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ON STRATIFICATIONS OF DERIVED MODULE CATEGORIES

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ABSTRACT. Some structural results about quotients and tensor products of hereditary respectively quasi-hereditary algebras are presented. They are related to properties of stratifications of derived module categories. The concept of derived-simplicity for an algebra is introduced and studied.

The work of E. Cline, B. Parshall and L. L. Scott on the Kazhdan-Lusztig conjecture and the Lusztig conjecture naturally led to the purely algebraic theory of quasi-hereditary algebras [1], [4], [6]. Their study of the recollement situation for perverse sheaves and its algebraic analog expressed in the derived category of the category of finitely generated modules over a finite dimensional algebra suggested quite strongly the set of axioms defining quasi-hereditary algebras. Namely these algebras have the nice property that their derived module category allows a stratification with subsequent quotients isomorphic to derived vector space categories.

In this note, we present some results suggested by L. L. Scott on quasi-hereditary algebras and relate them with stratifications of derived module categories. The quotient algebra of a hereditary algebra by an arbitrary two-sided ideal is quasi-hereditary. This was first obtained by L. L. Scott answering positively a question of J. Alperin. Secondly, tensor products are studied, and it is shown that the tensor product of two quasi-hereditary algebras is again quasi-hereditary. This answers a question of B. Parshall and L. L. Scott.

In order to understand examples of algebras whose derived category does not allow a stratification by derived module categories all of the form $D^b(f), f$ a finite dimensional skew field over the ground field k, we introduce the notion of derived-simplicity for an algebra, expressing the fact that its derived module category does not allow a non-trivial stratification induced by two other algebras. A result involving the global dimension shows that the derived module category of the path algebra B over k which is as follows given by its quiver with relations

$$a \underset{\beta}{\overset{\alpha}{\rightleftarrows}} b, \quad \alpha\beta = \beta\alpha = 0,$$

does not allow a stratification with subsequent quotients of the form $D^b(f)$, although B is the quotient of the quasi-hereditary algebra given by the same quiver and the relation $\beta\alpha = 0$. Stratificability with subsequent quotients of the form $D^b(f)$ is therefore not inherited by taking quotients, as it is true for hereditary algebras. Further investigations

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moreover show that B even is derived-simple. I want to express my gratitude to L. L. Scott for many valuable suggestions, hints and remarks.

1. **Structural results for algebras and stratifications.** All algebras are supposed to be finite dimensional over a field k which is arbitrary if not stated otherwise. "$\otimes$" always means the tensor product over k. All modules considered are supposed to be finitely generated left modules. For a finite dimensional k-algebra A, we denote by $D^b(A)$ the derived category of the category of finitely generated A-modules.

There are obvious generalizations to semiprimary rings in the sense of [2] which we do not emphasize for sake of simplicity.

We recall the basic definitions.

DEFINITION. Let A be a finite dimensional k-algebra. We call a two-sided ideal J of A *semi-heredity* if J satisfies

(i) J is projective as a left A-module.

(ii) $J = J^2$ is idempotent.

J is called a *heredity ideal* of A if we have additionally

(iii) J Rad $AJ = 0$.

The algebra A is called *quasi-hereditary* if there exists a so-called *heredity chain* for A, being an ascending chain of ideals in A

$$0 = J_0 \subsetneq J_1 \subsetneq \cdots \subsetneq J_n = A,$$

for some $n \in \mathbf{N}$, such that J_i/J_{i-1} is a heredity ideal of the quotient algebra A/J_{i-1} for each $i = 1, \ldots, n$.

As described in [4], this definition was suggested by the results of [1], because it has the consequence, that a quasi-hereditary algebra A has the property that its derived module category $D^b(A)$ allows successively a stratification with subsequent quotients all of the simple form $D^b(f)$, f a finite dimensional skew field over k.

LEMMA 1.1. *Let A be a hereditary algebra and I a two-sided ideal of A. Then the quotient algebra A/I is quasi-hereditary. In particular, $D^b(A/I)$ admits a stratification with subsequent quotients of the form $D^b(f)$.*

PROOF. Let $J = J_1$ be the socle of A and choose a heredity chain for A as follows

$$0 \subsetneq J_1 \subsetneq \cdots \subsetneq J_n = A.$$

Since A/J is again hereditary, we may by induction on the dimension of A assume that I does not contain J. Also by induction we then have that

$$(A/J)/((I+J)/J) \cong (A/I)/((I+J)/I)$$

is quasi-hereditary. Therefore, it essentially remains to show that the ideal $\bar{J} = (J+I)/I$ is a projective module over $\bar{A} = A/I$. For this, let $\bar{T}$ be an indecomposable direct summand of $\bar{J}$. Then $\bar{T}$ is a simple A-module which can be identified with a simple direct summand T of J intersecting I trivially. Let e be a primitive idempotent of A such that $T \cong Ae$. Then there exists an element $a \in A$ with $T = Aea$, and therefore $e \notin I$. With $\bar{e} = e + I$ in $\bar{A}$, we have $\bar{T} = T \cong Ae \cong \bar{A} \cdot \bar{e}$, showing that $\bar{T}$ is projective over $\bar{A}$. ∎

