu), 1)$, $\nu = 1, \ldots, n$, and $L \geqslant |l_{\mu,\nu}(\sigma_\mu(\alpha_\nu))|$, $\mu = 1, \ldots, D$, $\nu = 1, \ldots, n$, then we can use $A_\nu = \max(A, L)$, $\nu = 1, \ldots, n$, and

$$E = \min\left(\max(e^A, e^L), \max\left(\frac{eA}{L}, e\right)\right).$$

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With this choice,

$$\max(c_1^{-1}A, e) \leqslant E \leqslant c_2 A$$

and thus

$$c_3^{-1}\max(\log A, 1) \leqslant \log E \leqslant c_3 \max(\log A, 1)$$

where c_1, c_2, c_3 are numbers, greater than 1, depending only on L. Now (5) takes the form

$$|q_\nu| \leqslant C_1 A^{n-1}\max(\log A, 1), \qquad \nu = 1, \ldots, n,$$

where C_1 is an effectively computable number that depends only on n and L.

Remark 2. Using inequality (3), one can easily reformulate the theorem as well as its consequence indicated in the first remark, in terms of the classical height.

Remark 3. When α_ν is an algebraic integer, we have

$$\max_{\mu=1,\ldots,D}|l_{\mu,\nu}(\sigma_\mu(\alpha_\nu))| \geqslant \max_{\mu=1,\ldots,D}\big|\log|\sigma_\mu(\alpha_\nu)|\big| \geqslant D^{-1}\sum_{\mu=1}^{D}\big|\log|\sigma_\mu(\alpha_\nu)|\big|$$
$$\geqslant D^{-1}\sum_{\mu=1}^{D}\max(\log|\sigma_\mu(\alpha_\nu)|, 0) = h(\alpha_\nu),$$

so then

$$\max\Big(h(\alpha_\nu), \max_{\mu=1,\ldots,D}|l_{\mu,\nu}(\sigma_\mu(\alpha_\nu))|, 1\Big) = \max\Big(\max_{\mu=1,\ldots,D}|l_{\mu,\nu}(\sigma_\mu(\alpha_\nu))|, 1\Big).$$

The auxiliary results we shall use are either very well known or contained in Waldschmidt (1980), except for two that we formulate explicitly:

Lemma 2. *Suppose $\alpha_1, \ldots, \alpha_n$ algebraic and put $D := [\mathbf{Q}(\alpha_1, \ldots, \alpha_n): \mathbf{Q}]$. Then there exists a primitive element α for $\mathbf{Q}(\alpha_1, \ldots, \alpha_n)$ of the form*

$$\alpha = e_1\alpha_1 + \cdots + e_n\alpha_n,$$

where $e_1, \ldots, e_n$ are non-negative integers satisfying $e_1 + \cdots + e_n \leqslant D^2$.

Proof. From Mignotte and Waldschmidt (1977), Lemme 3, by induction.

Lemma 3. *Let $\xi_1, \ldots, \xi_n$ be algebraic numbers. Put $\mathbf{K} := \mathbf{Q}(\xi_1, \ldots, \xi_n)$, $D := [\mathbf{K}:\mathbf{Q}]$. Let $\sigma_1, \ldots, \sigma_D$ denote the $\mathbf{Q}$-isomorphisms of $\mathbf{K}$ into $\mathbf{C}$. Let $P \in \mathbf{Z}[X_1, \ldots, X_n]$ have degree at most K_ν in X_ν for $\nu = 1, \ldots, n$. Then either $P(\xi_1, \ldots, \xi_n) = 0$ or*

$$\prod_{\mu=1}^{D}|\sigma_\mu(P(\xi_1, \ldots, \xi_n))| \geqslant \exp\Big(-D\sum_{\nu=1}^{n}K_\nu h(\xi_\nu)\Big).$$

PROOF. Write

$$P = \sum_{k_1=0}^{K_1} \cdots \sum_{k_n=0}^{K_n} p(k_1, \dots, k_n) X_1^{k_1} \cdots X_n^{k_n};$$

then

$$\begin{aligned} \eta &:= \prod_{\mu=1}^{D} \sigma_\mu(P(\xi_1, \dots, \xi_n)) \\ &= \prod_{\mu=1}^{D} \sum_{k_1=0}^{K_1} \cdots \sum_{k_n=0}^{K_n} p(k_1, \dots, k_n) \sigma_\mu^{k_1}(\xi_1) \cdots \sigma_\mu^{k_n}(\xi_n). \end{aligned}$$

