

ON HOLOMORPHIC DIFFERENTIALS OF SOME ALGEBRAIC  
FUNCTION FIELD OF ONE VARIABLE OVER  $\mathbf{C}$

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We give holomorphic differentials of some algebraic function field  $K$  of complex dimension one which is a generalisation of a hyperelliptic field.

1. INTRODUCTION

Let  $K$  be an algebraic function field of one variable over  $\mathbf{C}$ . Then  $K$  is generated over  $\mathbf{C}$  by two generic points  $x$  and  $y$  of a plane curve defined by some equation  $f(X, Y) = 0$ , if  $f$  is the irreducible polynomial vanishing on  $(x, y)$ . Here  $x$  is transcendental over  $\mathbf{C}$  and  $y$  is algebraic over the rational function field  $F = \mathbf{C}(x)$ . In the sense of the theory of Riemann surfaces one can identify  $K$  with the field of meromorphic functions on a compact Riemann surface ([1, 6, 7, 8]). Thus all the statements parallel one another.

Let  $\Omega$  be the set of differentials (= differential one forms) of  $K$ . Then  $\Omega$  is a one dimensional vector space over  $K$  ([3, 5, 8]), which allows us to express it as  $\Omega = K \cdot dx$ . If  $w = y dx$  is a differential of  $K$  and  $\mathfrak{P}$  a prime divisor of  $K$ , we define the order of  $w$  at  $\mathfrak{P}$  by  $v_{\mathfrak{P}}(w) = v_{\mathfrak{P}}(y dx/dt)$  where  $t$  is a local uniformising parameter in  $K_{\mathfrak{P}}$  (the completion of  $K$  with respect to the  $\mathfrak{P}$ -adic topology). As is well known,  $K_{\mathfrak{P}}$  is isomorphic to the field  $\mathbf{C}((t))$  of formal power series and hence  $v_{\mathfrak{P}}$  is the standard valuation on  $\mathbf{C}((t))$ . Note that  $v_{\mathfrak{P}}(w)$  is independent of the choice of  $t$  and  $v_{\mathfrak{P}}(w) = 0$  for almost all prime divisors  $\mathfrak{P}$ . We call  $w \in \Omega$  a *holomorphic differential* (or a *differential of the first kind*) if  $v_{\mathfrak{P}}(w) \geq 0$  for all  $\mathfrak{P}$ . If  $\Omega_1$  is the set of holomorphic differentials and  $g$  is the genus of  $K$ , then  $\Omega_1$  is a  $g$ -dimensional vector space over  $\mathbf{C}$  ([1, 2, 3, 6, 8]). Therefore we can describe all the holomorphic differentials when we are given an algebraic function field  $K$ .

In this paper we find holomorphic differentials of  $K = \mathbf{C}(x, y)$  for which  $x$  and  $y$  satisfy the following equation :

$$(1) \quad y^n = A(x - a_1)^{n_1}(x - a_2)^{n_2} \dots (x - a_\ell)^{n_\ell}$$

where all  $a_i \in \mathbf{C}$  are distinct,  $A \in \mathbf{C}^\times$ ,  $n \geq 2$ ,  $n_j \geq 1$  ( $j = 1, 2, \dots, \ell$ ) and  $(n, n_1, \dots, n_\ell) = 1$ . Observe that  $(n, n_1, \dots, n_\ell)$  is the greatest common divisor of

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the integers  $n, n_1, \dots, n_\ell$ . In particular,  $K$  is called a hyperelliptic field provided that  $n = 2$ ,  $\ell \geq 5$  and  $n_1 = \dots = n_\ell = 1$ , in which case one can give an explicit basis  $\{w_1, \dots, w_g\}$  for  $\Omega_1$  as follows ([2, 3, 8]):

$$w_j = \frac{x^{j-1} dx}{y}, \quad 1 \leq j \leq g = \left\lfloor \frac{\ell - 1}{2} \right\rfloor$$

where  $y = \sqrt{A} \sqrt{(x - a_1) \dots (x - a_\ell)}$  and  $\lfloor r \rfloor$  denotes the largest integer  $\leq r$ .

### 2. PRELIMINARIES

To start with we need to show that the polynomial

$$(*) \quad Y^n - A(X - a_1)^{n_1} \dots (X - a_\ell)^{n_\ell}$$

is irreducible over  $\mathbb{C}$ .

