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Compositio Math. **148** (2012), 907–920.

[doi:10.1112/S0010437X11007469](https://doi.org/10.1112/S0010437X11007469)



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# Quantifying residual finiteness of arithmetic groups

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## ABSTRACT

The normal residual finiteness growth of a group quantifies how well approximated the group is by its finite quotients. We show that any  $S$ -arithmetic subgroup of a higher rank Chevalley group  $G$  has normal residual finiteness growth  $n^{\dim(G)}$ .

## 1. Introduction

The quantification of residual finiteness, begun in [Bou10], seeks to describe how well a residually finite group is approximated by its finite quotients. This is measured by the normal residual finiteness growth of the group. During a geometry seminar at Yale University in December 2009, G. D. Mostow asked the following question.

*Question 1.1* (Mostow). Does asymptotic information of residual finiteness characterize arithmetic subgroups of a given linear algebraic group?

This paper presents a first major step towards answering this question, by showing that in a fixed Chevalley group  $G$ , all  $S$ -arithmetic subgroups share the same normal residual finiteness growth, and moreover this growth is  $n^{\dim(G)}$ . Note that for us, a Chevalley group will be a split simple algebraic group that is not necessarily simply connected.

To state our results more precisely, we need some notation. Let  $\Gamma$  be a finitely generated, residually finite group, and let  $X$  be a finite generating set for  $\Gamma$ . For  $\gamma \in \Gamma$ , let  $\|\gamma\|_X$  denote the word length of  $\gamma$  with respect to  $X$ . Define

$$D_\Gamma(\gamma) := \min\{|Q| : Q \text{ is a finite quotient of } \Gamma, \text{ where } \gamma \neq 1\}$$

and

$$F_{\Gamma,X}(n) := \max\{D_\Gamma(\gamma) : \|\gamma\|_X \leq n, \gamma \neq 1\}.$$

The function  $F_{\Gamma,X}$  is called the *normal residual finiteness growth function*. It is known that the asymptotic behavior of  $F_{\Gamma,X}$  is independent of  $X$  (see §2). The asymptotic growth of this function is called the *normal residual finiteness growth* of  $\Gamma$ .

The main results of this paper characterize the normal residual finiteness growth of  $S$ -arithmetic groups in Chevalley groups. We use the term  $S$ -arithmetic subgroup of  $G$  to denote any subgroup of  $G(\mathbb{C})$  which is commensurable with  $G(\mathcal{O}_{K,f})$ , where  $K \subset \mathbb{C}$  is a number field,  $\mathcal{O}_K$  is its ring of integers, and  $f \in \mathcal{O}_K \setminus \{0\}$ . The ingredients used include the structure theory of split semisimple group schemes, results on the congruence subgroup problem, Moy–Prasad filtrations, Selberg’s lemma, the prime number theorem, and the Cebotarev density theorem. Furthermore, we use in an essential way the results of Lubotzky–Mozes–Raghunathan [LMR00].

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Received 16 September 2010, accepted in final form 12 May 2011, published online 19 March 2012.

*2010 Mathematics Subject Classification* 20E26 (primary), 11F06, 22E40, 20H05 (secondary).

*Keywords:* arithmetic groups, normal residual finiteness growth, residual finiteness.

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**THEOREM 1.2.** *Let  $G$  be a Chevalley group of rank at least two,  $K$  be a number field, and  $f \in \mathcal{O}_K \setminus \{0\}$ . If  $\Gamma$  is a finitely generated subgroup of  $G(\mathbb{C})$  with the property that  $\Gamma \cap G(\mathcal{O}_{K,f})$  is of finite index in  $G(\mathcal{O}_{K,f})$ , then its normal residual finiteness growth is bounded below by  $n^{\dim(G)}$ .*

It is interesting to ask whether an analogous result holds in rank one. So far, the normal residual finiteness growth of a non-abelian free group has been bounded below by  $n^{2/3}$  (see [KM11]).

**THEOREM 1.3.** *Let  $G$  be a Chevalley group,  $K$  be a number field, and  $f \in \mathcal{O}_K \setminus \{0\}$ . If  $\Gamma$  is a finitely generated subgroup of  $G(\mathbb{C})$  with the property that  $\Gamma \cap G(\mathcal{O}_{K,f})$  is of finite index in  $\Gamma$ , then its normal residual finiteness growth is bounded above by  $n^{\dim(G)}$ .*

As a corollary of Theorems 1.2 and 1.3, we have the following result.

**COROLLARY 1.4.** *Let  $G$  be a Chevalley group of rank at least two. Then the normal residual finiteness growth of every  $S$ -arithmetic subgroup of  $G$  is precisely  $n^{\dim(G)}$ .*

This result is surprising, since, in general, if  $\Delta$  has finite index in  $\Gamma$ , we cannot hope for  $F_\Gamma \approx F_\Delta$  (see Example 2.5 at the end of §2). Instead, the most general result in this direction is  $F_\Gamma(n) \preceq (F_\Delta(n))^{[\Gamma:\Delta]}$  (see [Bou10, Lemma 1.3]).

## 2. Preliminaries

Let  $\Gamma$  be a finitely generated, residually finite group. For  $\gamma \in \Gamma \setminus \{1\}$ , we define  $Q(\gamma, \Gamma)$  to be the set of finite quotients of  $\Gamma$  in which the image of  $\gamma$  is non-trivial. We say that these quotients detect  $\gamma$ . Since  $\Gamma$  is residually finite, this set is non-empty, and thus the natural number

$$D_\Gamma(\gamma) := \min\{|Q| : Q \in Q(\gamma, G)\}$$

is defined and positive for each  $\gamma \in \Gamma \setminus \{1\}$ . For a fixed finite generating set  $X \subset \Gamma$ , we define

$$F_{\Gamma,X}(n) := \max\{D_\Gamma(\gamma) : \gamma \in \Gamma, \|\gamma\|_X \leq n, \gamma \neq 1\}.$$

For two functions  $f, g : \mathbb{N} \rightarrow \mathbb{N}$ , we write  $f \preceq g$  if there exists a natural number  $M$  such that  $f(n) \leq Mg(Mn)$ , and we write  $f \approx g$  if  $f \preceq g$  and  $g \preceq f$ . We will also write  $f \succeq g$  for  $g \preceq f$  and in the case when  $f \approx g$  does not hold we write  $f \not\approx g$ .

It was shown in [Bou10] that if  $X, Y$  are two finite generating sets for the residually finite group  $\Gamma$ , then  $F_{\Gamma,X} \approx F_{\Gamma,Y}$ . Since we will only be interested in asymptotic behavior, we let  $F_\Gamma$  be the equivalence class (with respect to  $\approx$ ) of the functions  $F_{\Gamma,X}$  for all possible finite generating sets  $X$  of  $\Gamma$ . Sometimes, by abuse of notation,  $F_\Gamma$  will stand for some particular representative of this equivalence class, constructed with respect to a convenient generating set.

We will need to use the following auxiliary function in our proofs. For any natural number  $k$ , we define

$$D_\Gamma^k(\gamma) := D_\Gamma(\gamma^k) \quad \text{and} \quad F_{\Gamma,X}^k(n) := \max\{D_\Gamma^k(\gamma) : \gamma \in \Gamma, \|\gamma\|_X \leq n, \gamma^k \neq 1\}.$$

The next lemma, which is a consequence of Selberg’s lemma (see [Alp87]), reveals the potential utility of  $F_{\Gamma,X}^k$ .

**LEMMA 2.1.** *Let  $\Gamma$  be an infinite linear group generated by a finite set  $X$  and let  $k$  a natural number. Then  $F_{\Gamma,X} \approx F_{\Gamma,X}^k$ .*

*Proof.* The inequality  $F_{\Gamma,X}^k(n) \leq F_{\Gamma,X}(kn)$  is straightforward. It suffices to prove  $F_{\Gamma,X}(n) \leq F_{\Gamma,X}^k(n)$  for all but finitely many  $n$ . Let  $\gamma_n$  be an element such that  $D_\Gamma(\gamma_n) = F_{\Gamma,X}(n)$  and  $\|\gamma_n\|_X \leq n$ . If  $\gamma_n^k \neq 1$ , then  $D_\Gamma(\gamma_n) \leq D_\Gamma^k(\gamma_n)$ , giving  $F_{\Gamma,X}(n) \leq F_{\Gamma,X}^k(n)$ . The proof will be complete if we show that  $\gamma_n^k = 1$  holds for only finitely many  $n$ . Suppose otherwise; then, by Selberg’s lemma, there exists a finite-index normal subgroup  $\Delta$  of  $\Gamma$  that is torsion-free, and in particular  $\gamma_n \notin \Delta$  for infinitely many  $n$ . Since  $F_{\Gamma,X}(n)$  is non-decreasing in  $n$ , it must be bounded by  $[\Gamma : \Delta]$ , but this contradicts the infinitude of  $\Gamma$ .  $\square$

**COROLLARY 2.2.** *If  $\Gamma$  is an infinite linear group and  $X, Y$  are finite generating sets for  $\Gamma$ , then  $F_{\Gamma,X}^k \approx F_{\Gamma,Y}^k$ .*

As with the function  $F$ , we will denote the asymptotic equivalence class of  $F_{\Gamma,X}^k$  as  $X$  varies by  $F_\Gamma^k$ . The following example shows that the linearity assumption cannot be dropped from Lemma 2.1.