Put $d_\nu := \mathrm{dg}(\xi_\nu)$, $\nu = 1, \dots, n$. Then ξ_ν has exactly d_ν different conjugates, each of which occurs D/d_ν times among $\sigma_1(\xi_\nu), \dots, \sigma_D(\xi_\nu)$; thus each of the different conjugates of ξ_ν occurs in the above expression for η to a power at most DK_ν/d_ν. Let a_ν denote the leading coefficient of the minimal polynomial of ξ_ν; then Hilfssatz 17 of Schneider (1957) implies that

$$\mathrm{den}(\eta) \leqslant \prod_{\nu=1}^{n} a_\nu^{DK_\nu/d_\nu}.$$

However, as η is clearly invariant under conjugation and thus a rational number, we conclude that either $\eta = 0$ or

$$|\eta| \geqslant \prod_{\nu=1}^{n} a_\nu^{-DK_\nu/d_\nu}.$$

The lemma now follows from the trivial estimate

$$a_\nu \leqslant \exp(d_\nu h(\xi_\nu)), \qquad \nu = 1, \dots, n.$$

3. Proof of the main theorem

For abbreviation, we shall use

$$U := \Omega \log^n \Omega_0 \cdot \log^2 (\Omega_0 E) \log(D\Omega_0 B) \log^{-n-1} E$$

and

$$\Delta(z; R) = (z+1) \cdots (z+R)/R!, \qquad R \in \mathbf{N};$$

further, $\nu(R)$ denotes the least common multiple of $1, \dots, R$. By $c_4, c_5, \dots,$ we shall denote effectively computable real numbers greater than 1 that depend only on n. By x we shall denote some real number greater than 1; additional restrictions on the choice of x will be formulated at later stages of the proof.

I. Suppose that $b_n \neq 0$ and define $B := \max(|b_1|, \ldots, |b_n|)$. Also define

$$K_\nu := \left[x^{4n+4}\Omega_\nu \log^n \Omega_0 \cdot \log(\Omega_0 E) \log^{-n} E\right], \qquad \nu = 1, \ldots, n;$$

$$M := \left[x^{4n+6}\Omega \log^n \Omega_0 \cdot \log(\Omega_0 E) \log^{-n-1} E\right]\left[D \log(D\Omega_0 B)\right];$$

$$R := \left[D \log(D\Omega_0 B)\right];$$

$$S := \left[x^4 D \log(\Omega_0 E) \log(D\Omega_0 B) \log^{-1} E\right];$$

$$T := \left[x^{4n+8}\Omega \log^{n+1} \Omega_0 \cdot \log(\Omega_0 E) \log^{-n-1} E\right].$$

For $\nu = 1, \ldots, n$ we have $\log E \leqslant A_\nu$, so

$$\log^n E \leqslant \log^{n-1} E \cdot \log(\Omega_0 E) \leqslant \Omega_\nu \log(\Omega_0 E) \leqslant \Omega_\nu \log^n \Omega_0 \cdot \log(\Omega_0 E),$$

which implies $K_\nu \in \mathbf{N}$. By similar ways of reasoning, M/R, S and T are positive integers and $S/R > 1$ when x is more than some absolute constant.

The M functions

$$(\Delta(z + r_1; R))^{r_2}, \qquad r_1 = 0, \ldots, R-1, \quad r_2 = 1, \ldots, M/R,$$

will, in some arbitrary order, be denoted by $\Delta_0(z), \ldots, \Delta_{M-1}(z)$. We introduce the auxiliary functions

$$F_\mu(z) := \sum_{k_1=0}^{K_1-1} \cdots \sum_{k_n=0}^{K_n-1} \sum_{m=0}^{M-1} \sum_{d=0}^{D-1} p(\mathbf{k}, m, d)\sigma_\mu^d(\alpha)\Delta_m(z)$$
$$\times \exp\big((k_1 l_{\mu,1}(\sigma_\mu(\alpha_1)) + \cdots + k_n l_{\mu,n}(\sigma_\mu(\alpha_n)))z\big), \qquad \mu = 1, \ldots, D,$$

where α is a primitive element for $\mathbf{K}$ satisfying the conditions of Lemma 2, and where the $p(\mathbf{k}, m, d) = p(k_1, \ldots, k_n, m, d)$ are rational integers to be determined later.

