CONTINUOUS FAMILIES OF CURVES

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1. The present paper is an attempt to find the unifying principle of results obtained by different authors and dealing—in the original papers—with area-bisectors, chords, or diameters of planar convex sets, with outwardly simple planar line families, and with chords determined by a fixed-point free involution on a circle. The proofs in the general setting seem to be simpler and are certainly more perspicuous than many of the original ones. The tools required do not transcend simple continuity arguments and the Jordan curve theorem. The author is indebted to the referee for several helpful remarks.

Let C be a simple closed curve in the plane and D the bounded component of the complement of C. A family $\mathfrak{L} = \{L\}$ of simple open arcs will be called a *continuous family of curves* (in D) provided:

(i) Each $L \in \mathfrak{L}$ is contained in D, its end points are different and belong to C.

(ii) Each point $P \in C$ is an end point of one and only one curve $L = L(P)$ belonging to $\mathfrak{L}$.

(iii) If L_1 and L_2 are different curves in $\mathfrak{L}$, then $L_1 \cap L_2$ is a single point.

(iv) The curve $L(P)$ depends continuously on $P \in C$.

Using standard continuity arguments and the Jordan curve theorem, it is easily seen that each continuous family of curves $\mathfrak{L}$ has also the following properties:

(v) For different $L_1, L_2 \in \mathfrak{L}$, the end points of L_1 separate on C those of L_2.

(vi) A continuous involution is defined on C by assigning to each $P \in C$ the other end point P^* of $L(P)$.

(vii) The point $L_1 \cap L_2$ depends continuously on $L_1, L_2 \in \mathfrak{L}$.

A point $X \in D$ will be called a *multiple point* (*triple point*) of $\mathfrak{L}$ provided X belongs to at least two (three) different curves in $\mathfrak{L}$; the set of all multiple points of $\mathfrak{L}$ will be denoted by $M(\mathfrak{L})$, that of all triple points by $T(\mathfrak{L})$.

The following simple result is the source of many well-known properties of convex sets.

THEOREM 1. *Let $\mathfrak{L}$ be a continuous family of curves. With at most one exception, each curve in $\mathfrak{L}$ contains a triple point.*

Proof. Let $L_0 = L(P_0)$ be a curve in $\mathfrak{L}$, and let $X(P) = L_0 \cap L(P)$ be the

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continuous mapping of an open arc $A = (P_0, P_0^*)$ of C into L_0. Then clearly one of the following three cases arises:

(a) $X(P)$ is a strictly monotone function of $P \in A$.

(b) $X(P)$ is a monotone, but not strictly monotone, function of $P \in A$.

(c) $X(P)$ is not a monotone function of $P \in A$.

In case (a), we have a homeomorphism between A and the arc

$$\{X(P) | P \subset A\} \subset L_0,$$

and through each point of L_0 passes at most one curve $L \in \mathfrak{L}$ different from L_0. In case (b), there exists a closed arc $[P_1, P_2] \subset A$ such that $X(P)$ is constant for $P \in [P_1, P_2]$; thus there exists a point $X(P_1) \in L_0$ through which pass all the curves $L(P)$ for $P \in [P_1, P_2]$. In case (c), there exists an open arc $(X_1, X_2) \subset L_0$ such that each $X \in (X_1, X_2)$ is the image of at least two different points $P \in A$; thus a continuum of points of L_0 are triple points of $\mathfrak{L}$.

In order to establish the theorem we have to prove that $\mathfrak{L}$ contains at most one curve of type (a). Assume, on the contrary, that $L_0 = L(P_0)$ and

$$L_1 = L(P_1)$$

are both of type (a). Let P_2 be a point of A not belonging to the set

$$\{P_0, P_0^*, P_1, P_1^*\};$$

thus $L_2 = L(P_2)$ is different from both L_0 and L_1. Without loss of generality we may assume that the notation is as indicated schematically in Figure 1.