**LEMMA 1.** *Let  $D$  be a unique factorisation domain of characteristic 0 and  $F$  be the quotient field of  $D$ . For  $a \in D$ , put  $a = \pi_1^{e_1} \pi_2^{e_2} \dots \pi_\ell^{e_\ell}$  with  $\pi_j$  prime elements in  $D$ . If  $n$  is a positive integer such that  $(n, e_1, \dots, e_\ell) = 1$ , then the polynomial  $\varphi(Y) = Y^n - a$  over  $D$  is irreducible over  $F$ .*

**PROOF:** Let  $\xi$  be a primitive  $n$ -th root of unity in  $F$  and  $\alpha$  be an element of the algebraic closure  $\overline{F}$  of  $F$  for which  $\varphi(\alpha) = 0$ . Since  $F(\alpha)$  is a (finite) Galois extension of  $F$ , let  $G$  be the Galois group of  $F(\alpha)$  over  $F$  and  $m = [F(\alpha) : F]$ . It suffices to show that  $m = n$ .

Suppose  $m < n$ . Take an  $F$ -automorphism  $\sigma$  from  $G$ . Then

$$\sigma(\alpha)^n = \sigma(\alpha^n) = \sigma(a) = a;$$

hence  $\sigma(\alpha) = \xi^{\nu(\sigma)} \alpha$  for  $\nu(\sigma) \in \mathbb{Z}/n\mathbb{Z}$ . We derive from this an injective homomorphism  $\nu$  from  $G$  into the additive group  $\mathbb{Z}/n\mathbb{Z}$ . Thus we may view  $G$  as a subgroup of  $\mathbb{Z}/n\mathbb{Z}$  which is cyclic of order  $n$ . Let  $G = \langle \tau \rangle$ . Since  $F(\alpha)$  is Galois over  $F$ ,  $|G| = m$  divides  $n$ . We observe by the definition of  $\nu$  that  $\tau(\alpha) = \xi^{\nu(\tau)} \alpha$  implies  $\tau^m(\alpha) = \xi^{m\nu(\tau)} \alpha = \xi^{m\nu(\tau)} \alpha$ .

But,  $\tau^m(\alpha) = \alpha$ ; hence

$$(2) \quad m\nu(\tau) \equiv 0 \pmod{n}.$$

On the other hand, by (2),

$$(\tau(\alpha))^m = \xi^{m\nu(\tau)} \alpha^m = \alpha^m$$

and

$$(\tau(\alpha))^m = \tau(\alpha^m).$$

Therefore  $\alpha^m$  belong to  $F$ . Set  $\alpha^m = b$ . Then  $b^{n/m} = (\alpha^m)^{n/m} = \alpha^n = a$ .

Since  $m$  divides  $n$ , let  $b^{n/m} = b^r$ . Obviously  $r > 1$ . Now we have  $a = b^r$  with  $a = \pi_1^{e_1} \cdots \pi_\ell^{e_\ell}$ , from which we conclude that  $r$  is a common divisor of  $e_1, e_2, \dots$ , and  $e_\ell$ . Since  $r$  also divides  $n$ ,  $(n, e_1, \dots, e_\ell) > 1$ . This is a contradiction.  $\square$

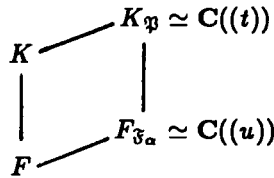
Lemma 1 asserts that the polynomial in (\*) is irreducible over  $\mathbb{C}$ .

**THEOREM 2** (Riemann-Hurwitz's formula). *Let  $K$  be an algebraic function field of one variable over  $\mathbb{C}$  of genus  $g$ , and  $F = \mathbb{C}(x)$  for  $x \in K - \mathbb{C}$  such that  $[K : F] = n$ . Let  $\mathfrak{P}$  be a prime divisor of  $K$  and  $e_{\mathfrak{P}}$  be the ramification index of  $\mathfrak{P}$ . Then*

$$\frac{1}{2} \sum_{\mathfrak{P}} (e_{\mathfrak{P}} - 1) = n + g - 1.$$

PROOF: ([2, 3, 4, 5, 8]).

Since  $\mathfrak{P}$  lies over some prime divisor  $\mathfrak{F}_\alpha$  of  $F$  ( $\alpha \in \widehat{\mathbb{C}}$  the extended complex plane), we have the following diagram :



where  $F_{\mathfrak{F}_\alpha}$  is the completion of  $F$  and  $u$  is a local uniformising parameter in  $F_{\mathfrak{F}_\alpha}$ . If  $e = e(\mathfrak{P}|\mathfrak{F}_\alpha)$  is the ramification index of  $\mathfrak{P}$  over  $\mathfrak{F}_\alpha$ , then  $u = t^e$ . Hence

$$u = \begin{cases} x - \alpha & \text{if } \alpha \neq \infty \\ 1/x & \text{if } \alpha = \infty, \end{cases}$$

implies that

$$(3) \quad v_{\mathfrak{P}}(dx) = \nu_{\mathfrak{P}} \left( \frac{dx}{dt} \right) = \begin{cases} e - 1 & \alpha \neq \infty \\ -e - 1 & \alpha = \infty. \end{cases}$$

### 3. MAIN THEOREM

Following the Riemann-Hurwitz's formula one can derive the genus  $g$  of  $K$  as follows

$$(4) \quad g = \frac{1}{2}n(\ell - 1) - \frac{1}{2} \left\{ \sum_{j=1}^{\ell} (n, n_j) + (n, N) \right\} + 1$$

where  $N = \sum_{j=1}^{\ell} n_j$ . We first consider the differential of the form  $w = dx/y$  with  $y = \sqrt[n]{A} \sqrt[n]{(x - a_1)^{n_1} \cdots (x - a_{\ell})^{n_{\ell}}}$ .