*Example 2.3.* Let  $\Gamma$  be the Lamplighter group  $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$ . Set  $\Delta = \bigoplus_{i \in \mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$  to be the base group of  $\Gamma$ , so  $\Gamma/\Delta \cong \mathbb{Z}$ . It is easy to see that for any generating set  $X$  of  $\Gamma$ , we have  $F_{\Gamma,X}^2(n) \approx F_{\mathbb{Z}}(n)$ . Thus,  $F_{\Gamma,X}^2(n) \approx \log(n)$  by [Bou10, Corollary 2.3]. We now prove that  $F_\Gamma(n) \succeq (\log(n))^2$ , so in particular  $F_\Gamma \not\approx F_{\Gamma,X}^2$ .

*Proof.* Let  $\delta_i \in \Delta$  be the element given by the  $i$ th Kronecker delta function. For  $k$  a natural number greater than 4, set  $\gamma_k := \delta_1 + \delta_{\text{lcm}(1, \dots, k)}$ . Let  $\phi : \Gamma \rightarrow P$  be a homomorphism to a finite quotient of  $\Gamma$  that realizes  $D_\Gamma(\gamma_k)$ . We first claim that if  $\delta_1 + \delta_{1+n} \in \ker \phi$  for  $n \in \mathbb{N}$ , then  $n \geq k$ . Indeed, a simple calculation shows that  $\delta_1 + \delta_{1+mn} \in \ker \phi$  for any  $m \in \mathbb{N}$ . If  $n \leq k$ , we have that  $\text{lcm}(1, \dots, k)$  is a multiple of  $n$ , so  $\delta_1 + \delta_{\text{lcm}(1, \dots, k)} \in \ker \phi$ , which is impossible.

Next we claim that the set  $S := \{(\delta_n, t) : n, t \in \{1, \dots, \lfloor k/4 \rfloor\}\} \subseteq \Gamma$  injects into  $P$  through  $\phi$ . Suppose not; then  $(\delta_n, t)(\delta_{n'}, t')^{-1} \in \ker \phi$  for  $t, t', n, n' \in \{1, \dots, \lfloor k/4 \rfloor\}$  with  $(\delta_n, t) \neq (\delta_{n'}, t')$ . Set  $\alpha = (\delta_n, t)(\delta_{n'}, t')^{-1} = (\delta_n + \delta_{n'+t-t'}, t - t')$ . If  $t - t' = 0$ , then, by our first claim,  $n = n'$  or  $\|n - n'\| \geq k$ . If  $n = n'$ , then  $\alpha = (0, 0)$ , while the latter possibility contradicts  $\|n - n'\| \leq k/2$ . If  $t - t' \neq 0$ , because  $\alpha \delta_i \alpha^{-1} \delta_i^{-1} \in \ker \phi$  for all  $i$ , we have  $\delta_{1+t-t'} + \delta_1 \in \ker \phi$ , where, by our first claim,  $|t - t'| \geq k$ ; however,  $|t - t'| \leq k/2$ . Our second claim is now shown.

Since  $S$  injects into  $P$ , we have  $|P| \geq \lfloor k/4 \rfloor^2$ . Fix a finite generating set  $X$  for  $\Gamma$ ; by the prime number theorem, there exists a natural number  $M$  such that  $\|\gamma_k\|_X \leq M3^k$ . Set  $k = \lfloor \log_3(n) \rfloor$ ; then, because  $F_\Gamma$  is increasing, we have, for sufficiently large  $n$ ,

$$F_\Gamma(Mn) \geq F_\Gamma(M3^k) \geq F_\Gamma(\|\gamma_k\|_X) \geq \lfloor k/4 \rfloor^2 \geq \frac{1}{32} \left[ \frac{\log(n)}{\log(3)} \right]^2. \quad \square$$

**LEMMA 2.4.** *Let  $\Gamma, \Delta$  be finitely generated and residually finite.*

- *If  $f : \Delta \rightarrow \Gamma$  is a homomorphism with finite kernel, then  $F_\Delta \preceq F_\Gamma$ .*
- *If in addition  $f$  is surjective, its kernel is central in  $\Delta$ , and  $\Gamma$  is linear, then  $F_\Delta \approx F_\Gamma$ .*

*Proof.* For the first assertion, factor  $f$  as  $\Delta \xrightarrow{j} \text{im}(f) \xrightarrow{i} \Gamma$ . By [Bou10, Lemma 1.1], we have  $F_{\text{im}(f)} \preceq F_\Gamma$ . By a similar argument one can show that  $F_\Delta \preceq F_{\text{im}(f)}$ .

Now consider the second assertion. We will show  $F_\Delta^k \succeq F_\Gamma$ , where  $k = |\ker(f)|$ . To that end, fix a finite generating set  $X$  for  $\Delta$  and use its image for  $\Gamma$ . Construct  $F_\Delta$  and  $F_\Gamma$  with respect to these generating sets. Let  $g \in \Delta$ ,  $g^k \neq 1$ . Since  $g^k = (zg)^k$  for all  $z \in \ker(f)$ , we see that  $\ker(f)N$

is a normal subgroup of  $\Delta$  not containing  $g$ . Thus,  $D_{\Delta}^k(g) \geq D_{\Gamma}(f(g))$  for all  $g \in \Delta$  with  $g^k \neq 1$ . We now need to handle torsion elements in  $\Gamma$ .

For each natural number  $n$ , let  $\gamma_n \in \Gamma$  be an element satisfying  $D_{\Gamma}(\gamma_n) = F_{\Gamma}(n)$  and  $\|\gamma_n\| \leq n$ . Since  $f$  is surjective, by our choice of generating sets, there exists  $g_n \in \Delta$  such that  $f(g_n) = \gamma_n$  and  $\|g_n\| \leq n$ . Then, if  $g_n^k = 1$  for infinitely many  $n$ , we have  $\gamma_n^k = 1$  for infinitely many  $n$ . Following the Selberg lemma application from Lemma 2.1, we see that  $\Gamma$  is finite, which is impossible. Thus,  $g_n^k \neq 1$  for all but finitely many  $n$ . For such  $n$ , we have  $D_{\Delta}^k(g_n) \geq D_{\Gamma}(f(g_n))$  and hence  $F_{\Delta}^k(n) \succeq F_{\Gamma}(n)$ . □

We finish the preliminaries section with an example that illustrates that normal residual finiteness growth of a group may be different from that of a finite index subgroup.

*Example 2.5.* Let  $Q$  be the subgroup of  $GL_2(\mathbb{Z})$  generated by

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let  $\Delta = \mathbb{Z} \times \mathbb{Z}$  and set  $\Gamma = \Delta \rtimes Q$ , where  $Q$  acts on  $\Delta$  via the standard action of  $GL_2(\mathbb{Z})$ . Because  $Q$  is finite,  $\Gamma$  contains  $\Delta$  as a subgroup of finite index. Further,  $F_{\Delta}(n) \approx \log(n)$  by [Bou10, Corollary 2.3]. We now prove that  $F_{\Gamma}(n) \succeq (\log(n))^2$ .

*Proof.* Let  $X$  be a generating set for  $\Gamma$  containing  $(1, 0)$  and  $(0, 1)$  in  $\Delta$ . Set  $\gamma_k$  to be  $(\text{lcm}(1, \dots, k), 0) \in \Delta$ . By the prime number theorem, there exists a natural number  $M$  such that  $\|\gamma_k\|_X \leq M3^k$ . Let  $\phi : \Gamma \rightarrow P$  be a homomorphism to a finite quotient of  $\Gamma$  that realizes  $D_{\Gamma}(\gamma_k)$  and set  $V = \ker \phi \cap \Delta$ . We first construct a subgroup of  $V$  of the form  $d\mathbb{Z} \times d\mathbb{Z}$  for some natural number  $d$ . Consider the intersection of  $V$  with  $\mathbb{Z} \times 0$ . This is a subgroup of  $\mathbb{Z}$  and hence is isomorphic to  $d\mathbb{Z}$  for some natural number  $d$ . Thus, we have  $d\mathbb{Z} \times 0 \subset V$ , and conjugating by  $B$  we also find  $0 \times d\mathbb{Z}$  is in  $V$ .

