Proceedings of the Edinburgh Mathematical Society (2005) **48**, 295–336 ©
DOI:10.1017/S0013091503000506 Printed in the United Kingdom

A PROBABILISTIC REPRESENTATION FOR THE VORTICITY OF A THREE-DIMENSIONAL VISCOUS FLUID AND FOR GENERAL SYSTEMS OF PARABOLIC EQUATIONS

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(Received 12 June 2003)

Abstract A probabilistic representation formula for general systems of linear parabolic equations, coupled only through the zero-order term, is given. On this basis, an implicit probabilistic representation for the vorticity in a three-dimensional viscous fluid (described by the Navier–Stokes equations) is carefully analysed, and a theorem of local existence and uniqueness is proved. The aim of the probabilistic representation is to provide an extension of the Lagrangian formalism from the non-viscous (Euler equations) to the viscous case. As an application, a continuation principle, similar to the Beale–Kato–Majda blow-up criterion, is proved.

Keywords: Navier–Stokes equations; parabolic systems; probabilistic representation; Feynman–Kac formula; vorticity field; blow-up criterion

2000 *Mathematics subject classification:* Primary 76D05
Secondary 35A20

1. Introduction

Consider the Navier–Stokes equations in $[0,T] \times \mathbb{R}^3$

$$\left.\begin{aligned} \partial_t u + (u\cdot\nabla)u + \nabla p &= \nu\Delta u + f,\\ \operatorname{div} u &= 0,\\ u(0,x) &= u_0(x). \end{aligned}\right\} \tag{1.1}$$

These equations describe, in *Eulerian coordinates*, the evolution of a viscous incompressible Newtonian fluid, where u is the velocity field, p the pressure, f the body force and $\nu > 0$ the kinematic viscosity. The vorticity field $\xi = \operatorname{curl} u$ satisfies the equation

$$\partial_t \xi + (u\cdot\nabla)\xi = \nu\Delta\xi + (\xi\cdot\nabla)u + g, \tag{1.2}$$

with $g = \operatorname{curl} f$. As we shall remark later on, the *stretching* term $(\xi \cdot \nabla)u$ can be written in the form

$$(\xi \cdot \nabla)u = \mathcal{D}_u \xi,$$

where $\mathcal{D}_u = \frac{1}{2}(\nabla u + \nabla u^{\mathrm{T}})$, which better describes the action of the deformation tensor $\mathcal{D}_u$ on ξ. The analysis of the vorticity field is a fundamental issue related to questions like the possible emergence of singularities (see, for example, [**3**, **10**]), or the description of three-dimensional (3D) structures (see, for example, [**9**]).

The *Lagrangian* formulation of the fluid dynamics is important when analysing the vorticity field. Wide use has been made of it in the inviscid case ($\nu = 0$) (see [**28**]). Our aim is to extend this approach to the viscous case. Strictly speaking, even in the presence of viscosity, fluid particles (we mean infinitesimal portions of fluid, not the single molecules) move according to the deterministic law

$$\dot{X}(t) = u(t, X(t)).$$

However, a *virtual* Lagrangian dynamic of the particles of the form

$$\mathrm{d}X(t) = u(t, X(t))\,\mathrm{d}t + \sqrt{2\nu}\,\mathrm{d}W_t \tag{1.3}$$

(where W_t is an auxiliary 3D Brownian motion) allows us to describe the evolution of quantities which not only are transported by the fluid, but have a diffusive character. The vorticity has this property, as do many scalars or fields possibly spreading into the fluid. Roughly speaking, we prove the representation formula

$$\xi(t, x) = \boldsymbol{E}[V(t, 0)\xi_0(X(0))] + \int_0^t \boldsymbol{E}[V(t, s)g(s, X(s))]\,\mathrm{d}s$$

where $\boldsymbol{E}[\cdot]$ denotes the mean value with respect to the Wiener measure, ξ_0 is the vorticity at time zero, $X(s)$ is the solution of Equation (1.3) with final condition $X(t) = x$ and $V(r, s)$ is the solution of the 3×3 matrix equation

$$\left.\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}r} V(r, s) &= \mathcal{D}_u(r, X(r))V(r, s), \quad r \in [s, t], \\ V(s, s) &= I. \end{aligned}\right\} \tag{1.4}$$