It follows from (3) that

$$(5) \quad \nu_{\mathfrak{P}}(dx) = \begin{cases} 1 - 1 = 0 & \text{if } \alpha \neq \infty \text{ and } a_j, \\ \frac{n}{\binom{n}{n_j}} - 1 & \text{if } \alpha = a_j, \\ -\frac{n}{\binom{n}{N}} - 1 & \text{if } \alpha = \infty. \end{cases}$$

Next, we obtain from (1) that

$$n\nu_{\mathfrak{P}}(y) = \nu_{\mathfrak{P}}((x - a_1)^{n_1}) + \cdots + \nu_{\mathfrak{P}}((x - a_{\ell})^{n_{\ell}})$$

because  $\nu_{\mathfrak{P}}(y) = \nu_{\mathfrak{P}}(y)$  and  $\nu_{\mathfrak{P}}(A) = 0$ . Since all  $(x - a_j)$  belong to the field  $F$ ,

$$(6) \quad n\nu_{\mathfrak{P}}(y) = n_1 e_{\mathfrak{P}} \nu_{\mathfrak{F}_{\alpha}}((x - a_1)) + \cdots + n_{\ell} e_{\mathfrak{P}} \nu_{\mathfrak{F}_{\alpha}}((x - a_{\ell}))$$

where  $\nu_{\mathfrak{F}_{\alpha}}$  is the discrete (exponential) valuation of  $F$ . Thus, by (6),

$$(7) \quad \nu_{\mathfrak{P}}(y) = \begin{cases} \frac{1}{n} \{n_1 \cdot 1 \cdot 0 + \cdots + n_{\ell} \cdot 1 \cdot 0\} = 0, & \alpha \neq \infty \text{ and } a_j; \\ \frac{1}{n} \{0 + \cdots + 0 + n_j \cdot \frac{n}{\binom{n}{n_j}} \cdot 1 + 0 + \cdots + 0\} = \frac{n_j}{\binom{n}{n_j}}, & \alpha = a_j; \\ \frac{1}{n} \{n_1 \cdot \frac{n}{\binom{n}{N}} \cdot -1 + \cdots + n_{\ell} \cdot \frac{n}{\binom{n}{N}} \cdot -1\} = -\frac{N}{\binom{n}{N}}, & \alpha = \infty. \end{cases}$$

Since  $\nu_{\mathfrak{P}}(w) = \nu_{\mathfrak{P}}(dx) - \nu_{\mathfrak{P}}(y)$ , it follows from (5) and (7) that for a prime divisor  $\mathfrak{P}$  over  $\mathfrak{F}_{\alpha}$

$$\nu_{\mathfrak{P}}(w) = \begin{cases} 0 & \text{if } \alpha \neq \infty \text{ and } a_j, \\ \frac{n - (n_j + \binom{n}{n_j})}{\binom{n}{n_j}} & \text{if } \alpha = a_j, \\ \frac{N - (n + \binom{n}{N})}{\binom{n}{N}} & \text{if } \alpha = \infty. \end{cases}$$

Therefore,  $dx/y$  is holomorphic if and only if  $n \geq n_j + \binom{n}{n_j}$  ( $1 \leq j \leq \ell$ ) and  $N \geq n + \binom{n}{N}$ . We readily see from this that the necessary conditions are

$$(8) \quad n > n_j \quad (j = 1, 2, \dots, \ell) \quad \text{and} \quad N > n.$$

For instance, look at the case  $n = 3$  and  $\ell = 3$ . Then by (4)

$$g = 3 - \frac{1}{2} \{(3, n_1) + (3, n_2) + (3, n_3) + (3, n_1 + n_2 + n_3)\} + 1,$$

and hence, by the restriction (8), we come up with the following table :

$n_1$	$n_2$	$n_3$	validity	$g$
1	1	1	×	1
1	1	2	0	2
1	2	2	0	2
2	2	2	0	1

In the first case  $dx/y$  is not a holomorphic differential; then what is the basis element of  $\Omega_1$ ? In the other cases  $dx/y \in \Omega_1$ , moreover in the last case every holomorphic differential is a constant multiple of  $dx/y$ . But, there still remains a question in the second and the third case; what is another basis element  $w$  independent of  $dx/y$ ?