As

$$\sum_{\nu=1}^{n} k_\nu l_{\mu,\nu}(\sigma_\mu(\alpha_\nu)) = b_n^{-1} \sum_{\nu=1}^{n-1} \big((k_\nu b_n - k_n b_\nu) l_{\mu,\nu}(\sigma_\mu(\alpha_\nu))\big)$$

we have

$$F_\mu^{(t)}(z) = \sum_{|\tau|=t} \frac{t!}{\tau_0! \cdots \tau_{n-1}!} l_{\mu,1}^{\tau_1}(\sigma_\mu(\alpha_1)) \cdots l_{\mu,n-1}^{\tau_{n-1}}(\sigma_\mu(\alpha_{n-1})) F_{\mu,\tau}(z), \tag{6}$$

where the summation ranges over all n-tuples $\tau = (\tau_0, \ldots, \tau_{n-1})$ of non-negative integers satisfying $|\tau| := \tau_0 + \cdots + \tau_{n-1} = t$, and where

$$F_{\mu,\tau}(z) := \sum_{k_1=0}^{K_1-1} \cdots \sum_{k_n=0}^{K_n-1} \sum_{m=0}^{D-1} \sum_{d=0}^{D-1} p(\mathbf{k}, m, d)\sigma_\mu^d(\alpha)\Delta_m^{(\tau_0)}(z) b_n^{-\tau_1 - \cdots - \tau_{n-1}}$$
$$\times \prod_{\nu=1}^{n-1} (k_\nu b_n - k_n b_\nu)^{\tau_\nu} \times \exp\left\{\sum_{\nu=1}^{n} k_\nu l_{\mu,\nu}(\sigma_\mu(\alpha_\nu)) z\right\}.$$

Now consider the system of linear equations

$$(\nu(R))^{\tau_0} b_n^{\tau_1+\cdots+\tau_{n-1}} F_{1,\tau}(s) = 0, \qquad s = 0, \ldots, S-1, |\tau| \leqslant T-1,$$

in the $K_1 \cdots K_n MD$ unknowns $p(\mathbf{k}, m, d)$. The coefficients of these linear equations are polynomial expressions in $\alpha, \alpha_1, \ldots, \alpha_n$ with rational integer coefficients; see Lemma 2.4 of Waldschmidt (1980). We can bound the lengths of the polynomials involved by

$$3^{R\tau_0}\tau_0! \left(\frac{s+2R}{R}\right)^M (2e)^M B^{\tau_1+\cdots+\tau_{n-1}} \prod_{\nu=1}^{n-1} (K_\nu + K_n)^{\tau_\nu}$$
$$\leqslant \exp\bigl(c_4 x^{4n+9} DU \log \Omega_0 / \log(\Omega_0 E)\bigr).$$

The degrees in α of these polynomials are less than D and the degrees in each α_ν are less than $K_\nu S \leqslant x^{4n+8}DU/A_\nu$. The absolute logarithmic height of α_ν is at most A_ν and the absolute logarithmic height of α is, by Lemma 2.7 of Waldschmidt (1980) and by Lemma 2, at most

$$2n(n+1)\log D + \log n + \sum_{\nu=1}^{n} h(\alpha_\nu) \leqslant c_5 \log D + c_5 \max(A_1, \ldots, A_n)$$
$$\leqslant 2c_5 \Omega \log^2(\Omega_0 E) \log(D\Omega_0 B) \log^{-n-1} E,$$

the last inequality because $\log^{-n+1} E \geqslant \Omega^{-1}\max(A_1, \ldots, A_n)$. The total number of equations is less than

$$ST^n \leqslant x^{4n^2+8n+4} D\,\Omega^n \log^{n^2+n}\Omega_0 \cdot \log^{n+1}(\Omega_0 E) \log(D\Omega_0 B) \log^{-n^2-n-1} E,$$

while the number of unknowns is $K_1 \cdots K_n MD \geqslant xDST^n$ when x exceeds a certain absolute constant. According to Lemma 2.1 of Waldschmidt (1980), there is a non-trivial choice for the $p(\mathbf{k}, m, d)$ such that $F_{1,\tau}(s) = 0$, $s = 0, \ldots, S-1$, $|\tau| \leqslant T-1$, while $P := \max|p(\mathbf{k}, m, d)|$ is at most $\exp(c_5 x^{4n+9} DU)$. Note that the numbers $F_{2,\tau}(s), \ldots, F_{D,\tau}(s)$ are conjugate to $F_{1,\tau}(s)$ for each τ and s, so that the same choice of $p(\mathbf{k}, m, d)$ gives us