The present paper is devoted to explaining the formula in detail, and using it to prove a local-in-time existence and uniqueness result. As an application of the formula, we also show a continuation principle similar to the criterion of Beale *et al*. [**3**] (we actually give a different proof of the variant presented in [**29**]). This paper is in a sense the continuation of that of the first author (see [**7**]), where the two-dimensional (2D) case has been considered. In the 2D case the stretching term $\mathcal{D}_u\xi$ is zero, so $V(r, s) = I$. The vorticity is purely transported and diffused, allowing for a global-in-time control which yields global existence and uniqueness results. In Busnello [**7**], the probabilistic formula is used to prove such a result, related to the deterministic work of Ben-Artzi [**4**], following a suggestion of M. Friedlin (personal communication). Here, in the 3D case, a nonlinear mechanism is present in the equation for $V(r,s)$, which may produce an increase of ξ, by the first equation above, and blow-up may appear. Similarly to the Eulerian case ($\nu = 0$), the Lagrangian structure of the equations allows one to see rather easily the influence of certain quantities (the deformation tensor $\mathcal{D}_u$ here) on this hypothetical blow-up procedure, and prove a corresponding continuation principle.

Around these ideas, let us also recall that global existence for (1.1) is known only at the level of weak solutions, but we have to work at a higher level of regularity to deal with the vorticity. In certain function spaces, global existence (and uniqueness) are known for sufficiently small data; in principle the probabilistic formulation could lead to such results, but we have found some obstacles, so a probabilistic proof of such a result remains an open problem (except for the completely different approach of Le Jan and Sznitman [**26**]).

At the technical level, Girsanov transformation is used in a basic part of the work, and the Bismut–Elworthy–Li formula is also used to treat by probabilistic methods the Biot–Savart law, which reconstructs u from ξ (necessary to solve (1.3)). In the 3D case the Biot–Savart law and its probabilistic representation are

$$u(t,x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{(x-y)\times\xi(y)}{|x-y|^3}\,\mathrm{d}y = \frac{1}{2}\int_0^\infty\frac{1}{s}\boldsymbol{E}[\xi(t,x+W_s)\times W_s]\,\mathrm{d}s.$$

The layout of the paper is as follows. In § 2 we state the precise representation formula, the local existence and uniqueness result for the Navier–Stokes equation and the continuation principle, with the main lines of their proofs. However, the full proof of the representation formula, the local result and the continuation result are based on three main items that we postpone to the next three sections:

(i) a general representation formula for linear systems of parabolic equations, which is given in § 3;

(ii) the probabilistic representation of Biot–Savart law and a number of estimates on it, which is given in § 4;

(iii) a series of estimates for the expected values appearing in the formula for the vorticity, which is given in § 5.

We have chosen this ordering to highlight the results for the Navier–Stokes equation at the beginning, for the reader who is not interested in the long list of estimates and preliminaries necessary to prove the main theorem. About item (i) above we remark that we use a method due to Krylov (in the scalar case) that introduces new variables in order to eliminate the zero-order terms of the parabolic equation. Such a method in the case of systems coupled through the zero-order part is particularly interesting because it reduces the original system to a decoupled one. The representation proved in § 3 can be applied, in principle, to several other systems of equations appearing in fluid dynamics, like the equation for u itself (but the term ∇p appears in the right-hand side), the equation for the magnetization variable (see, for example, [**9**]) and the equation for the transport of passive scalars.

Concerning the literature on the subject, at an advanced state of the present work we became aware of the interesting papers by Esposito *et al.* [**13**] and by Esposito and Pulvirenti [**12**], where a somewhat similar representation formula was introduced; this paper differs from others in that we also treat probabilistically the Biot–Savart law, we use different probabilistic tools, we analyse in detail the general case of systems of probabilistic equations in order to understand rigorously the equivalence with the probabilistic representation and we prove the local existence and uniqueness result in different function spaces (in particular, for a class of less smooth initial conditions).

There is also a paper by Rapoport [**30**] dealing with a general class of equations on manifolds which, in particular, throw light on the differential geometric structure of the formula. The probabilistic representation of systems of parabolic equations has been treated in the literature under certain assumptions (see [**16**, **22**, **31**]).