Next we claim that the index of  $d\mathbb{Z} \times d\mathbb{Z}$  in  $V$  is at most 4: let  $(a, b) \in V$ . Then  $(2a, 0) = (a, b) + A(a, b)A^{-1} \in V$ , and similarly  $(2b, 0) \in V$ , so  $2a, 2b \in d\mathbb{Z}$ , and hence  $2(a, b) \in d\mathbb{Z} \times d\mathbb{Z}$ , which shows that every element of  $V/d\mathbb{Z} \times d\mathbb{Z}$  has order (at most) two. But  $V$  is a free abelian group of rank two, so  $V/d\mathbb{Z} \times d\mathbb{Z}$  is generated by two elements, and the claim follows. We conclude that  $d^2 = [\Delta : d\mathbb{Z} \times d\mathbb{Z}] = [\Delta : V][V : d\mathbb{Z} \times d\mathbb{Z}] \leq 4[\Delta : V]$ , giving  $|P| \geq \frac{1}{4}d^2$ .

Finally, since  $\gamma_k \notin \ker(\phi)$ , we must have that  $d \geq k$ . Hence,  $F_{\Gamma}(M3^k) \geq D_{\Gamma}(\gamma_k) \geq \frac{1}{4}k^2$ . Set  $k = \lfloor (\log_3(n)) \rfloor$ ; then, because  $F_{\Gamma}$  is increasing, we have, for sufficiently large  $n$ ,

$$F_{\Gamma}(Mn) \geq F_{\Gamma}(M3^k) \geq \frac{1}{4}k^2 \geq \frac{1}{16} \left( \frac{\log(n)}{\log(3)} \right)^2,$$

giving  $F_{\Gamma}(n) \succeq (\log(n))^2$ , as desired. □

### 3. Lower bounds

Let  $G$  be a Chevalley group, that is, a split simple group scheme defined over  $\mathbb{Z}$ , and let  $\mathfrak{g}$  be its Lie algebra. Note that we do not assume that  $G$  is simply connected. For a natural number  $m$ , we put  $G(m) = G(\mathbb{Z}/m\mathbb{Z})$ . For a while, we will focus attention on the powers of a single prime  $p$ , and to lighten the notation we put  $G_k = G(\mathbb{Z}/p^k\mathbb{Z})$ .

Recall from [SGA3, exp. 1, 2.3.3+2.3.6] the definition of the center  $\mathcal{Z}(G)$  of  $G$ . It is the subfunctor of  $G$ , which assigns to each scheme  $S$  the following subgroup of  $G(S)$ :

$$\mathcal{Z}(G)(S) := \{g \in G(S) \mid \forall S' \rightarrow S : \text{Ad}(g)|_{G(S')} = \text{id}_{G(S')}\},$$

where  $\text{Ad}(g)|_{G(S')}$  denotes the automorphism of  $G(S')$  provided by conjugation by the image of  $g$  under the natural map  $G(S) \rightarrow G(S')$ .

It is shown in [SGA3, exp. 22, 4.1.8] that the functor  $\mathcal{Z}(G)$  is representable by a closed  $\mathbb{Z}$ -subgroup scheme of  $G$ , which is finite and diagonalizable. As such,  $\mathcal{Z}(G)$  is a product of finitely many group schemes, each isomorphic to  $\mu_n$  for some  $n$ , where  $\mu_n$  is the group scheme of  $n$ th roots of unity. In particular,  $\mathcal{Z}(G)$  is etale over  $\mathbb{Z}[\text{ord}(\mathcal{Z}(G))^{-1}]$ . See [SGA3, exp. 8, 2.1].

From the definition, it is obvious that  $\mathcal{Z}(G)(S) \subset Z(G(S))$ . We will show that there exists  $f \in \mathbb{Z} \setminus \{0\}$  such that if  $S$  lies over  $\text{Spec}(\mathbb{Z}_f)$ , then  $\mathcal{Z}(G)(S) = Z(G(S))$ . The main ingredient in this proof is the following lemma, which asserts the existence of a strongly regular section of the split maximal torus in  $G$  over  $\text{Spec}(\mathbb{Z}_f)$ .

LEMMA 3.1. *Let  $T \subset G$  be a split maximal torus. There exist  $f \in \mathbb{Z} \setminus \{0\}$  and a point  $s \in T(\mathbb{Z}_f)$  such that*

$$\text{Cent}(s, G \times \text{Spec}(\mathbb{Z}_f)) = T \times \text{Spec}(\mathbb{Z}_f).$$

*Proof.* Consider the closed subscheme of  $T$  given by

$$\bigcup_{\alpha \in R(T, G)} \ker(\alpha) \cup \bigcup_{w \in W} T^w,$$

where  $R(T, G)$  is the set of roots of  $T$  in  $G$  and  $W = \text{Norm}(G, T)/T$  is the Weyl group. Let  $U$  be its complement in  $T$ . Then  $U \rightarrow T$  is an open immersion, which when composed with an isomorphism  $T \cong \mathbb{G}_m^r$  and the open immersion  $\mathbb{G}_m^r \rightarrow \mathbb{A}_{\mathbb{Z}}^r$  provides an open immersion  $U \rightarrow \mathbb{A}_{\mathbb{Z}}^r$ . Since  $\mathbb{A}^r(\mathbb{Q})$  is dense in  $\mathbb{A}^r(\overline{\mathbb{Q}})$ , it follows that  $U(\mathbb{Q}) \neq \emptyset$ . As  $U$  is of finite type, any map  $\text{Spec}(\mathbb{Q}) \rightarrow U$  factors as  $\text{Spec}(\mathbb{Q}) \rightarrow \text{Spec}(\mathbb{Z}_f) \rightarrow U$  for some  $f$ . Thus, we have a point  $s : \text{Spec}(\mathbb{Z}_f) \rightarrow U$ . We claim that this point satisfies the statement of the lemma. To lighten notation, let us base change to  $\text{Spec}(\mathbb{Z}_f)$ . Consider the centralizer  $H := \text{Cent}(s, G)$ . It is a closed subscheme of  $G$ , hence affine and of finite type over  $\mathbb{Z}_f$ , and contains  $T$ . By generic flatness, we may assume that  $H$  is flat, after possibly changing  $f$ . By the choice of  $s$ , all fibers of  $H$  and  $T$  coincide. By [SGA3, exp. 10, 4.9],  $H$  is a torus and, since  $T$  is a maximal torus, it follows that  $H = T$ .  $\square$

COROLLARY 3.2. *There exists  $f \in \mathbb{Z} \setminus \{0\}$  such that for all schemes  $S \rightarrow \text{Spec}(\mathbb{Z}_f)$ , we have*

$$\mathcal{Z}(G)(S) = Z(G(S)).$$

*Proof.* The inclusion  $\subset$  is obvious from the definition of  $\mathcal{Z}(G)$  and we now have to show the converse. Choose  $f$  and  $s \in T(\mathbb{Z}_f)$  as in the above lemma. Let  $S \rightarrow \text{Spec}(\mathbb{Z}_f)$  and  $x \in Z(G(S))$ . If  $s_S \in T(S)$  denotes the image of  $s$  under  $T(\mathbb{Z}_f) \rightarrow T(S)$ , then

$$x \in \text{Cent}(s_S, G_S)(S) = T(S).$$

We claim that for every root  $\alpha \in R(T_S, G_S)$ , we have  $\alpha(x) = 1$ . Assume by way of contradiction that this were not the case. Let  $u_\alpha : \mathbb{G}_{a,S} \rightarrow G_S$  be the root subgroup corresponding to  $\alpha$ , and  $y = u_\alpha(1)$ . Then  $y \in G(S)$  is a point not centralized by  $x$ , contrary to the assumptions. It follows that

$$x \in \bigcap_{\alpha \in R(T_S, G_S)} \ker(\alpha)(S) = \mathcal{Z}(G)(S),$$

where the last equality is [SGA3, exp. 22, 4.1.6].  $\square$

**COROLLARY 3.3.** *There exists a finite set of primes  $P$  such that  $|Z(G_k)|$  divides  $\text{ord}(\mathcal{Z}(G))$  for all primes  $p \notin P$ . In particular, if  $m$  is an integer coprime to the elements of  $P$ , then the order of every element of  $Z(G(m))$  divides  $\text{ord}(\mathcal{Z}(G))$ .*