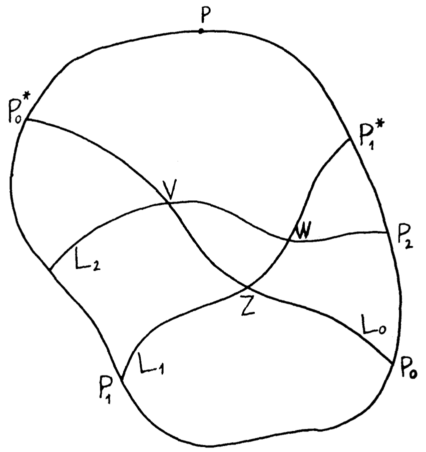

FIGURE 1

Now consider a point P belonging to that open arc (P_0^*, P_1^*) of A that does not contain P_2. Since L_0 is of type (a), the point $L_0 \cap L(P)$ belongs to the open arc (P_0, Z) of L_0; similarly the point $L_1 \cap L(P)$ belongs to the open arc (P_1, Z) of L_1. But this is impossible since (by the Jordan curve theorem)

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$L(P)$ must intersect the closed subarc of $L_0 \cup L_1$ containing P_0^*, V, Z, W, P_1^* and thus either $L_0 \cap L(P)$ or $L_1 \cap L(P)$ would consist of at least two points.

This completes the proof of Theorem 1.

It is to be noted that $\mathfrak{L}$ may indeed contain one exceptional curve L_0 of type (a), even if all the curves in $\mathfrak{L}$ are straight-line segments and D is convex. For example (see Figure 2), let C be a circle, $L_0 = L(P_0)$ a diameter of C,

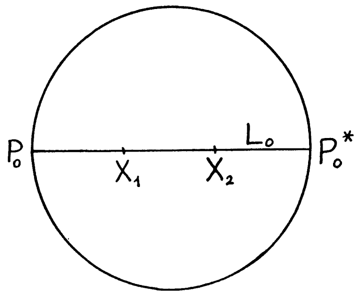

FIGURE 2

(X_1, X_2) any open subinterval of L_0, and ϕ any homeomorphism of the open half-circle (P_0, P_0^*) onto (X_1, X_2). Then L_0 and the chords of C determined by the segments $[P, \phi(P)]$ for $P \in (P_0, P_0^*)$ yield a continuous family of curves (segments) such that L_0 is of type (a).

Denoting the empty set by $\emptyset$ and the closure of a set A by $\bar{A}$, we have the following easy consequences of Theorem 1:

COROLLARY 1. *$L \cap \overline{T(\mathfrak{L})} \neq \emptyset$ for every $L \in \mathfrak{L}$.*

COROLLARY 2. *If $T(\mathfrak{L})$ consists of a single point T, then T belongs to each curve in $\mathfrak{L}$, and $M(\mathfrak{L}) = T(\mathfrak{L})$.*

COROLLARY 3. *$T(\mathfrak{L})$ contains either a point through which pass $\aleph$ different curves of $\mathfrak{L}$, or $\aleph$ different points.*

2. Many examples of continuous families of curves have been considered in the literature in connection with properties of planar convex sets.

If D is a non-empty bounded open convex set in the plane, the family of all chords of D that bisect the area of D, as well as the family of all chords that bisect the perimeter of D, are continuous families of curves. Corollary 2 implies that if the set of triple points of the family of (area, or perimeter) bisectors is reduced to a single point T, then all of them pass through T. The fact that T is the centre of symmetry of D is then easily established for either of the two familes. This characterization of centrally symmetric convex sets by area-bisectors is due to Zarankiewicz **(14)**; other proofs were given by Menon **(9)** and Piegat **(10)**.

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If D is, moreover, smooth and strictly convex, the mid-points of parallel chords of D form a simple curve; the totality of these curves (corresponding to chords of different directions) form a continuous family. In this case, Corollary 2 implies at once that if the set of triple points is reduced to a single point, the set D is centrally symmetric. This sharpens a result of Viet **(13**, Satz 2**)**.

For a smooth and strictly convex D, the (unique) longest chords of various directions form a continuous family; the carrier-lines of those chords are the *outwardly simple line families* of Hammer and Sobczyk **(7)**. For additional examples of continuous families of curves defined by geometric properties of convex sets see Zindler **(15)** and Grünbaum **(6)**. Corollary 3 implies for all these families the same type of alternative established by Steinhaus **(12)** in some related problems.

3. We turn now to a more detailed study of the sets $M(\mathfrak{L})$ and $T(\mathfrak{L})$ of multiple and triple points of a continuous family $\mathfrak{L}$. We shall consider only *non-trivial* families, i.e., families $\mathfrak{L}$ for which $T(\mathfrak{L})$ is not reduced to a single point. The structure of the sets $M(\mathfrak{L})$ and $T(\mathfrak{L})$ is, in general, rather complicated; varied examples have been given by Ceder **(2)**, Goldberg **(5)**, Hammer–Sobczyk **(7)**, and Smith **(11)**.