Finally, among the literature on probabilistic analysis of partial differential equations there are possible connections with the geometric approach of Gliklikh [**21**], with investigations on the vortex method in three dimensions by Meleard (see, for example, [**15**], where a similar probabilistic representation has been developed), and more closely with [**1**], where a probabilistic representation for the velocity u is employed.

Concerning the huge literature on the deterministic analysis of the Navier–Stokes equations, more refined results of local existence and uniqueness have been proved in a great amount of function spaces. Two almost-up-to-date collections of results can be found in the review papers by Cannone [**8**] and von Wahl [**32**]. The solutions we find live in rather classical spaces, similar to those considered by Majda and Bertozzi [**28**] (mainly for the Euler equation). In a different direction, an intense recent research aimed at finding the sharpest critical space (for example, of Besov type) where to solve the equations (see [**8**] for more details). At present we cannot work in such a direction since we have to solve the stochastic equation (1.3) and the linear equation (1.4), so we need a greater regularity of u. However, the analysis of stochastic equations with rough coefficients is under development, so we hope to get results in other function spaces by the probabilistic representation formula in the future.

Our results are also related to the vorticity approach to the Navier–Stokes equations. On such topics, one can refer to the work of Giga and Miyakawa [**18**], which deals with very rough vorticity, and to the papers by Giga, Miyakawa and Osada [**19**], Kato [**23**] and Giga [**17**], concerning the 2D approach. Again, at present we consider more classical spaces in order to solve Equations (1.3) and (1.4) by well-established techniques, but we believe that a generalization may be possible.

1.1. A heuristic interpretation of the probabilistic formula for the vorticity

We find that the probabilistic representation for the vorticity given above, apart from the rigorous results it may produce, gives a new mental image of the motion of the fluid in the 3D physical space and the associated transformations of the vorticity field: transport and diffusion, stretching by the cumulative action of the deformation tensor along the particle motion. Such a mental image is well known for the Euler equation; here we try to develop an extension in the presence of viscosity. Let us first recall it in the non-viscous case and then proceed to the viscous case.

1.1.1. *Evolution of the vorticity in the non-viscous case*

Let us first recall the Lagrangian formulation of the Euler equations. Let $\xi(t,x)$ be the value of the vorticity at time t and point $x \in \mathbb{R}^3$. The material point that occupies position x at time $t = 0$ moves according to the law

$$\dot{X}(t) = u(t, X(t)),$$
$$X(0) = x,$$

where u is the velocity field of the fluid. From the Eulerian description of the evolution of ξ,

$$\partial_t \xi + (u \cdot \nabla)\xi = \mathcal{D}_u \xi + g,$$

we deduce the Lagrangian formulation

$$\frac{\mathrm{d}}{\mathrm{d}t}\xi(t, X(t)) = \mathcal{D}_u(t, X(t))\xi(t, X(t)) + g(t, X(t)), \tag{1.5}$$

which gives us

$$\xi(t, X(t)) = V(t, 0)\xi_0(x) + \int_0^t V(t, s) g(s, X(s))\, \mathrm{d}s, \tag{1.6}$$

where

$$\left.\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}r}V(r, s) = \mathcal{D}_u(r, X(r))V(r, s), \quad r \in [s, t], \\ &V(s, s) = I. \end{aligned}\right\} \tag{1.7}$$

Take $g = 0$ for simplicity (the general case is similar); Equations (1.5) and (1.6) say that the initial vorticity $\xi_0(x)$ at point x is transported along the path $X(t)$, and during this motion it is modified by the deformation tensor. For instance, the vorticity is stretched when it is sufficiently aligned with the expanding directions of $\mathcal{D}_u$; of course the relative position of ξ with respect to the expanding and contracting (remember that $\operatorname{Tr} \mathcal{D}_u = 0$) directions of $\mathcal{D}_u$ changes in time, so $\xi(t, X(t))$ may undergo a complicated evolution with stretching, rotations and contractions. Heuristic reasoning and numerical experiments show a predominance of the stretching mechanism, which could yield a blow-up of $\xi(t, X(t))$ in finite time, for a certain initial point x.