LEMMA 1.2. *Let A and B be two algebras with chains of ideals*

$$0 = J_0 \subsetneq J_1 \subsetneq \cdots \subsetneq J_n = A \quad \text{and}$$

$$0 = L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_m = B$$

such that all quotients J_i/J_{i-1} and L_j/L_{j-1} are semi-heredity ideals in A/J_{i-1} and B/L_{i-1} for $i = 1, \ldots, n$ and $j = 1, \ldots, m$ respectively. Then the following chain of ideals

$$0 \subsetneq J_1 \otimes L_1 \subsetneq J_1 \otimes L_2 \subsetneq \cdots \subsetneq J_1 \otimes B \subsetneq J_2 \otimes (L_1 + J_1) \otimes B$$

$$\subsetneq J_2 \otimes L_2 \otimes + J_1 \otimes B \subsetneq \cdots\cdots \subsetneq A \otimes B$$

has the same property with respect to $A \otimes B$.

PROOF. By induction on n, the statement is true for $(A/J_1) \otimes B$, and it suffices to show that $(J_1 \otimes L_j)/(J_1 \otimes L_{j-1})$ is a semi-heredity ideal of $A \otimes B/(J_1 \otimes L_{j-1})$. Clearly, these ideals are idempotent. Moreover, $(J_1 \otimes L_j)/(J_1 \otimes L_{j-1}) \cong J_1 \otimes (L_j/L_{j-1})$ is projective as a left $A \otimes B/(J_1 \otimes L_{j-1})$-module for $j = 1, \ldots, m$ by the following reasoning. L_j/L_{j-1} is a direct summand of finitely many copies of B/L_{j-1}. Furthermore,

$$J_1 \otimes (B/L_{j-1}) \cong J_1 \otimes B/(J_1 \otimes L_{j-1}) \cong J_1 \otimes B/((J_1 \otimes L_{j-1})(J_1 \otimes B))$$

$$\cong (A \otimes B/(J_1 \otimes L_{j-1})) \otimes_{A \otimes B} J_1 \otimes B.$$

$J_1 \otimes B$ being $A \otimes B$-projective implies that the last expression is a direct summand of finitely many copies of

$$(A \otimes B/(J_1 \otimes L_{j-1})) \otimes_{A \otimes B} A \otimes B \cong A \otimes B/(J_1 \otimes L_{j-1}),$$

as was to be shown. ∎

1.3. We next answer a question of B. Parshall and L. L. Scott.

PROPOSITION. *Let A and B be two quasi-hereditary algebras over a perfect field k. Then the tensor product algebra $A \otimes B$ is again quasi-hereditary.*

PROOF. We keep the notation of Lemma 1.2 and assume that the given chains of ideals are heredity chains for A and B respectively. Again by induction, $A/J_1 \otimes B$ is quasi-hereditary, and with the result of Lemma 1.2, it suffices to verify property (iii) of the definition. Since k is perfect, for $j = 1, \ldots, m$ we have

$$J_1 \otimes L_j \operatorname{Rad}(A \otimes B) J_1 \otimes L_j = J_1 \otimes L_j((\operatorname{Rad} A) \otimes B + A \otimes (\operatorname{Rad} B)) J_1 \otimes L_j$$

$$= J_1 \otimes L_j(A \otimes (\operatorname{Rad} B)) J_1 \otimes L_j = J_1 \otimes (L_j \operatorname{Rad} B\, L_j)$$

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which is contained in $J_1 \otimes L_{j-1}$. ∎

As an application of this Proposition, let us note that under reasonable conditions as stated e.g. in [4, Theorem 5.17], the finite dimensional algebra associated with perverse sheaves on a direct product of stratified spaces is the tensor product of the algebras associated with each of the factors. Namely once one knows that the tensor product is quasi-hereditary, the Comparison Theorem [4, Theorem 5.8] can be used to derive this result.