*Proof.* The second statement is an immediate consequence of the first, since  $Z(G(m)) = \prod_{p^k \parallel m} Z(G_k)$ . To prove the first, let  $P$  be the set of primes  $p$  for which  $\mathcal{Z}(G)(\mathbb{Z}/p^k\mathbb{Z})$  is a proper subgroup of  $Z(G_k)$ . According to Corollary 3.2, the set  $P$  is finite. For a prime  $p$  not in  $P$ , we then have  $Z(G_k) = \mathcal{Z}(G)(\mathbb{Z}/p^k\mathbb{Z})$ . As already remarked,  $\mathcal{Z}(G)$  is a finite product of the  $\mu_n$ . Since  $(\mathbb{Z}/p^k\mathbb{Z})^\times$  is cyclic, the number  $|\mu_n(\mathbb{Z}/p^k\mathbb{Z})|$  divides  $n$ . The statement now follows.  $\square$

**LEMMA 3.4.** *The natural projection  $\mathbb{Z}/p^k\mathbb{Z} \rightarrow \mathbb{Z}/p^{k-1}\mathbb{Z}$  induces a surjective homomorphism*

$$G_k \rightarrow G_{k-1}.$$

*For all but finitely many primes  $p$ , this homomorphism restricts to an isomorphism*

$$Z(G_k) \rightarrow Z(G_{k-1}).$$

*Proof.* The first claim follows directly from the infinitesimal lifting property of smoothness. For the second claim, let  $p$  be a prime which does not divide  $\text{ord}(\mathcal{Z}(G))$  and for which  $Z(G_k) = \mathcal{Z}(G)(\mathbb{Z}/p^k\mathbb{Z})$  for all  $k$ . By Corollaries 3.2 and 3.3, these are all but finitely many primes. Then  $\mathcal{Z}(G)$  is étale over  $\mathbb{Z}_{(p)}$  and this implies the bijectivity of the second map.  $\square$

**COROLLARY 3.5.** *Assume that  $G$  is simply connected. Then, for all but finitely many  $p$ ,*

$$Z(G_k/Z(G_k)) = \{1\}.$$

*Proof.* We prove this by induction on  $k$ . The base case is  $k = 1$ , which is known, since  $G(\mathbb{F}_p)/Z(G(\mathbb{F}_p))$  is simple. For the induction step, let  $k > 1$ . Let  $z \in G_k$  be an element which is central in  $G_k/Z(G_k)$ . Then, for all  $g \in G_k$ ,  $z_g := gzg^{-1}z^{-1} \in Z(G_k)$ . Under the surjection  $G_k \rightarrow G_{k-1}$ , the element  $z$  maps to an element  $\bar{z}$  with the same property. Applying the induction hypothesis, we see that  $\bar{z} \in Z(G_{k-1})$ . This implies that  $\bar{z}_g = 1$ . Lemma 3.4 now implies  $z_g = 1$  and the statement follows.  $\square$

For  $0 \leq i \leq k$ , let  $G_k^i := \ker(G_k \rightarrow G_i)$ . This provides a descending filtration

$$G_k = G_k^0 \supseteq G_k^1 \supseteq \cdots \supseteq G_k^k = \{1\}.$$

We fix a closed embedding  $G \rightarrow \text{SL}_m$  defined over  $\mathbb{Z}$ . This yields an embedding of Lie algebras  $\mathfrak{g} \rightarrow \mathfrak{sl}_m$  defined over  $\mathbb{Z}$ . We identify  $G$  and  $\mathfrak{g}$  with their respective images. Clearly,  $G_k^i = [1 + p^i M_m(\mathbb{Z}/p^k\mathbb{Z})] \cap G_k$ , and an element  $1 + p^i x \in G_k^i$  belongs to  $G_k^{i+1}$  if and only if  $x \equiv 0 \pmod p$ .

The following lemma is a well-known result from the theory of Moy–Prasad filtrations [MP94].

**LEMMA 3.6** (Moy–Prasad). (i)  $[G_k^i, G_k^j] \subset G_k^{i+j}$ .

(ii) For  $1 \leq i \leq k - 1$ , the map

$$G_k^i/G_k^{i+1} \rightarrow \mathfrak{g}(\mathbb{F}_p), \quad 1 + p^i x \mapsto x \pmod p$$

*induces an isomorphism of groups, which is equivariant with respect to the action of  $G(\mathbb{F}_p)$  on both sides by conjugation.*

*Remark 3.7.* In particular, one sees inductively that each  $G_k^i$  for  $i > 0$  is a  $p$ -group.

LEMMA 3.8. *There exist positive constants  $c, C$  such that for all prime powers  $m = p^k$ ,*

$$cm^{\dim(G)} \leq |G(m)| \leq Cm^{\dim(G)}.$$

*Proof.* In the case  $k = 1$ , the lemma follows from [Ste68, Theorem 25, § 9]. The general case reduces to this, because according to Lemma 3.6 we have  $|G_k| = p^{(k-1)\dim(G)}|G(\mathbb{F}_p)|$ .  $\square$

LEMMA 3.9. *For all but finitely many  $p$ , the Lie algebra  $\mathfrak{g}(\mathbb{F}_p)$  has no center, and the adjoint action of  $G(\mathbb{F}_p)/Z(G(\mathbb{F}_p))$  on  $\mathfrak{g}(\mathbb{F}_p)$  is faithful and irreducible.*

*Proof.* This is a well-known classical result. See for example [His84, Hog82].  $\square$

LEMMA 3.10. *Assume that  $p$  is sufficiently large, and  $0 \leq i \leq k - 2$ . For every  $g \in G_k^i \setminus G_k^{i+1}Z(G_k)$ , there exists  $h \in G_k^1$  such that  $hgh^{-1}g^{-1} \in G_k^{i+1} \setminus G_k^{i+2}Z(G_k)$ .*

*Proof.* Note first that by Lemma 3.4,  $G_k^{i+1} \cap (G_k^{i+2}Z(G_k)) = G_k^{i+2}$ . Hence, it is enough to find  $h$  such that  $hgh^{-1}g^{-1} \notin G_k^{i+2}$ .

Write  $h = 1 + py$  with some  $y \in M_m(\mathbb{Z}/p^k\mathbb{Z})$  to be determined. We will make use of the following computation: for any  $x \in M_m(\mathbb{Z}/p^k\mathbb{Z})$ , we have:

$$\begin{aligned} (1 + py)x(1 + py)^{-1} &= (x + pyx)(1 + py)^{-1} \\ &= (x + pxy - p[x, y])(1 + py)^{-1} \\ &= (x - p[x, y](1 + py)^{-1}), \end{aligned}$$

where  $[x, y] = xy - yx$ .

First assume that  $i = 0$ . Then, using the above computation, we see that

$$hgh^{-1}g^{-1} = 1 - p[g, y](1 + py)^{-1}g^{-1}.$$

Clearly, the right-hand side belongs to  $G_k^1$ , and to show that it does not belong to  $G_k^2$  it is enough by Lemma 3.6 to show that the reduction mod  $p$  of the matrix  $[g, y](1 + py)^{-1}g^{-1} \in M_m(\mathbb{Z}/p^k\mathbb{Z})$  is non-zero. Call this reduction  $T$ . It belongs to  $\mathfrak{g}(\mathbb{F}_p)$ . Using the formula

$$(1 + py)^{-1} = \sum_{j=0}^{k-1} (-py)^j,$$

we compute that  $T = [\bar{g}, \bar{y}]\bar{g}^{-1} = \bar{g}\bar{y}\bar{g}^{-1} - \bar{y}$ . By Lemma 3.4, the preimage of  $Z(G(\mathbb{F}_p))$  under  $G_k \rightarrow G(\mathbb{F}_p)$  is  $G_k^1Z(G_k)$ . Thus, by assumption, the image  $\bar{g}$  of  $g$  in  $G(\mathbb{F}_p)/Z(G(\mathbb{F}_p))$  is non-trivial and, by Lemma 3.9, there exists  $\bar{y} \in \mathfrak{g}(\mathbb{F}_p)$  such that  $\bar{g}\bar{y}\bar{g}^{-1} \neq \bar{y}$ . According to Lemma 3.6, there exists  $h = 1 + py \in G_k^1$  corresponding to this  $\bar{y}$ . This completes the proof in the case  $i = 0$ .