To have complete answers we consider the differentials of the form

$$(9) \quad w = \frac{\prod_{j=1}^{\ell} (x - a_j)^{k_j} dx}{y^m}$$

where  $m \geq 1, k_j \geq 0$  and  $y = \sqrt[\ell]{A} \sqrt[(n_1)]{(x - a_1)^{n_1} \dots (x - a_{\ell})^{n_{\ell}}}$ .

**THEOREM 3.** A differential  $w$  in (9) is holomorphic if and only if  $n(k_j + 1) \geq mn_j + (n, n_j)$  ( $1 \leq j \leq \ell$ ) and  $mN \geq \left(\sum_{j=1}^{\ell} k_j + 1\right)n + (n, N)$ .

PROOF: For a prime divisor  $\mathfrak{P}$  over  $\mathfrak{F}_\alpha$ ,

$$\begin{aligned} v_{\mathfrak{P}}(w) &= v_{\mathfrak{P}}(dx) + \sum_{j=1}^{\ell} k_j \nu_{\mathfrak{P}}((x - a_j)) - m \nu_{\mathfrak{P}}(y) \\ &= v_{\mathfrak{P}}(dx) + \sum_{j=1}^{\ell} k_j e_{\mathfrak{P}} \nu_{\mathfrak{F}_\alpha}((x - a_j)) - m \nu_{\mathfrak{P}}(y). \end{aligned}$$

Here

$$(10) \quad \sum_{j=1}^{\ell} k_j e_{\mathfrak{P}} \nu_{\mathfrak{F}_\alpha}((x - a_j)) = \begin{cases} 0 & \text{if } \alpha \neq \infty \text{ and } a_j, \\ k_j \cdot \frac{n}{(n, n_j)} & \text{if } \alpha = a_j, \\ \sum_{j=1}^{\ell} k_j \cdot \frac{-n}{(n, N)} & \text{if } \alpha = \infty. \end{cases}$$

Therefore, by (5), (7) and (10)

$$v_{\mathfrak{P}}(w) = \begin{cases} 0, & \alpha \neq \infty \text{ and } a_j, \\ \frac{n(k_j+1) - (mn_j + (n, n_j))}{(n, n_j)}, & \alpha = a_j, \\ \frac{mN - \left(\left(\sum_{j=1}^{\ell} k_j + 1\right)n + (n, N)\right)}{(n, N)}, & \alpha = \infty, \end{cases}$$

from which we have the theorem. □

4. APPLICATIONS

In Section 3 we see that  $dx/y$  is not a holomorphic differential when  $n = 3$  and  $n_1 = n_2 = n_3 = 1$ . Applying Theorem 3, however, we can get appropriate basis element to this case. Since  $N = n_1 + n_2 + n_3 = 3$ ,  $(n, n_j) = 1$  and  $(n, N) = 3$ ,  $w$  in (9) is holomorphic if and only if  $3(k_1 + 1) \geq m + 1$ ,  $3(k_2 + 1) \geq m + 1$ ,  $3(k_3 + 1) \geq m + 1$  and  $3m \geq 3(k_1 + k_2 + k_3 + 1) + 3$ , or equivalently  $3k_1 \geq m - 2$ ,  $3k_2 \geq m - 2$ ,  $3k_3 \geq m - 2$  and  $m \geq k_1 + k_2 + k_3 + 2$  ( $m > 1$ ).

Taking smallest possible integers,  $m = 2$  and  $k_1 = k_2 = k_3 = 0$ . Therefore every holomorphic differential in this case is a constant multiple of  $dx/y^2$  with  $y = \sqrt[3]{A} \sqrt[3]{(x - a_1)(x - a_2)(x - a_3)}$ .

Likewise, in the second and third cases holomorphic bases are

and 
$$\left\{ \frac{dx}{y}, \frac{(x - a_3) dx}{y^2} \right\} \quad \left( y = \sqrt[3]{A} \sqrt[3]{(x - a_1)(x - a_2)(x - a_3)^2} \right)$$

$$\left\{ \frac{dx}{y}, \frac{(x - a_2)(x - a_3) dx}{y^2} \right\} \quad \left( y = \sqrt[3]{A} \sqrt[3]{(x - a_1)(x - a_2)^2(x - a_3)^2} \right),$$

respectively.

In the case of a hyperelliptic field  $K$ , we can derive a different basis for  $\Omega_1$  from the one described in Section 1 such as  $\{w'_1, \dots, w'_g\}$  with

$$w'_k = \frac{\prod_{j=1}^{\ell} (x - a_j)^{k-1} dx}{y^{2k-1}} \quad (1 \leq k \leq g)$$

and

$$y = \sqrt{A} \sqrt{(x - a_1) \cdots (x - a_{\ell})}.$$

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