If we want to know $\xi(\bar{t}, \bar{x})$ at a certain time $\bar{t}$ and position $\bar{x}$, we have to solve the backward equation

$$\left.\begin{aligned} &\dot{X}(t) = u(t, X(t)), \quad t \in [0, \bar{t}], \\ &X(\bar{t}) = \bar{x}, \end{aligned}\right\} \tag{1.8}$$

to find the initial position $x = X(0)$ which moves to $\bar{x}$ at time $\bar{t}$; then

$$\xi(\bar{t}, \bar{x}) = V^{\bar{t},\bar{x}}(\bar{t}, 0)\xi_0(X^{\bar{t},\bar{x}}(0)) + \int_0^{\bar{t}} V^{\bar{t},\bar{x}}(\bar{t}, s) g(s, X^{\bar{t},\bar{x}}(s))\, \mathrm{d}s, \tag{1.9}$$

where we have denoted by $X^{\bar{t},\bar{x}}(\cdot)$ the solution of (1.8), to stress the dependence of the final condition $\bar{x}$ at time $\bar{t}$, and by $V^{\bar{t},\bar{x}}(r,s)$ the corresponding solution of Equation (1.7).

1.1.2. *Path integral modification in the viscous case*

In the viscous case the position $X(t)$ of a material point still evolves under the deterministic equation $\dot{X}(t) = u(t, X(t))$. However, the vorticity carried by the fluid particle at time $t = 0$ is not simply transported along its motion and modified by the action of the tensor $\mathcal{D}_u$; a diffusion of ξ takes place. To describe it, let us introduce a *virtual* evolution of fluid particles,

$$\mathrm{d}X(t) = u(t, X(t))\,\mathrm{d}t + \sqrt{2\nu}\,\mathrm{d}W_t,$$

where W_t is 3D Brownian motion. In a sense, in the non-viscous case we considered only one trajectory $X^x(t)$, for every given initial position x. On the other hand, in the viscous case, we introduce an infinite family of trajectories, parametrized by the initial condition x and the noise path W_t. Let us describe the transformations to which the vorticity is subject, in this new framework.

Let $(\Omega, \mathcal{F}, \boldsymbol{P})$ be the Wiener space of 3D continuous curves starting at zero, with $\boldsymbol{P}$ being the Wiener measure, and let $W(t,\omega) = \omega(t)$ be the canonical process: the 3D Brownian motion. Let us decompose the initial vorticity $\xi_0(x)$ at every point x with respect to the measure $\boldsymbol{P}$:

$$\xi_0(x) = \int_\Omega \xi_0(x)\boldsymbol{P}(\mathrm{d}\omega).$$

The vector $\xi_0(x)P(\mathrm{d}\omega)$ will be the infinitesimal component of $\xi_0(x)$ that will travel along the path $X(t,\omega)$. The vector $\xi_0(x)\boldsymbol{P}(\mathrm{d}\omega)$ is also subject to the action $V(t,s)$ of $\mathcal{D}_u$ along the path $X(t,\omega)$ (we should write $V^x(t,s,\omega)$ to emphasize the dependence on x and ω). Therefore, given the realization ω of the Brownian motion, $\xi_0(x)\boldsymbol{P}(\mathrm{d}\omega)$ transforms at time t into the vector

$$V(t,0)\xi_0(x)\boldsymbol{P}(\mathrm{d}\omega)$$

(we consider here the case $g = 0$ for the sake of brevity).

We have now reached the following picture: we have a family of virtual particle motions $X^x(t,\omega)$, parametrized by x and ω, in place of the single true particle motion $X^x(t)$. We then decompose $\xi_0(x)$ with respect to $\boldsymbol{P}$ and observe, for every x and ω, the evolution $V(t,0)\xi_0(x)\boldsymbol{P}(\mathrm{d}\omega)$ of the infinitesimal component $\xi_0(x)\boldsymbol{P}(\mathrm{d}\omega)$ along the path $X^x(t,\omega)$.