Now assume  $i > 0$ . We write  $g = 1 + p^i x$  for some  $x \in M_m(\mathbb{Z}/p^k\mathbb{Z})$  whose reduction mod  $p$  belongs to  $\mathfrak{g}(\mathbb{F}_p)$ . Then

$$\begin{aligned} (1 + py)(1 + p^i x)(1 + py)^{-1}(1 + p^i x)^{-1} &= (1 + p^i(1 + py)x(1 + py)^{-1})(1 + p^i x)^{-1} \\ &= (1 + p^i x - p^{i+1}[x, y](1 + py)^{-1})(1 + p^i x)^{-1} \\ &= 1 - p^{i+1}[x, y](1 + py)^{-1}(1 + p^i x)^{-1}. \end{aligned}$$

Again,  $hgh^{-1}g^{-1} \in G_k^{i+1}$ , and we want to choose  $y$  so that this element does not belong to  $G_k^{i+2}$ . By Lemma 3.6, this is equivalent to the demand that the reduction mod  $p$  of the element

$$[x, y](1 + py)^{-1}(1 + p^i x)^{-1} \in M_m(\mathbb{Z}/p^k\mathbb{Z})$$

be non-trivial. Using the formula for  $(1 + p^i x)^{-1}$  analogous to that used above for  $(1 + py)^{-1}$ , we compute that this element is equal mod  $p$  to  $[x, y]$ . Now we consider the image of  $[x, y] \in M_m(\mathbb{F}_p)$ . Of course, this is just the bracket of the images of  $x$  and  $y$  in  $M_m(\mathbb{F}_p)$ . But these images, and hence their bracket, lie in  $\mathfrak{g}(\mathbb{F}_p)$ . Again, as in the case  $i = 0$ , specifying  $h$  is equivalent to choosing the class of  $y$  in  $\mathfrak{g}(\mathbb{F}_p)$  in such a way that its bracket with the class of  $x$  is non-trivial. Since the Lie algebra  $\mathfrak{g}(\mathbb{F}_p)$  has no center, the class of  $x$  is non-central, and so an appropriate  $y$  exists.  $\square$

**PROPOSITION 3.11.** *Assume that  $p$  is sufficiently large and  $G$  is simply connected. Then every normal subgroup  $N < G_k$  which contains  $Z(G_k)$  equals  $G_k^i Z(G_k)$  for some  $i$ .*

*Proof.* We will first prove under the assumption  $k > 1$  by descending induction on  $i$  the following statement.

$$\forall 0 \leq i < k : N \cap [G_k^i \setminus G_k^{i+1} Z(G_k)] \neq \emptyset \Rightarrow G_k^i \subset N.$$

The base case is when  $i = k - 1 > 0$ . Then the isomorphism of Lemma 3.6 identifies  $G_k^i$  with  $\mathfrak{g}(\mathbb{F}_p)$  and  $N \cap G_k^i$  with an invariant subspace of  $\mathfrak{g}(\mathbb{F}_p)$ . By assumption, this space is non-trivial and, by Lemma 3.9, it is all of  $\mathfrak{g}(\mathbb{F}_p)$ , hence  $N \cap G_k^i = G_k^i$ . For the induction step, assume  $i \geq 0$ . Let  $g \in N \cap [G_k^i \setminus G_k^{i+1} Z(G_k)]$ . Use Lemma 3.10 to obtain  $h \in G_k^1$  such that  $hgh^{-1}g^{-1} \in G_k^{i+1} \setminus G_k^{i+2} Z(G_k)$ . Then  $hgh^{-1}g^{-1} \in N$ , and we may apply the induction hypothesis to conclude  $G_k^{i+1} \subset N$ . Now look at the normal subgroup  $(N \cap G_k^i)/G_k^{i+1}$  of  $G_k^i/G_k^{i+1}$ . If  $i > 0$ , then we have the isomorphism  $G_k^i/G_k^{i+1} \rightarrow \mathfrak{g}(\mathbb{F}_p)$  and the image of that normal subgroup is a non-trivial invariant subspace. If  $i = 0$ , then we have the isomorphism  $G_k^i/G_k^{i+1} \rightarrow G(\mathbb{F}_p)$  and the image of that normal subgroup is a normal subgroup of  $G(\mathbb{F}_p)$  which properly contains  $Z(G(\mathbb{F}_p))$ . In both cases, we conclude that  $(N \cap G_k^i)/G_k^{i+1} = G_k^i/G_k^{i+1}$ , and hence  $N \cap G_k^i = G_k^i$ . This completes the induction.

Now we show how the proposition follows from the above statement. The case  $k = 1$  is trivial, since  $G_1/Z(G_1)$  is simple. Thus, assume  $k > 1$ . If  $N = Z(G_k)$ , there is nothing to prove. Otherwise, there exists a unique smallest index  $i$  such that  $G_k^i \setminus G_k^{i+1} Z(G_k)$  contains an element of  $N$ . By the above statement,  $Z(G_k)G_k^i \subset N$ , but by minimality of  $i$  this must in fact be an equality.  $\square$

**PROPOSITION 3.12.** *Let  $N$  be a natural number, and  $H = \ker[G(\mathbb{Z}) \rightarrow G(N)]$ . If  $G$  is simply connected, then, for any  $m$  coprime to  $N$ , the projection  $G(\mathbb{Z}) \rightarrow G(m)$  maps  $H$  surjectively onto  $G(m)$ .*

*Proof.* We begin with the special case  $N = 1$ ; then  $H = G(\mathbb{Z})$ . Since  $G$  is smooth, the natural projection  $G(\mathbb{Z}_p) \rightarrow G(\mathbb{Z}/p^k\mathbb{Z})$  is surjective for all primes  $p$  and all natural numbers  $k$ , and hence the natural projection  $G(\widehat{\mathbb{Z}}) \rightarrow G(m)$  is surjective for all natural numbers  $m$ . By strong approximation [PR94], the inclusion  $G(\mathbb{Z}) \rightarrow G(\widehat{\mathbb{Z}})$  has dense image. Thus, the natural projection  $G(\mathbb{Z}) \rightarrow G(m)$  is surjective.

For the general case, we have  $G(Nm) \cong G(N) \times G(m)$  and, by the first part of the proof, the projection  $G(\mathbb{Z}) \rightarrow G(N) \times G(m)$  is surjective. The preimage in  $G(\mathbb{Z})$  of the subgroup  $1 \times G(m)$  of  $G(N) \times G(m)$  is precisely  $H$ , and maps surjectively onto  $G(m)$ .  $\square$

**PROPOSITION 3.13.** *Assume that the rank of  $G$  is at least two. Let  $u : \mathbb{G}_a \rightarrow G$  be a root subgroup, and  $X$  a finite generating set for  $G(\mathbb{Z})$ . Then there exists a positive constant  $M$  such that for any positive  $z \in \mathbb{Z}$ ,*

$$\|u(z)\|_X \leq M \log(z).$$

*Proof.* Composing  $u$  with the chosen closed embedding  $G \rightarrow \mathrm{SL}_m$ , and then further with the natural inclusion  $\mathrm{SL}_m \rightarrow M_m$ , we obtain a morphism of  $\mathbb{Z}$ -schemes

$$u' : \mathbb{A}_{\mathbb{Z}}^1 \rightarrow \mathbb{A}_{\mathbb{Z}}^{m^2},$$

which is given by the collection  $\{u'_{i,j}\}$  of  $m^2$ -many polynomials in one variable with integral coefficients. Let  $k = \max \deg(u'_{i,j}) + 1$ . Then there exists a positive constant  $C$  such that  $u'_{i,j}(z) \leq Cz^k$  for all positive integers  $z$  and all  $i, j$ . Thus,  $\|u'(z)\| \leq Cz^k$  for all  $z \in \mathbb{N}$ , where  $\|\cdot\|$  is the maximum norm on  $M_m(\mathbb{R})$ . The result now follows from [LMR00, Theorem A].  $\square$

We are now ready to prove our main lower bound. In the proof, we are going to use the fact that if  $G$  is simply connected and has rank at least two, then  $G(\mathbb{Z})$  has the congruence subgroup property. We refer the reader to [PR94, ch. 9.5] for a discussion of this property. Also, recall that a subgroup of  $G(m)$  is called *essential* if it does not contain the kernel of the natural map  $G(m) \rightarrow G(r)$  for any  $r|m$  with  $r < m$ .