(7) $\qquad F_{\mu,\tau}(s) = 0, \qquad s = 0, \ldots, S-1, |\tau| \leqslant T-1, \mu = 1, \ldots, D.$

II. Define $J := [\log(x^3 T^{n-1})]$. For $j = 0, \ldots, J$ we put $S_j := [e^j S]$, $T_j := T - j[T/2J]$. Then, by our special choice of the $p(\mathbf{k}, m, d)$, we have for $j = 0, \ldots, J$

(8) $\qquad F_{\mu,\tau}(s) = 0, \qquad s = 0, \ldots, S_j - 1, |\tau| \leqslant T_j - 1, \mu = 1, \ldots, D.$

This is proved by induction; for $j = 0$ the assertion is precisely (7). Now suppose that (8) holds for some $j \leqslant J-1$. Then, if $|\tau| \leqslant T_{j+1} - 1$ and $r \leqslant [T/2J]$, we have

$$F_{\mu,\tau}^{(r)} = \sum_{|\rho|=r} \frac{r!}{\rho_0! \cdots \rho_{n-1}!} l_{\mu,1}^{\rho_1}(\sigma_\mu(\alpha_1)) \cdots l_{\mu,n-1}^{\rho_{n-1}}(\sigma_\mu(\alpha_{n-1})) F_{\mu,\tau+\rho}(z),$$

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where the summation ranges over all n-tuples $\boldsymbol{\rho} = (\rho_0, \ldots, \rho_{n-1})$ of nonnegative integers satisfying $|\boldsymbol{\rho}| := \rho_0 + \cdots + \rho_{n-1} = r$, and where $\boldsymbol{\tau} + \boldsymbol{\rho} := (\tau_0 + \rho_0, \ldots, \tau_{n-1} + \rho_{n-1})$. Clearly $|\boldsymbol{\tau} + \boldsymbol{\rho}| \leqslant T_j - 1$ and thus, by the induction hypothesis,

$$F_{\mu,\boldsymbol{\tau}}^{(r)}(s) = 0, \qquad s = 0, \ldots, S_j - 1, |\boldsymbol{\tau}| \leqslant T_{j+1} - 1, \mu = 1, \ldots, D, r \leqslant [T/2J].$$

By Lemma 7 of Cijsouw (1974),

$$\max_{|z| \leqslant S_{j+1}} |F_{\mu,\boldsymbol{\tau}}(z)| \leqslant 2 \max_{|z| \leqslant ES_{j+1}} |F_{\mu,\boldsymbol{\tau}}(z)| \cdot \left(\frac{2}{E}\right)^{S_j[T/2J]}. \tag{9}$$

If $|\boldsymbol{\tau}| \leqslant T_{j+1} - 1$, some computations show that

$$\max_{|z| \leqslant ES_{j+1}} |F_{\mu,\boldsymbol{\tau}}(z)| \leqslant \exp\left(c_6 e^j x^{4n+9} DU\right)$$

and

$$\left(\frac{2}{E}\right)^{S_j[T/2J]} \leqslant \exp\left(-c_7^{-1} e^j x^{4n+11} DU\right);$$

therefore, if we choose $x > c_6 c_7$, substitution in (9) gives

$$\max_{|z| \leqslant S_{j+1}} |F_{\mu,\boldsymbol{\tau}}(z)| \leqslant \exp\left(-c_8^{-1} e^j x^{4n+10} DU\right)$$

thus in particular

$$|F_{\mu,\boldsymbol{\tau}}(s)| \leqslant \exp\left(-c_8^{-1} e^j x^{4n+10} DU\right),$$
$$s = 0, \ldots, S_{j+1} - 1, |\boldsymbol{\tau}| \leqslant T_{j+1} - 1, \mu = 1, \ldots, D.$$