Now, $V(t,0)\xi_0(x)\boldsymbol{P}(\mathrm{d}\omega)$ is not the total value of the vorticity field at time t and point $\bar{x} = X(t,\omega)$, but only the contribution due to the ω-evolution started from position x: other initial positions x' and other ω'-evolutions will reach the point $\bar{x}$ at time t, and we have to add all of these contributions. Therefore, to compute $\xi(\bar{t},\bar{x})$ at a certain time $\bar{t}$ and point $\bar{x}$, we have to solve the backward stochastic equation

$$\left.\begin{aligned} &\mathrm{d}X(t) = u(t, X(t))\,\mathrm{d}t + \sqrt{2\nu}\,\mathrm{d}W_t, \quad t \in [0,\bar{t}], \\ &X(\bar{t}) = \bar{x} \end{aligned}\right\}$$

to find the various positions $X(0,\omega)$ which move to $\bar{x}$ at time $\bar{t}$ under different noise paths $W(t,\omega)$. Each ω add a contribution to $\xi(\bar{t},\bar{x})$ given by

$$V^{\bar{t},\bar{x}}(\bar{t},0;\omega)\xi_0(X^{\bar{t},\bar{x}}(0;\omega))\boldsymbol{P}(\mathrm{d}\omega)$$

(see (1.9) and (1.5)), so the total vorticity $\xi(\bar{t},\bar{x})$ is given by

$$\xi(\bar{t},\bar{x}) = \boldsymbol{E}[V^{\bar{t},\bar{x}}(\bar{t},0)\xi_0(X^{\bar{t},\bar{x}}(0))].$$

This is the probabilistic representation.

2. Main result on the probabilistic representation for the vorticity

2.1. Some definitions and notation

First we recall some classical spaces, like the space $L^p(\mathbb{R}^3,\mathbb{R}^3)$ of 3D vector fields whose p-power is summable, with norm

$$\|f\|_p = \left(\int_{\mathbb{R}^3} |f(x)|^p \,\mathrm{d}x\right)^{1/p},$$

the space $C_b^k(\mathbb{R}^3,\mathbb{R}^3)$ of k-times differentiable vector fields, with norm

$$\|g\|_{C_b^k} = \sum_{|\beta|\leqslant k} \|D^\beta g\|_\infty,$$

and finally the space $C_b^{k,\alpha}(\mathbb{R}^3,\mathbb{R}^3)$ of vector fields whose kth-order derivatives are Hölder-continuous with exponent α, with norm

$$\|g\|_{C_b^{k,\alpha}} = \|g\|_{C_b^k} + [g]_{k+\alpha},$$

where

$$[g]_{k+\alpha} = \sum_{|\beta|=k} \sup_{x,y\in\mathbb{R}^3} \frac{|D^\beta g(x) - D^\beta g(y)|}{\|x-y\|^\alpha}.$$

Next we define the spaces where our problem will be set. The velocity field of Navier–Stokes equations will be in the space

$$\mathcal{U}^\alpha(T) = \{u \in C([0,T]; C_b^1(\mathbb{R}^3,\mathbb{R}^3)) \cap L^\infty(0,T; C_b^{1,\alpha}(\mathbb{R}^3,\mathbb{R}^3)) \mid \operatorname{div} u(t) = 0\}, \tag{2.1}$$

endowed with the norm

$$\sup_{0\leqslant t\leqslant T} \operatorname{ess} \|u(t)\|_{C_b^{1,\alpha}},$$

while the vorticity will be in the space

$$\mathcal{V}^{\alpha,p}(T) = C([0,T]; C_b(\mathbb{R}^3,\mathbb{R}^3) \cap L^p(\mathbb{R}^3,\mathbb{R}^3)) \cap L^\infty(0,T; C_b^\alpha(\mathbb{R}^3,\mathbb{R}^3)), \tag{2.2}$$

endowed with the norm

$$\sup_{0\leqslant t\leqslant T} \operatorname{ess} \|v(t)\|_{L^p\cap C_b^\alpha},$$

where $\|\psi\|_{C_b^\alpha\cap L^p} = \|\psi\|_p + \|\psi\|_{C_b^\alpha}$. We will use also the space

$$\mathcal{U}_M^\alpha(T) = \Big\{ u \in \mathcal{U}^\alpha(T) \;\Big|\; \sup_{t\leqslant T} \operatorname{ess} \|u(t)\|_{C_b^{1,\alpha}} \leqslant M \Big\}, \tag{2.3}$$

and the space

$$\mathcal{V}_L^{\alpha,p}(T) = \Big\{ \psi \in \mathcal{V}^{\alpha,p}(T) \;\Big|\; \sup_{t\leqslant T} \operatorname{ess} \|\psi(t)\|_{L^p\cap C_b^\alpha} \leqslant L \Big\}. \tag{2.4}$$