**THEOREM 3.14.** *Assume that the rank of  $G$  is at least two. Let  $K$  be a number field,  $f \in \mathcal{O}_K$ , and  $\Delta$  a finitely generated subgroup of  $G(\mathbb{C})$  with the property that  $\Delta \cap G(\mathcal{O}_{K,f})$  is of finite index in  $G(\mathcal{O}_{K,f})$ . Then*

$$F_{\Delta}(n) \succeq n^{\dim(G)}.$$

*Proof.* Let  $G_{\mathrm{sc}}$  be the simply connected cover of  $G$ , and  $p : G_{\mathrm{sc}}(\mathcal{O}_{K,f}) \rightarrow G(\mathcal{O}_{K,f})$  the natural map. Then  $\Delta_{\mathrm{sc}} := p^{-1}(\Delta \cap G(\mathcal{O}_{K,f}))$  is of finite index in  $G_{\mathrm{sc}}(\mathcal{O}_{K,f})$  and the map  $p : \Delta_{\mathrm{sc}} \rightarrow \Delta$  has finite kernel. By Lemma 2.4, we may assume for the rest of the proof that  $G = G_{\mathrm{sc}}$  and  $\Delta \subset G(\mathcal{O}_{K,f})$ .

Since  $\Delta$  is of finite index in  $G(\mathcal{O}_{K,f})$ , so is  $\Delta \cap G(\mathbb{Z})$  of finite index in  $G(\mathbb{Z})$ . By virtue of the congruence subgroup property of  $G(\mathbb{Z})$ , we can find a principal congruence subgroup  $\Delta' \subset \Delta \cap G(\mathbb{Z})$ . Applying again Lemma 2.4, we may assume for the rest of the proof that  $\mathcal{O}_{K,f} = \mathbb{Z}$  and that  $\Delta$  is a principal congruence subgroup of  $G(\mathbb{Z})$ .

Let  $N = \mathrm{ord}(\mathcal{Z}(G))$ . By Lemma 2.1, it suffices to find a lower bound for  $F_{\Delta}^N$ . Loosely speaking, we will see that working with  $F_{\Delta}^N$  instead of  $F_{\Delta}$  will aid us in ignoring certain central elements in finite images of  $\Delta$ .

We first construct candidates that are poorly approximated by finite quotients. Let  $X$  and  $Y$  be finite generating sets for  $G(\mathbb{Z})$  and  $\Delta$ , respectively. Let  $S$  be the set of primes  $p$  for which at least one of the following conditions fails.

- $|Z(G_k)|$  divides  $N$ .
- If  $Z(G_k) \trianglelefteq N \trianglelefteq G_k$ , then  $N = G_k^i Z(G_k)$  for some  $i$ .
- The projection  $G(\mathbb{Z}) \rightarrow G_k$  maps  $\Delta$  surjectively onto  $G_k$ ,

where, as before,  $G_k = G(\mathbb{Z}/p^k\mathbb{Z})$ . By Corollary 3.3 and Propositions 3.11 and 3.12, this set is finite. Put  $\alpha = \prod_{p \in S} p$  and  $r_k = \alpha^k \mathrm{lcm}(1, \dots, k)$ . Let  $u : \mathbb{G}_a \rightarrow G$  be a root subgroup, and  $B_k = u(r_k)$ . Since  $u$  is defined over  $\mathbb{Z}$ , we have  $B_k \in G(\mathbb{Z})$ , hence  $A_k := B_k^{[G(\mathbb{Z}) : \Delta]} \in \Delta$ . The elements  $A_k$  will be our candidates for achieving lower bounds for  $F_{\Delta}^N$ .

Next we bound the word length of  $A_k$ , that is, the function  $k \mapsto \|A_k\|_Y$ . By Proposition 3.13, there exists a natural number  $M$  such that

$$\|A_k\|_X \leq M \log(\mathrm{lcm}(1, \dots, k)\alpha^k).$$

Hence, by the prime number theorem, we may find a potentially different natural number  $M$  so that  $\|A_k\|_X \leq Mk$ . Finally, since  $G(\mathbb{Z})$  is quasi-isometric to  $\Delta$ , we have that

$$\|A_k\|_Y \leq Mk \tag{1}$$

for some other natural number  $M$ .

The remainder of the proof is devoted to finding a lower bound for the cardinality of any finite quotient  $Q = \Delta/H$  which detects  $A_k^N$ , in particular to the quotient realizing  $D_\Delta^N(A_k)$ . We start by taking one such quotient  $Q$ . Since we are looking for a lower bound of the cardinality of  $Q$ , we may replace it by either a subgroup or a quotient of it, and we will do so repeatedly in the following.

By the congruence subgroup property for  $G(\mathbb{Z})$ , there exists a natural number  $m$  such that the kernel of the projection  $\phi : G(\mathbb{Z}) \rightarrow G(m)$  lies in  $H$ . Let  $\Delta', H'$ , and  $A'_k$  be the images of  $\Delta, H$ , and  $A_k$ , respectively, in  $G(m)$ . By the Chinese remainder theorem, we may write  $G(m) = A \times B$ , where

$$A = \prod_{\substack{p^j \parallel m \\ p \in S}} G(p^j) \quad \text{and} \quad B = \prod_{\substack{p^j \parallel m \\ p \notin S}} G(p^j)$$

and  $p^j \parallel m$  means that  $j$  is the greatest power of  $p$  which divides  $m$ .

We know  $(A'_k)^N \neq 1$ . For any  $c \in Z(B)$ , we have  $\text{ord}(c) \mid N$  (see Corollary 3.3 and the choices of  $S$  and  $N$ ). Thus, we have  $(cA'_k)^N = (A'_k)^N$  for any  $c \in Z(B)$ , which implies  $cA'_k \notin H'$ . Hence,  $A'_k \notin H'Z(B)$ . Letting  $A''_k, \Delta'',$  and  $H''$  be the images of  $A'_k, \Delta',$  and  $H'$  in  $A \times B/Z(B)$ , respectively, we have that  $A''_k \notin H''$ . Further,  $[\Delta'' : H''] \leq [\Delta' : H']$ , since  $\Delta''/H''$  is an image of  $Q = \Delta'/H'$ .

We claim that any quotient of  $B/Z(B)$  is centerless: indeed, by the choice of  $S$ , for every  $p \notin S$ , Lemma 3.4 and Corollary 3.5 imply that all quotients of  $G(p^j)/Z(G(p^j))$  are centerless. By [LL, 1.4], every normal subgroup of  $B/Z(B)$  is a product of normal subgroups of the factors of  $B/Z(B)$ , and the statement follows.

Recall that  $\Delta$  was assumed to be a principal congruence subgroup of  $G(\mathbb{Z})$ . By Proposition 3.12,  $G(\mathbb{Z})$  projects onto  $A \times B/Z(B)$ . Hence,  $\Delta''$  is normal in  $A \times B/Z(B)$  and, applying [LL, 1.3, 1.4], we see that  $\Delta'' = \Delta_1 \times \Delta_2$ , where

$$\Delta_1 = \pi_1(\Delta'') \quad \text{and} \quad \Delta_2 = \pi_2(\Delta''),$$

where  $\pi_1$  and  $\pi_2$  are the natural projection maps of  $A \times B/Z(B)$  onto  $A$  and  $B/Z(B)$ , respectively.

By the choice of  $S$ , we have  $\Delta_2 = B/Z(B)$ . The subgroup  $H''$  is normal in  $\Delta'' = \Delta_1 \times \Delta_2$  and, since  $\Delta_2$  has no center, [LL, Corollary 1.4] applies again giving  $H'' = H_1 \times H_2$ , where  $H_1 = \pi_1(H'')$  and  $H_2 = \pi_2(H'')$ . Now, since  $A''_k \notin H_1 \times H_2$ , we have two cases:  $\pi_1(A''_k) \notin \pi_1(H'')$  or  $\pi_2(A''_k) \notin \pi_2(H'')$ . In both cases, we claim that there exists a natural number  $M$ , independent of  $k$ , such that  $M|Q| \geq k^d$ , where  $d := \dim(G)$ .