1.4. Let $e_0 = 0, e_1, \ldots, e_n = 1_A$ be a set of idempotents in a finite dimensional k-algebra A, such that the ideals $J_i = Ae_iA$ form a strictly increasing chain

$$0 = J_0 \subsetneq J_1 \cdots \subsetneq J_n = A.$$

Let $\overline{e_i} = e_i + J_{i-1}$ be the residue class of e_i in A/J_{i-1}, and put $A_i = \overline{e_i}(A/J_{i-1})\overline{e_i}$. Assume also that $\mathrm{Ext}^k_{A/J_{i-1}}(A/J_i, A/J_i) = 0$ and $hd_{A/J_{i-1}}A/J_i < \infty$ for all $i = 1, \ldots, n$ and all $k > 0$. Assume also that the homological dimensions $hd_{A_i}({}_{A_i}\overline{e_i}(A/J_{i-1}))$ and $hd_{A_i}((A/J_{i-1})\overline{e_i}_{A_i})$ both are finite for $i = 1, \ldots, n-1$. Then by [4, (2.7)], $D^b(A)$ admits a stratification with subsequent quotients $D^b(A_i), i = 1, \ldots, n$.

THEOREM. *Assume that the above situation is given for a k-algebra A and that similar hypothesis hold for a k-algebra B, idempotents $f_1, \ldots f_m$ in B, $L_j = Bf_jB, \bar{f_j} = f_j + J_{i-1}$ and $B_j = \bar{f_j}(B/L_{j-1})\bar{f_j}, j = 1, \ldots, m$. Then for $A \otimes B$, the derived module category $D^b(A \otimes B)$ admits a stratification with subsequent quotients $D^b(A_i \otimes B_j), i = 1, \ldots, n$ and $j = 1, \ldots, m$.*

PROOF. This is trivial for $n = m = 1$. Otherwise, we may assume that $n > 1$. By induction on $m + n$, $A/J_1 \otimes B$ admits a stratification with subsequent quotients $D^b(A_i \otimes B_j), i = 2, \ldots, n$ and $j = 1, \ldots, m$. We have $\mathrm{Ext}^k_{A \otimes B}(A/J_1 \otimes B, A/J_1 \otimes B) \cong \mathrm{Ext}^k_A(A/J_1, A/J_1) \otimes B = 0$ for all $k > 0$ and $hd_{A \otimes B}(A/J_1 \otimes B) = hd_A(A/J_1) < \infty$. Moreover, the homological dimensions of $e_1A \otimes B$ and $Ae_1 \otimes B$ over $A_1 \otimes B$ are finite. Therefore, $D^b(A \otimes B)$ admits a stratification with quotients $D^b(A_1 \otimes B)$ and $D^b(A/J_1 \otimes B)$. Using induction again, we are done. ∎

1.5. If A and B above have finite global dimension, then the conclusion of Theorem 1.4 holds, provided that J_i/J_{i-1} and L_j/L_{j-1} are semi-heredity ideals in A/J_{i-1} and B/L_{j-1} for $i = 1, \ldots, n-1$ and $j = 1, \ldots, m-1$ respectively.

2. **Derived-Simplicity.**

DEFINITION. Let k be an arbitrary field, and let A be a finite dimensional k-algebra. We shall call A derived-simple, provided that there are no finite dimensional k-algebras A' and A'' such that there exists with the notation of [4, § 2] a recollement of the form

$$D^b(A') \leftrightarrows D^b(A) \leftrightarrows D^b(A'').$$

By a Grothendieck group argument, it is clear that any local k-algebra is derived-simple. By Rickard's Morita theory for derived module categories [5], it is also easy to see that for two local k-algebras A and A', their derived module categories $D^b(A)$ and $D^b(A')$ are equivalent as triangulated categories if and only if A and A' are isomorphic.

LEMMA 2.1. *Let A be a finite dimensional k-algebra, and assume that $D^b(A)$ admits a stratification with subsequent quotients $D^b(A_1), \dots, D^b(A_n)$ for finite dimensional k-algebras $A_1, \dots, A_n$. Then A has finite global dimension if and only if* $\operatorname{gldim} A_i < \infty$ *for all $i = 1, \dots, n$.*

PROOF. Since $D^b(A_i)$ can be fully embedded into $D^b(A)$, it is clear that $\operatorname{gldim} A < \infty$ implies that $\operatorname{gldim} A_i < \infty$ for $i = 1, \dots, n$.

For the converse direction, it is by induction on n sufficient to show the following. Given two algebras A' and A'' of finite global dimension and a recollement $D^b(A') \leftrightarrows D^b(A) \leftrightarrows D^b(A'')$, then $\operatorname{gldim} A < \infty$.