2.2. Probabilistic representation for the vorticity

The formulation of the 3D Navier–Stokes equations

$$\left.\begin{aligned} \partial_t u - \nu\Delta u + (u\cdot\nabla)u + \nabla P &= f, \\ \operatorname{div} u &= 0, \\ u(0,x) &= u_0(x), \\ \lim_{|x|\to\infty} u(t,x) &= 0, \end{aligned}\right\} \tag{2.5}$$

can be given in terms of the vorticity field $\xi = \operatorname{curl} u$ as

$$\begin{aligned} \partial_t\xi - \nu\Delta\xi + (u\cdot\nabla)\xi - (\xi\cdot\nabla)u &= g, \\ \xi(0,x) &= \xi_0(x), \\ \xi &= \operatorname{curl} u, \\ \operatorname{div} u &= 0, \\ \lim_{|x|\to\infty} u(t,x) &= 0, \end{aligned}$$

where $g = \operatorname{curl} f$. We shall write the term $(\xi\cdot\nabla)u$ as $(\nabla u)\xi$. Moreover, the same term can be written as $\mathcal{D}_u\xi$, where $\mathcal{D}_u$ is the *deformation tensor*, the symmetric part of ∇u,

$$\mathcal{D}_u = \tfrac{1}{2}(\nabla u + \nabla u^{\mathrm{T}}),$$

since

$$(\nabla u)\xi - \mathcal{D}_u\xi = \tfrac{1}{2}(\nabla u - \nabla u^{\mathrm{T}})\xi = \xi\times\xi = 0.$$

As we explained intuitively in § 1.1 and we shall describe rigorously in the remainder of the paper, using the representation formula of Theorem 4.4 and the generalized Feynman–Kac formula of Theorem 3.12, the formulation of Navier–Stokes equations can be given in the following way:

$$\left.\begin{aligned} \xi(t,x) &= \boldsymbol{E}[U_t^{x,t}\xi_0(X_t^{x,t})] + \int_0^t \boldsymbol{E}[U_s^{x,t} g(t-s, X_s^{x,t})]\,\mathrm{d}s, \\ u(t,x) &= \frac{1}{2}\int_0^\infty \frac{1}{s}\boldsymbol{E}[\xi(t, x+W_s)\times W_s]\,\mathrm{d}s, \end{aligned}\right\} \tag{2.6}$$

where the *Lagrangian paths* $(X_s^{x,t})_{0\leqslant s\leqslant t}$ are processes solutions of the following stochastic differential equations

$$\left.\begin{aligned} \mathrm{d}X_s^{x,t} &= -u(t-s, X_s^{x,t})\,\mathrm{d}s + \sqrt{2\nu}\,\mathrm{d}W_s, \quad s\leqslant t,\\ X_0^{x,t} &= x, \end{aligned}\right\}$$

and the *deformation matrices* $(U_s^{x,t})_{0\leqslant s\leqslant t}$ are the solutions to the following differential equations with random coefficients

$$\left.\begin{aligned} \mathrm{d}U_s^{x,t} &= U_s^{x,t}\mathcal{D}_u(t-s, X_s^{x,t})\,\mathrm{d}s, \quad s\leqslant t,\\ U_0^{x,t} &= I. \end{aligned}\right\}$$

Here $\mathcal{D}_u$ is either ∇u or the deformation tensor (the name deformation matrices of $U_s^{x,t}$ refers to the latter case). Note that, with respect to § 1, we have made a time-reversion which simplifies the mathematical analysis.

A sufficiently regular solution of the classical formulation (2.5) is a solution of (2.6) and vice versa. The main aim of this section is to show that, under suitable conditions, problem (2.6) has a unique local-in-time solution.

Theorem 2.1. *Given $p\in[1,\frac{3}{2})$, $\alpha\in(0,1)$ and $T>0$, let $\xi_0\in C_b^\alpha(\mathbb{R}^3,\mathbb{R}^3)\cap L^p(\mathbb{R}^3,\mathbb{R}^3)$ and $g\in\mathcal{V}^{\alpha,p}(T)$, and set*

$$\varepsilon_0 = \|\xi_0\|_{C_b^\alpha\cap L^p} + \int_0^T \|g(s)\|_{C_b^\alpha\cap L^p}\,\mathrm{d}s.$$

There then exists $\tau\in(0,T]$, depending only on ε_0, such that there is a unique solution $u\in\mathcal{U}^\alpha(\tau)$, with $\xi\in\mathcal{V}^{\alpha,p}(\tau)$, of problem (2.6).