We first handle the case  $\pi_1(A''_k) \notin \pi_1(H'')$ . Write  $A = G(m_0)$ , and let  $r$  be the smallest natural number such that the kernel of the natural map  $\phi : G(m_0) \rightarrow G(r)$  is contained in  $\pi_1(H'')$ . Then  $\phi(\pi_1(A''_k)) \notin \phi(\pi_1(H''))$  and  $\phi(\pi_1(H''))$  is essential or trivial. Since the image of  $A_k$  in  $G(r)$  is non-trivial,  $r$  does not divide  $\alpha^k$ . But any prime dividing  $r$  also divides  $\alpha$  (recall the choices of  $A, r,$  and  $\alpha$ ), hence  $p^k \mid r$  for some  $p \in S$ . In the case  $\phi(\pi_1(H''))$  is essential, [LS03, Proposition 6.1.2] gives  $C|G(r) : \phi(\pi_1(H''))| \geq r \geq p^k$ , where  $C$  is a natural number that only depends on  $G$ . If  $\phi(\pi_1(H''))$  is trivial, we get the better bound  $C|G(r)| \geq C|G(p^k)| \geq p^{kd}$  by

Lemma 3.8, where  $C$  is again a natural number that depends only on  $G$ . Set  $M' = C[G(\mathbb{Z}) : \Delta]$ . Since  $[G(r) : \phi(\pi_1(\Delta''))] \leq [G(\mathbb{Z}) : \Delta]$ , we have

$$M'[\Delta'' : H''] \geq C[G(r) : \phi(\pi_1(\Delta''))][\phi(\pi_1(\Delta'')) : \phi(\pi_1(H''))] = C[G(r) : \phi(\pi_1(H''))] \geq p^k.$$

There exists a natural number  $M''$  such that  $M''p^k \geq k^d$  for all  $p \in S$  and  $k \in \mathbb{N}$ . Setting  $M = M'M''$ , we see that

$$M[\Delta'' : H''] \geq M''p^k \geq k^d.$$

Since  $|Q| \geq [\Delta'' : H'']$ , the claim is shown.

Next we handle the case  $\pi_2(A''_k) \notin \pi_2(H'')$ . By repeated use of [LL, Corollary 1.4], there exists a natural projection  $\phi : A \times B/Z(B) \rightarrow G_k/Z(G_k)$  with  $\phi(A''_k) \notin \phi(H'')$  and  $G_k = G(p^k)$ , where  $p \notin S$ . By Proposition 3.11 and the normality of  $H_2$  in  $\Delta_2 = B/Z(B)$ , we have  $\phi(H'') = G^i_k/Z(G_k)$  for some  $i$ , hence the image of  $\phi(A''_k)$  through the natural projection onto  $G_i/Z(G_i)$  is non-trivial. Further,  $Q$  maps onto  $G_i/Z(G_i)$ .

From the estimate  $M'|G_i| \geq p^{di}$  (Lemma 3.8), where  $M'$  is a natural number, and the fact that  $p^i$  does not divide  $\text{lcm}(1, \dots, k)$ , we obtain  $p^i \geq k$  and, thus,

$$M'|G_i| \geq p^{id} \geq k^d.$$

Finally, since  $|G_i|/|Z(G_i)| \leq |Q|$  and  $|Z(G_i)| \leq N$  (by the choice of  $S$ ), the claim holds with  $M = M'N$ .

The inequality  $M|Q| \geq k^d$  in tandem with inequality (1) gives some natural number  $M$  such that  $MF_{\Delta}^N(k) \geq k^d$ , finishing the proof of the theorem.  $\square$

### 4. Upper bounds

In this section,  $G$  continues to be a Chevalley group. Our main upper bound result is a corollary of the following three propositions.

PROPOSITION 4.1. *Let  $L$  be a number field with ring of integers  $\mathcal{O}_L$ . Then*

$$F_{\mathcal{O}_L}(n) \approx \log(n).$$

Moreover, the finite quotients of the form  $\mathbb{Z}/p\mathbb{Z} \cong \mathcal{O}_L/\mathfrak{p}$ , where  $p$  is a prime number that splits completely in  $\mathcal{O}_L$ ,  $\mathfrak{p}|p\mathcal{O}_L$ , are enough to obtain the upper bound.

*Proof.* The fact  $F_{\mathcal{O}_L}(n) \geq \log(n)$  follows immediately from [Bou10, Theorem 2.2] and Lemma 2.4. We just need to show that the upper bound is obtainable from quotients  $\mathcal{O}_L/\mathfrak{q}$ .

Let  $S = \{b_1, \dots, b_k\}$  be an integral basis for  $\mathcal{O}_L$ , and fix a non-trivial  $g$  in  $\mathcal{O}_L$  with  $\|g\|_S = n$ . Then  $g = \sum_{i=1}^n a_i b_i$ , where  $a_i \in \mathbb{Z}$  and  $|a_i| \leq n$ . Since  $g \neq 0$ , there exists  $k$  such that  $a_k \neq 0$ . By the Cebotarëv density theorem, the set  $P$  of all primes in  $\mathbb{Z}$  that split in  $\mathcal{O}_L$  has non-zero natural density in the set of all primes. We claim that there exist  $C > 0$ , which does not depend on  $n$ , and a prime  $q$  such that  $(q)$  splits in  $\mathcal{O}_L$  and  $q \leq C \log(n)$  and  $a_k \not\equiv 0 \pmod q$ . Indeed, enumerate  $P = \{q_1, q_2, \dots\}$ . Let  $q_{r+1}$  be the first prime in  $P$  such that  $a_k \not\equiv 0 \pmod{q_{r+1}}$ . Then  $q_1 \cdots q_r$  divides  $a_k$  and, by the prime number theorem and positive density of  $P$ , we have that  $q_{r+1} \leq Mr \log(r)$  for some  $M > 0$ , depending only on  $L$ . A similar calculation shows that there exists  $M' > 0$  such that  $q_1 \cdots q_r \geq e^{M'r \log(r)}$ . Hence,  $q_{r+1} \leq C \log(a_k)$ , where  $C > 0$  depends only on  $L$ . The claim is shown.

Write  $(q) = \mathfrak{q}_1 \cdots \mathfrak{q}_c$  with  $|\mathcal{O}_L/\mathfrak{q}_i| = q$ . Since  $q$  does not divide  $a_k$  and, since the integral basis  $S$  gets sent to a  $\mathbb{F}_q$ -basis of  $\mathcal{O}_L/(q)$ , we have that  $g \neq 1$  in  $\mathcal{O}_L/(q)$ . Hence, there exists one  $\mathfrak{q}_i$  with  $g \neq 1$  in  $\mathcal{O}_L/\mathfrak{q}_i$ . As the cardinality of  $\mathcal{O}_L/\mathfrak{q}_i$  is equal to  $q$ , which is no greater than  $C \log(n)$ , we have the desired upper bound.  $\square$

PROPOSITION 4.2. *Let  $\Gamma$  be a finitely generated subgroup of  $G(\mathcal{O}_{L,f})$ , where  $L$  is a number field and  $f \in \mathbb{Z}$ . Then*

$$F_\Gamma \preceq n^{\dim(G)}.$$

*Proof.* Recall that we have fixed a closed embedding  $G \rightarrow \mathrm{SL}_m$  and are identifying  $G$  with its image. Let  $\mathcal{X}$  be a finite set of generators for  $\Gamma$  as a semigroup. Let  $S$  be an integral basis for  $\mathcal{O}_L$ . We claim that there exists  $\lambda > 0$  such that for any  $A \in \Gamma$  with  $\|A\|_{\mathcal{X}} = n$  and any non-zero coefficient  $a \in \mathcal{O}_{L,f}$  of  $A - I$ , we have

$$\|f^k a\|_S \leq \lambda^n,$$

where  $k$  is the least natural number such that  $f^k a \in \mathcal{O}_L$ .

To prove the claim, let  $a' = a + 1$  or  $a' = a$  according to whether  $a$  is a diagonal coefficient or not. Thus,  $a'$  is a coefficient of  $A$ . Let  $K$  be the least natural number such that for all  $X \in \mathcal{X}$ ,  $f^K X \in M_m(\mathcal{O}_L)$ . Because  $A$  is a product of exactly  $n$  elements of  $\mathcal{X}$ , we have  $f^{nK} A \in M_m(\mathcal{O}_L)$  and, in particular,  $k < nK$ . Then

$$\|f^k a\|_S \leq \|f^{nK} a\|_S \leq \|f^{nK} a'\|_S + f^{nK} \|1\|_S.$$

This reduces the above claim to the following. There exists  $\mu > 0$  such that for any  $A \in \Gamma$  with  $\|A\|_{\mathcal{X}} = n$  and any non-zero coefficient  $a \in \mathcal{O}_{L,f}$  of  $A$ , we have

$$\|f^{nK} a\|_S \leq \mu^n.$$

We claim that if  $\alpha$  denotes the maximum of  $\|st\|_S$ , where  $s, t$  range over the elements of  $S$ , and  $\beta$  denotes the maximum of  $\|f^K x\|_S$ , where  $x$  ranges over all entries of all elements of  $\mathcal{X}$ , then  $\mu := m\alpha\beta$  satisfies the last statement. To see this, consider first the case  $A = XY$  with  $X, Y \in \mathcal{X}$ . The entries of  $A$  are scalar products of the rows of  $X$  and the columns of  $Y$ . Thus, we are led to study  $\|x \cdot y\|_S$  for  $x, y \in \mathcal{O}_L^m$ , where  $\cdot$  denotes scalar product. Clearly, we have  $\|x \cdot y\|_S \leq m \max\{\|x_i y_i\|_S : 1 \leq i \leq m\}$ . In terms of the basis  $S$ , we can write

$$x_i = \sum_{s \in S} \lambda_{x,i,s} s \quad \text{and} \quad y_i = \sum_{s \in S} \lambda_{y,i,s} s,$$

where the  $\lambda$  belong to  $\mathbb{Z}$ . One computes

$$\|x_i y_i\|_S \leq \|x_i\|_S \|y_i\|_S \max\{\|st\|_S : s, t \in S\}.$$

This formula and induction on  $n$  complete the proof of the claim.