Let a continuous family $\mathfrak{L}$ be given; we shall call *triangle* any open domain whose boundary consists of arcs of three different curves of $\mathfrak{L}$, or of arcs of two different curves of $\mathfrak{L}$ and an arc of the boundary of D; those arcs are the *sides* of the triangle. The *vertices* of a triangle are the end points of the three sides; we shall denote a triangle by its three vertices. Thus the four triangles in Figure 3 are (P_0, P_1, X), (P_0, P_1^*, X), (P_0^*, P_1, X), and (P_0^*, P_1^*, X).

LEMMA 1. *Let $L(P_0)$ and $L(P_1)$ be different curves in $\mathfrak{L}$, let*

$$X = L(P_0) \cap L(P_1),$$

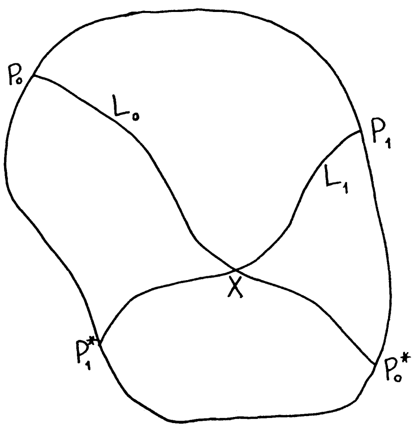

FIGURE 3

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and let (P_0, P_1) *be the arc of* C *with end points* P_0 *and* P_1 *which does not contain* P_0^* *and* P_1^*. *Then the triangle* (P_0, P_1, X) *is contained in the set*

$$\bigcup_{P \in (P_0, P_1)} L(P).$$

Proof. Let $Y \in (P_0, P_1, X)$. Consider the non-empty set S consisting of all $P \in (P_0, P_1)$ such that $Y \in (P_0, P, L(P_0) \cap L(P))$. Since $\mathfrak{L}$ is continuous, S is an open subset of (P_0, P_1) and $P_0 \notin S$. Let $R \notin S$, $R \in \bar{S}$. Then

$$Y \in \operatorname{bd}(P_0, R, L(P_0) \cap L(R)),$$

since otherwise R would belong to S. Thus $Y \in L(R)$ and the proof of Lemma 1 is completed.

Lemma 1 clearly implies the two-dimensional case of a theorem of Forrester **(4)** on fixed-point free involutions.

LEMMA 2. *If* $L_0, L_1, L_2 \in \mathfrak{L}$ *are different and do not have a point in common, then, in the notation of Figure* 4, *the triangle* (X_0, X_1, X_2) *is contained in* $T(\mathfrak{L})$, *while its sides belong to* $M(\mathfrak{L})$.

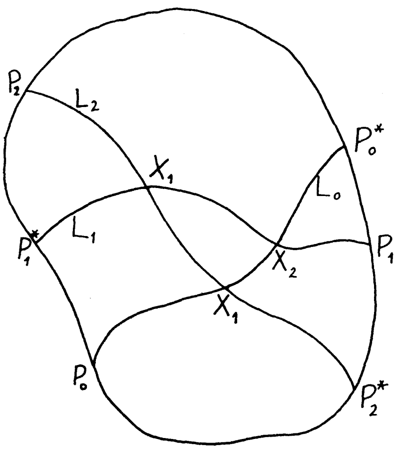

FIGURE 4

Proof. By Lemma 1, each $Y \in (X_0, X_1, X_2)$ is on a curve $L(R_0)$ with $R_0 \in (P_1, P_2^*)$, on another $L(R_1)$ with $R_1 \in (P_2, P_0^*)$, and on a third $L(R_2)$ with $R_2 \in (P_0, P_1^*)$. These lines are clearly different and thus

$$(X_0, X_1, X_2) \subset T(\mathfrak{L}).$$

Note that $L(R_0)$ meets the arc $(X_1, X_2) \subset L_0$, and similarly for $L(R_1)$ and $L(R_2)$. Again by Lemma 1, any point of L_0 that is between X_1 and X_2 belongs to some $L(R_0)$ with $R_0 \in (P_1, P_2^*)$ besides being on L_0. This clearly implies that the sides of (X_0, X_1, X_2) are contained in $M(L)$.