Therefore

$$\prod_{\mu=1}^{D} |(\nu(R))^{\tau_0} b_n^{\tau_1 + \cdots + \tau_{n-1}} F_{\mu,\boldsymbol{\tau}}(s)| \leqslant \exp\left(-c_9^{-1} e^j x^{4n+10} D^2 U\right), \tag{10}$$
$$s = 0, \ldots, S_{j+1} - 1, |\boldsymbol{\tau}| \leqslant T_{j+1} - 1.$$

Now for these values of s and $\boldsymbol{\tau}$, the expression $(\nu(R))^{\tau_0} b_n^{\tau_1 + \cdots + \tau_{n-1}} F_{1,\boldsymbol{\tau}}(s)$ is a polynomial in $\alpha, \alpha_1, \ldots, \alpha_n$ with rational integer coefficients; the degree of this polynomial with respect to α is at most D and its degree in α_ν is at most $K_\nu S_{j+1}$. Thus, by Lemma 3 above, either $F_{1,\boldsymbol{\tau}}(s) = \cdots = F_{D,\boldsymbol{\tau}}(s) = 0$ or

$$\prod_{\mu=1}^{D} |(\nu(R))^{\tau_0} b_n^{\tau_1 + \cdots + \tau_{n-1}} F_{\mu,\boldsymbol{\tau}}(s)| \geqslant \exp\left(-c_{10} e^j x^{4n+8} D^2 U\right). \tag{11}$$

Combining (10) and (11), and choosing $x^2 > c_9 c_{10}$, gives $F_{1,\boldsymbol{\tau}}(s) = \cdots = F_{D,\boldsymbol{\tau}}(s) = 0$ for $s = 0, \ldots, S_{j+1} - 1$, $|\boldsymbol{\tau}| \leqslant T_{j+1} - 1$. This completes the proof of (8).

III. Taking $j = J$, $\mu = 1$ in (8) yields

$$F_{1,\boldsymbol{\tau}}(s) = 0, \qquad s = 0, \ldots, S_J - 1, |\boldsymbol{\tau}| \leqslant T_J - 1.$$

Substitution in (6) shows that

$$F_1^{(t)}(s) = 0, \qquad s = 0, \ldots, S_J - 1, t = 0, \ldots, T_J - 1.$$

Thus the number of zeros of F_1 in $W := \{z \in \mathbf{C}: |z| \leq S_J - 1\}$ is at least

$$\begin{aligned} S_J T_J &\geq c_{11}^{-1} x^3 S T^n \\ &\geq c_{12}^{-1} x^{4n^2+8n+7} D\Omega^n \log^{n^2+n}\Omega_0 \cdot \log^{n+1}(\Omega_0 E)\log(D\Omega_0 B)\log^{-n^2-n-1}E. \end{aligned}$$

However, if F_1 is not identically zero, the corollary of Theorem 1 of Tijdeman (1971) implies that the number of zeros of F_1 in W does not exceed

$$\begin{aligned} &3K_1 \cdots K_n M + 4S_J(A_1K_1 + \cdots + A_nK_n)/E \\ &\quad \leq c_{13} x^{4n^2+8n+6} D\Omega^n \log^{n^2+n}\Omega_0 \cdot \log^{n+1}(\Omega_0 E)\log(D\Omega_0 B)\log^{-n^2-n-1}E. \end{aligned}$$

In this case, comparison of the two estimates for the number of zeros of F_1 in W yields a contradiction if we choose $x > c_{12}c_{13}$. Thus F_1 is identically zero. As the $p(\mathbf{k}, m, d)$ are not all zero, the polynomials $\Delta_m(z)$ are linearly independent and α has degree D, it follows that two of the frequencies of F_1 must be equal. This shows that there exist two non-identical n-tuples $(k_1, \ldots, k_n)$ and $(k'_1, \ldots, k'_n)$ of non-negative integers that satisfy

$$k_1 l_{1,1}(\alpha_1) + \cdots + k_n l_{1,n}(\alpha_n) = k'_1 l_{1,1}(\alpha_1) + \cdots + k'_n l_{1,n}(\alpha_n)$$

while

$$\max(k_\nu, k'_\nu) \leq K_\nu - 1 \leq x^{4n+4}\Omega_\nu \log^n \Omega_0 \cdot \log(\Omega_0 E)\log^{-n}E, \qquad \nu = 1, \ldots, n.$$

Taking $q_\nu := k_\nu - k'_\nu$ for $\nu = 1, \ldots, n$ proves the theorem with $C = x^{4n+4}$.