To complete the proof of the proposition, let  $A \in \Gamma$  be such that  $\|A\|_{\mathcal{X}} \leq n$ . Let  $a$  be a non-zero entry of  $A - I$  and  $k$  the least integer with  $f^k a \in \mathcal{O}_L$ . According to Proposition 4.1 and the claim above, there exist a natural number  $M$ , independent of  $n$ , and a homomorphism  $\phi : \mathcal{O}_L \rightarrow \mathbb{F}_p$  such that  $p < Mn$  and  $\phi(f^k a) \neq 0$ . For all but finitely many primes  $p$ , we have that  $\phi(f)$  is non-zero in  $\mathbb{F}_p$ . Hence, we may assume that  $\phi$  extends to a homomorphism  $\phi : \mathcal{O}_{L,f} \rightarrow \mathbb{F}_p$  and  $\phi(a) \neq 0$ . The image of  $A$  under the induced map  $G(\mathcal{O}_{L,f}) \rightarrow G(\mathbb{F}_p)$  is non-trivial. Further, according to Lemma 3.8, there exists  $M' > 0$  such that  $|G(\mathbb{F}_p)| \leq M' p^{\dim(G)}$ . Hence,  $|G(\mathbb{F}_p)| \leq M'(Mn)^{\dim(G)}$ .  $\square$

PROPOSITION 4.3. *Let  $K \subset \mathbb{C}$  be a number field,  $b \in \mathcal{O}_K \setminus \{0\}$ , and  $\Gamma \subset G(\mathbb{C})$  a finitely generated subgroup, such that  $G(\mathcal{O}_{K,b}) \cap \Gamma$  is of finite index in  $\Gamma$ . Then there exist a finite extension  $L \subset \mathbb{C}$  of  $K$ , an element  $f \in \mathbb{Z} \setminus \{0\}$ , and a homomorphism  $\Gamma \rightarrow G(\mathcal{O}_{L,f})$  with finite kernel.*

*Proof.* Let  $S \subset \Gamma$  be a finite generating set. There exists a field  $F \subset \mathbb{C}$ , finitely generated over  $K$ , such that  $S \subset G(F)$ . Let  $t_1, \dots, t_n$  be a transcendence basis for  $F/K$ . The extension  $F/K(t_1, \dots, t_n)$  is finitely generated and algebraic, hence finite. Let  $a \in F$  be a primitive element for that extension. Thus,  $F = K(t_1, \dots, t_n, a)$ . The ring  $\mathcal{O}_{K,b}[t_1, \dots, t_n]$  is a free polynomial algebra over  $\mathcal{O}_{K,b}$  with field of fractions  $K(t_1, \dots, t_n)$ . There exists  $s \in \mathcal{O}_K[t_1, \dots, t_n]$  such that the coefficients of the minimal polynomial of  $a$  over  $K(t_1, \dots, t_n)$  lie in the localization  $\mathcal{O}_{K,b}[t_1, \dots, t_n]_s$ . Thus, the element  $a$  is integral over  $\mathcal{O}_{K,b}[t_1, \dots, t_n]_s$  and the ring  $\mathcal{O}_{K,b}[t_1, \dots, t_n]_s[a] \subset F$  has  $F$  as its field of fractions. Thus, there exists  $r \in \mathcal{O}_{K,b}[t_1, \dots, t_n]_s[a]$ , such that if we put  $R = \mathcal{O}_{K,b}[t_1, \dots, t_n]_s[a]_r$ , then  $S \subset G(R)$ , and consequently  $\Gamma \subset G(R)$ .

We can find a homomorphism of  $\mathcal{O}_{K,b}$ -algebras

$$\phi : \mathcal{O}_{K,b}[t_1, \dots, t_n] \rightarrow \mathcal{O}_{K,b}$$

such that  $\phi(s) \neq 0$ . Then  $\phi$  extends to a homomorphism

$$\phi : \mathcal{O}_{K,b}[t_1, \dots, t_n]_s \rightarrow \mathcal{O}_{K,b\phi(s)}.$$

There exists a finite extension  $L \subset \mathbb{C}$  of  $K$  such that the composition of  $\phi$  with the natural inclusion  $\mathcal{O}_{K,b\phi(s)} \rightarrow K$  extends to a homomorphism

$$\phi : \mathcal{O}_{K,b}[t_1, \dots, t_n]_s[a] \rightarrow L.$$

The element  $\phi(a) \in L$  is integral over  $\mathcal{O}_{K,b\phi(s)}$ , and hence belongs to  $\mathcal{O}_{L,b\phi(s)}$ . Thus, in fact, we obtain a homomorphism

$$\phi : \mathcal{O}_{K,b}[t_1, \dots, t_n]_s[a] \rightarrow \mathcal{O}_{L,b\phi(s)}.$$

We consider  $\phi(r) \in \mathcal{O}_{L,b\phi(s)}$ . Perturbing  $\phi$  slightly if necessary, we may assume that  $\phi(r) \neq 0$ . In this way we obtain a homomorphism of  $\mathcal{O}_K$ -algebras

$$\phi : R \rightarrow \mathcal{O}_{L,b\phi(rs)}.$$

The algebra homomorphism  $\mathcal{O}_L \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow L$  given by multiplication is an isomorphism. Since  $\mathbb{Q} = \varinjlim_{f \in \mathbb{Z}} \mathbb{Z}_f$ , we conclude that

$$L \cong \varinjlim_{f \in \mathbb{Z}} \mathcal{O}_L \otimes_{\mathbb{Z}} \mathbb{Z}_f \cong \varinjlim_{f \in \mathbb{Z}} \mathcal{O}_{L,f}.$$

Thus, there exists some  $f \in \mathbb{Z}$  such that  $[b\phi(rs)]^{-1} \in \mathcal{O}_{L,f}$ . Composing  $\phi$  with the inclusion  $\mathcal{O}_{L,b\phi(rs)} \rightarrow \mathcal{O}_{L,f}$ , we finally arrive at a homomorphism of  $\mathcal{O}_{K,b}$ -algebras

$$\phi : R \rightarrow \mathcal{O}_{L,f}.$$

It induces a group homomorphism  $\phi_* : G(R) \rightarrow G(\mathcal{O}_{L,f})$  which fits into the commutative diagram.

$$\begin{array}{ccc} G(R) & \xrightarrow{\phi_*} & G(\mathcal{O}_{L,f}) \\ & \searrow & \swarrow \\ & G(\mathcal{O}_{K,b}) & \end{array}$$

The restriction of  $\phi_*$  to  $\Gamma$  is the desired homomorphism: its kernel has trivial intersection with  $G(\mathcal{O}_{K,b})$ , that is, it avoids a finite-index subgroup of  $\Gamma$ , and hence must be finite.  $\square$

**COROLLARY 4.4.** *Let  $\Gamma \subset G(\mathbb{C})$  be a finitely generated subgroup. Assume that there exist a finite extension  $K \subset \mathbb{C}$  of  $\mathbb{Q}$  and  $b \in \mathcal{O}_K \setminus \{0\}$  such that  $G(\mathcal{O}_{K,b}) \cap \Gamma$  is of finite index in  $\Gamma$ . Then*

$$F_\Gamma(n) \preceq n^{\dim(G)}.$$

*Proof.* This follows immediately from Proposition 4.3, Lemma 2.4, and Proposition 4.2.  $\square$

#### ACKNOWLEDGEMENTS

It is our pleasure to thank G. D. Mostow for suggesting this pursuit. We are grateful to Martin Kassabov, Alexander Lubotzky, Madhav Nori, and Gopal Prasad for helpful mathematical conversations, and wish to thank Alexander Premet for providing us with a useful reference. Further, we are thankful to Benson Farb for giving us comments on an earlier draft.

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