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The particular case of Lemma 2, in which $\mathfrak{L}$ is the family of area-bisectors of a convex domain D, was established by Piegat **(10)**. The case in which $\mathfrak{L}$ is the family of longest chords of D is due to Hammer and Sobczyk **(7)**. For the curves formed by the mid-points of parallel chords, Lemma 2 was proved by Ceder **(3)** even without the restriction that the convex set D be smooth and strictly convex.

LEMMA 3. *Each $X \in M(\mathfrak{L})$ is in the boundary of a triangle formed by curves in $\mathfrak{L}$ (and thus, by Lemma 2, is contained in $\overline{T(\mathfrak{L})}$); thus $M(\mathfrak{L}) \subset \overline{T(\mathfrak{L})}$.*

Proof. Let L_1, L_2 be distinct lines through X and let P be a point of C such that $X \notin L(P)$. Then L_1, L_2, and $L(P)$ determine a triangle having X as a vertex.

Let $\sim$ denote set-difference, and μ the 2-dimensional Lebesgue measure.

THEOREM 2. *For any continuous family of curves $\mathfrak{L}$, $\mu(M(\mathfrak{L}) \sim T(\mathfrak{L})) = 0$; if $\mathfrak{L}$ is non-trivial, then* int $T(\mathfrak{L}) \neq \emptyset$.

Proof. Let T denote the union of all the triangles determined by curves in $\mathfrak{L}$; if $\mathfrak{L}$ is non-trivial, such triangles exist and $T \neq \emptyset$. By Lemma 2, $T \subset T(\mathfrak{L})$; on the other hand, by Lemma 3, $\bar{T} \supset M(\mathfrak{L}) \supset T(\mathfrak{L})$. Since T is open, we have $\mu(T) = \mu(\bar{T})$ and int $T(\mathfrak{L}) \neq \emptyset$, while bd $T \supset (M(\mathfrak{L}) \sim T(\mathfrak{L}))$ implies that $\mu(M(\mathfrak{L}) \sim T(\mathfrak{L})) = 0$. If $\mathfrak{L}$ is trivial, then $M(\mathfrak{L}) = T(\mathfrak{L})$ is a single point. This completes the proof of Theorem 2.

If $\mathfrak{L}$ is the family of longest chords of a convex domain (their extensions forming an outwardly simple line family), Theorem 2 is a special case of a result of Hammer and Sobczyk **(7)** stating that the set of points contained in an even number or in infinitely many different members of $\mathfrak{L}$ has measure 0. It may be conjectured that this result remains valid for general continuous families of curves.

In the statement and proof of the next result we need some additional notions.

Let $K \subset D$; we shall say that K is $\mathfrak{L}$-*convex* if $L \cap K$ is connected (or empty) for each $L \in \mathfrak{L}$. K is *polygonally* $\mathfrak{L}$-*connected* provided that for each pair X, Y of points in K there is a curve in K consisting of a finite number of arcs of curves in $\mathfrak{L}$ connecting X and Y. If the minimal necessary number of arcs is less than or equal to n for each X, $Y \in K$, we shall say that K is an $L_n(\mathfrak{L})$-set; see Horn–Valentine **(8)** for the corresponding generalization of convex sets to L_n-sets and Bruckner–Bruckner **(1)** for additional results and references. Note that a polygonally $\mathfrak{L}$-connected set is connected; an $\mathfrak{L}$-convex set is not necessarily connected.

A triangle is *maximal* provided it is not properly contained in another triangle. Clearly any triangle determined by curves in $\mathfrak{L}$ is contained in a maximal triangle of the same type.

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THEOREM 3. *$M(\mathfrak{L})$ is $\mathfrak{L}$-convex and an $L_2(\mathfrak{L})$-set; $T(\mathfrak{L})$ is a polygonally $\mathfrak{L}$-connected set.*

Proof. If $X, Y \in L \cap M(\mathfrak{L})$, let L_1 and L_2 be curves in $\mathfrak{L}$ different from L and containing X and Y respectively. The $\mathfrak{L}$-convexity of $M(\mathfrak{L})$ follows at once from Lemma 2 applied to the triangle $(X, Y, L_1 \cap L_2)$. Now let $X_0, X_1 \in M(\mathfrak{L})$. In order to show that $M(\mathfrak{L})$ is an $L_2(\mathfrak{L})$-set, we may assume that X_i is on $L_i \in \mathfrak{L}$, where $X_i \notin L_{1-i}$, $i = 0, 1$. Let $X = L_0 \cap L_1$. Since $M(\mathfrak{L})$ is $\mathfrak{L}$-convex, the arc (X_i, X) of L_i is in $M(\mathfrak{L})$. This shows that $M(\mathfrak{L})$ is an $L_2(\mathfrak{L})$-set. (This proof is similar to the proof of Theorem 3.3 of Horn and Valentine **(8)**.)