We use the notation of [4, § 2], and we also consider the objects of mod A in $D^b(A)$ as complexes concentrated in degree zero. Let S and T be two simple A-modules and denote by $S^{\cdot}$ and $T^{\cdot}$ the corresponding complexes in $D^b(A)$. By [4, (2.1)(d)], there are distinguished triangles

$$S' = i_! i^! S^{\cdot} \longrightarrow S^{\cdot} \longrightarrow S'' = j_* j^* S^{\cdot} \longrightarrow \quad \text{and}$$
$$T'' = j_! j^! T^{\cdot} \longrightarrow T^{\cdot} \longrightarrow T' = i_* i^* T^{\cdot} \longrightarrow$$

in $D^b(A)$. We abbreviate $\operatorname{Hom}^n_{D^b(A)}(-,-)$ by ${}^n(-,-)$, similarly for $D^b(A')$ and $D^b(A'')$. These distinguished triangles give rise to long exact sequences

$$\cdots \longrightarrow {}^n(T^{\cdot},\, S') \longrightarrow {}^n(T^{\cdot},\, S^{\cdot}) \longrightarrow {}^n(T^{\cdot},\, S'') \longrightarrow {}^{n+1}(T^{\cdot}, S') \longrightarrow \cdots,$$
$$\cdots \longrightarrow {}^n(T',\, S') \longrightarrow {}^n(T^{\cdot},\, S') \longrightarrow {}^n(T'',\, S') \longrightarrow {}^{n+1}(T', S') \longrightarrow \cdots \quad \text{and}$$
$$\cdots \longrightarrow {}^n(T',\, S'') \longrightarrow {}^n(T^{\cdot},\, S'') \longrightarrow {}^n(T'',\, S'') \longrightarrow {}^{n+1}(T', S'') \longrightarrow \cdots.$$

Since the global dimensions of A' and A'' are finite, the full embedding of $D^b(A')$ and $D^b(A'')$ into $D^b(A)$ forces that ${}^n(T', S') = 0 = {}^n(T'', S'')$ for $n >> 0$. By the adjointness properties (a) and by (b) of [4, (2.1)], we also have ${}^n(T'', S') = {}^n(i^* j_! j^! T^{\cdot}, i^! S^{\cdot}) = 0$, and ${}^n(T', S'') = {}^n(j^* i_* i^* T^{\cdot}, j^* S^{\cdot}) = 0$. Therefore, ${}^n(T^{\cdot}, S') = {}^n(T^{\cdot}, S'') = 0$ for $n >> 0$, and the first sequence yields $\operatorname{Ext}^n_A(T, S) = {}^n(T^{\cdot}, S^{\cdot}) = 0$ for $n >> 0$, as was to be shown. ∎

2.2. As a consequence of Lemma 2.1, the derived module category of the algebra B of the introduction cannot be stratified with subsequent quotients of the form $D^b(f)$ because $\operatorname{gldim} B = \infty$. As we shall see next, B even is derived-simple. So there are non-local derived simple k-algebras.

PROPOSITION. *The k-algebra B given as path algebra of the quiver with relations*

$$a \underset{\beta}{\overset{\alpha}{\rightleftarrows}} b, \quad \alpha\beta = \beta\alpha = 0$$

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is derived-simple.

PROOF. If B would not be derived-simple, by the recollement situation and Lemma 2.1, $D^b(B)$ would contain, say via a full embedding i_*, a full subcategory of the form $D^b(A)$ for a local k-algebra A of infinite global dimension. Let S be the simple A-module. Then $\mathrm{Ext}^n_A(S,S) \cong \mathrm{Hom}^n_{D^b(B)}(i_*S^{\cdot}, i_*S^{\cdot}) \neq 0$ for all $n > 0$, and clearly $i_*S^{\cdot}$ is indecomposable in $D^b(B)$. Let projB be the category of finitely generated projective B-modules, and let P and Q be the two non-isomorphic indecomposables in projB. Now $D^b(B)$ is equivalent to $K^{-,b}$(projB), the triangulated category of complexes modulo homotopy over projB which are bounded to the right and have bounded cohomology. It is easy to see that an indecomposable complex X in $K^{-,b}$(projB) is of the form

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots \longrightarrow P \longrightarrow Q \longrightarrow P \longrightarrow \cdots \longrightarrow 0 \longrightarrow \cdots \quad \text{or}$$

$$\cdots \longrightarrow P \longrightarrow Q \longrightarrow P \longrightarrow \cdots \longrightarrow 0 \longrightarrow \cdots .$$

It follows that $\mathrm{Hom}^n_{K^{-,b}(\mathrm{projB})}(X,X)$ vanishes for all odd n: contradiction.#

Further more involved examples of algebras of finite global dimension with two simple modules which are also derived simple have been recently found by Happel [3].

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