Proof. The theorem will be proved using a fixed point argument. Namely, we will show that there are suitable L, M and τ such that the map $\mathcal{BS}\circ\mathcal{NS}$, where $\mathcal{BS}:\mathcal{V}^{\alpha,p}(\tau)\to\mathcal{U}_M^\alpha(\tau)$ is defined as

$$\mathcal{BS}(\xi)(t,x) = \frac{1}{2}\int_0^\infty \frac{1}{s}\boldsymbol{E}[\xi(t,x+W_s)\times W_s]\,\mathrm{d}s,$$

and $\mathcal{NS}:\mathcal{U}_M^\alpha(\tau)\to\mathcal{V}^{\alpha,p}(\tau)$ is defined as

$$\mathcal{NS}(u)(t,x) = \boldsymbol{E}[U_t^{x,t}\xi_0(X_t^{x,t})] + \int_0^t \boldsymbol{E}[U_t^{x,t}g(t-s,X_s^{x,t})]\,\mathrm{d}s,$$

is a contraction. By the Banach fixed point theorem, the claim will follow.

First, in view of Corollary 4.5, $M\geqslant\tilde{C}L$. Using Proposition 5.5, we see that $\mathcal{NS}$ maps $\mathcal{U}_M^\alpha(\tau)$ to $\mathcal{V}^{\alpha,p}(\tau)$ if

$$\mathrm{e}^{3\tau M}(1+\tau M)\varepsilon_0 \leqslant L. \tag{2.7}$$

By means of Corollary 4.5 and Proposition 5.6, $\mathcal{BS}\circ\mathcal{NS}$ is a contraction if

$$\tilde{C}C(\nu,p)C_M(\tau)\varepsilon_0 < 1, \tag{2.8}$$

where $C(\nu,p)$ is a constant depending only on p and ν, and $\lim_{\tau\to0}C_M(\tau)=0$. Hence, it is sufficient to choose τ small enough in order to have both conditions (2.7) and (2.8) verified. □

Remark 2.2. As usual, the statement of the above theorem can be read in terms of small initial data. More precisely, for each fixed time T, there is a constant ε such that, if $\varepsilon_0 \leqslant \varepsilon$, there exists a unique solution $u \in \mathcal{U}^\alpha(T)$, with $\xi \in \mathcal{V}^{\alpha,p}(T)$, of problem (2.6).

2.3. A continuation principle

In this section, as an application of the probabilistic representation we have developed in the previous section, we prove a continuation principle for problem (2.5), similar to the celebrated blow-up criterion presented in [**3**]. We state it in the variant of Ponce [**29**], which says that a mild control of the magnitude of the deformation tensor gives global existence. Such a form of the principle is particularly suited for our representation (and indeed, the proof is almost straightforward), because the probabilistic formula tends to emphasize the role of the deformation tensor (see also Remark 2.4). Without loss of generality, we can assume that $g \equiv 0$.

Theorem 2.3 (continuation principle). *Assume that for each $T > 0$ there is an M_1 such that*

$$\int_0^T \|\mathcal{D}_u(s)\|_{L^\infty} \leqslant M_1.$$

Then, for each $T > 0$ there is an $M > 0$ such that $u \in \mathcal{U}^\alpha_M(T)$.

If for each $M > 0$ there is a maximal existence time T_M for the solution $u \in \mathcal{U}^\alpha_M(T_M)$ and $\lim_{M\to+\infty} T_M = T_*$, then

$$\lim_{t\uparrow T_*} \int_0^t \|\mathcal{D}_u(s)\|_{L^\infty} = +\infty.$$

Proof. From Theorem 2.1 we know that if the initial condition has finite $C^\alpha_b \cap L^p$ norm, then there is a unique local solution. Hence, it is possible to continue the solution as long as the norm $\|\xi(t)\|_{C^\alpha_b\cap L^p}$ is finite. By virtue of Lemma 2.5, such a claim is true if the integral

$$\int_0^t \|\nabla u(s)\|_{L^\infty}\,\mathrm{d}s$$

is bounded. As final step of the proof, we show that the integral above is bounded by $\int_0^t \|\mathcal{D}_u\|_{L^\infty}$. Indeed, since $\mathcal{D}_u$ is the symmetric part of ∇u and the entries of the antisymmetric part are given by the components of ξ, it is straightforward that

$$\|\nabla u\|_{L^\infty} \leqslant \|\mathcal{D}_u\|_{L^\infty} + \|\xi\|_{L^\infty}.$$