4. An application

Let $\alpha \neq 0, 1$ be an element of a number field K. It is known, that bounds can be given for the exponents q for which $\alpha = \zeta\beta^q$ where $\zeta, \beta \in K$ and ζ is a root of unity; see for example Schinzel (1978), Lemma 1. We propose to extend this kind of bounds to products $\alpha_1^{q_1} \cdot \cdots \cdot \alpha_n^{q_n}$ instead of α, in the case of a totally real field K.

COROLLARY. *Let K be a totally real field of degree D; let $\sigma_1, \ldots, \sigma_D$ denote the $\mathbf{Q}$-isomorphisms of K into $\mathbf{R}$. Let $\alpha_1 \cdots \alpha_n$ be multiplicatively independent elements of K and choose $R \geq 1$ such that $e^{-R} \leq |\sigma_\mu(\alpha_\nu)| \leq e^R$ for $\mu = 1, \ldots, D$ and $\nu = 1, \ldots, n$. When $q_1, \ldots, q_n \in \mathbf{Z}$, not all zero, and $\beta \in K$ such that*

(12) $$\alpha_1^{q_1} \cdots \alpha_n^{q_n} = \beta^q, \qquad (q_1, \ldots, q_n, q) = 1,$$

then

(13) $$|q| < C_2 A^n \log^{n+2}(A + Q)\log^{-n-1}(eA/R),$$

where

$$A = \max\Big(\max_{\nu=1,\ldots,n} h(\alpha_\nu), R, 1\Big)$$

and $Q = \max(|q_1|, \ldots, |q_n|)$. *Here* C_2 *is an effectively computable number, depending only on* n.

PROOF. By c_{14}, c_{15} we shall denote effectively computable real numbers that depend only on n. Suppose that there exist numbers $\beta \in K$ and $q \in \mathbf{Z}$ such that (12) holds and (13) does not hold. We define $A_\nu = A$ for $\nu = 1, \ldots, n$ and $A_{n+1} = nQA$. Further, $\Omega_0 \leqslant enA^{n+1}Q$ and $E = eA/R$ will be used; note that $\exp(A) \geqslant eA \geqslant eA/R$ and $\exp(nQA) \geqslant enA \geqslant eA/R$. It follows from the theorem that there exist numbers $r_1, \ldots, r_n$, $r \in \mathbf{Z}$, not all zero (so in particular $r \neq 0$), such that

$$\alpha_1^{r_1} \cdots \alpha_n^{r_n} = \beta^r \tag{14}$$

and

$$\begin{aligned} |r| &\leqslant c_{14}A^n \log^{n+1}(enA^{n+1}Q)\log(e^2nA^{n+2}Q/R)\log^{-n-1}(eA/R) \\ &\leqslant c_{14}A^n \log^{n+2}(e^2nA^{n+2}Q)\log^{-n-1}(eA/R) \\ &\leqslant c_{15}A^n \log^{n+2}(A+Q)\log^{-n-1}(eA/R). \end{aligned}$$

For $C_2 > c_{14}$ this implies $|r| < |q|$. From (12) and (14) we can eliminate β, obtaining

$$\alpha_1^{q_1r - r_1q} \cdots \alpha_n^{q_nr - r_nq} = 1.$$

By the multiplicative independence of $\alpha_1, \ldots, \alpha_n$, we get $q_\nu r - r_\nu q = 0$ for $\nu = 1, \ldots, n$. Thus, q is a divisor of $(q_1, \ldots, q_n)r$; since $(q_1, \ldots, q_n, q) = 1$ we have q/r, contradicting $|r| < |q|$.

It should be noted, that the $h(\alpha_\nu)$ are at most $B := \max_{\nu=1,\ldots,n} \log H(\alpha_\nu) + 1$, and that R can be chosen as B. So (13) implies

$$|q| < C_3B^n \log^{n+2}(B+Q)$$

where C_3 depends only on n.

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