In order to show that $T(\mathfrak{L})$ is polygonally $\mathfrak{L}$-connected, it is sufficient, in view of Lemmas 2 and 3, to prove the polygonal $\mathfrak{L}$-connectedness of the set $T \subset T(\mathfrak{L})$, T being, as above, the union of all the triangles determined by curves in $\mathfrak{L}$. We shall prove the polygonal $\mathfrak{L}$-connectedness of T by establishing the following two lemmas.

LEMMA 4. *Any two maximal triangles determined by curves in $\mathfrak{L}$ have a non-empty intersection.*

LEMMA 5. *Any triangle determined by curves in $\mathfrak{L}$ is an $L_2(\mathfrak{L})$-set.*

We shall first prove Lemma 5, using the notation indicated in Figure 5. Let

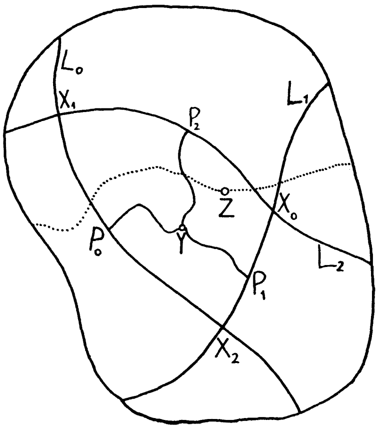

FIGURE 5

$Y \in (X_0, X_1, X_2) \subset T$ be on the curves $L^{(0)}$, $L^{(1)}$, $L^{(2)}$ (not shown in Figure 5), which, according to Lemma 2, intersect the three sides of (X_1, X_2, X_3) in points P , P_1, and P_2 respectively. If another point $Z \in (X_0, X_1, X_2)$ is on one of these curves, there is nothing to prove. Otherwise, consider the three domains into which (X_0, X_1, X_2) is divided by the arcs (Y, P_i) of $L^{(i)}$. Each of these domains is adjacent to two of the sides of the triangle (X_0, X_1, X_2). The curve L (dotted in Figure 5) that contains Z and passes through a point

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of that side of the triangle not adjacent to the domain containing Z must, by the Jordan curve theorem, intersect one of the arcs (Y, P_i). This completes the proof of Lemma 5.

Lemma 4 may be proved by deriving a contradiction from the assumption that there exists a pair of mutually disjoint maximal triangles formed by curves in $\mathfrak{L}$. In doing so one has to distinguish several different cases, according to the mutual position of the triangles. Since the procedure is similar in all the cases, we refrain from listing them all, and illustrate the idea by considering one of the possible configurations (Figure 6). Each of the two triangles (shaded)

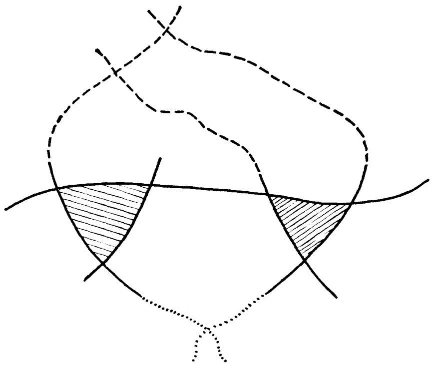

FIGURE 6

has one side on a certain curve in $\mathfrak{L}$, the two triangles being on the same side of that curve. If the extensions of the sides (indicated by the dotted lines) intersect, neither of the original triangles is maximal; if they do not intersect, then the dashed extensions must intersect, and the triangles are again not maximal.

Eliminating in this fashion all the possibilities, we prove Lemma 4, and with it also Theorem 3.

It should be noted that the above proof yields more information about $T(\mathfrak{L})$ than claimed in Theorem 3; in fact, *the set* T *is an* $L_4(\mathfrak{L})$-*set, and* $T(\mathfrak{L})$ *is an* L-$(\mathfrak{L})$-*set*.

It may be conjectured that $T(\mathfrak{L})$ is even an $L_2(\mathfrak{L})$ set.

Theorem 3 is a stronger version of Theorem 7 of Ceder **(2)**, which deals with outwardly simple line families; it overlaps with Ceder's main result **(3)**.

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