Finally, using Lemma 5.2,

$$|\xi(t,x)| = |\boldsymbol{E} U^{x,t}_t \xi_0(X^{x,t}_t)| \leqslant \|\xi_0\|_{L^\infty} \exp\left\{\int_0^t \|\mathcal{D}_u\|_{L^\infty}\,\mathrm{d}s\right\},$$

and hence the theorem follows. □

Remark 2.4. The original blow-up criterion given by Beale *et al.* [**3**] involved the estimate of the L^∞ norm of the vorticity, rather than the same norm of the deformation tensor, as in the above theorem. The double exponential estimate of Lemma 2.5 prevents us from deducing the original criterion from our representation.

We conclude the section by giving the technical lemma we have used in the proof of the above theorem.

Lemma 2.5. *We have*

$$\|\xi(t)\|_{C_b^\alpha \cap L^p} \leqslant F\biggl(\|\xi_0\|_{C_b^\alpha \cap L^p}, \int_0^t \|\nabla u(s)\|_\infty \, \mathrm{d}s\biggr)$$

for a function F, which is given explicitly in the proof.

Proof. First, using (5.3),

$$|\xi(t,x)| = |\boldsymbol{E} U_t^{x,t} \xi_0(X_t^{x,t})| \leqslant \exp\biggl\{\int_0^t \|\nabla u\|_\infty\biggr\}\|\xi_0\|_{L^\infty},$$

and, from formula (5.2),

$$\begin{aligned}\|\xi(t)\|_{L^p}^p &\leqslant \int_{\mathbb{R}^3} \boldsymbol{E}|U_t^{x,t}\xi_0(X_t^{x,t})|^p \, \mathrm{d}x \\ &\leqslant \exp\biggl\{p\int_0^t \|\nabla u\|\biggr\}\int_{\mathbb{R}^3} |\xi_0(X_t^{x,t})|^p \, \mathrm{d}x \\ &= \|\xi_0\|_{L^p}^p \exp\biggl\{p\int_0^t \|\nabla u\|_\infty\biggr\}.\end{aligned}$$

Finally, if x, $y \in \mathbb{R}^3$,

$$\begin{aligned}|\xi(t,x) - \xi(t,y)| &\leqslant \boldsymbol{E}|U_t^{x,t}\xi_0(X_t^{x,t}) - U_t^{y,t}\xi_0(X_t^{y,t})| \\ &\leqslant \boldsymbol{E}|U_t^{x,t}| \cdot |\xi_0(X_t^{x,t}) - \xi_0(X_t^{y,t})| + \boldsymbol{E}|\xi_0(X_t^{y,t})| \cdot |U_t^{x,t} - U_t^{y,t}| \\ &\leqslant \exp\biggl\{(1+\alpha)\int_0^t \|\nabla u\|\biggr\}\biggl([\xi_0]_\alpha + \|\xi_0\|_{L^\infty}\int_0^t [\nabla u(s)]_\alpha \, \mathrm{d}s\biggr)|x-y|^\alpha,\end{aligned}$$

and, since $[\nabla u]_\alpha \leqslant C[\xi]_\alpha$ (it is a classical singular integral operator estimate, see, for example, [**28**]), we get

$$[\xi(t)]_\alpha \leqslant \mathrm{e}^{(1+\alpha)A(t)}[\xi_0]_\alpha + C\|\xi_0\|_{L^\infty}\mathrm{e}^{(1+\alpha)A(t)}\int_0^t [\xi(s)]_\alpha \, \mathrm{d}s$$

and, by Gronwall's lemma, the proof of the estimate is complete. □

3. The Feynman–Kac formula for a deterministic system of parabolic equations

This section is devoted to the development of a probabilistic representation formula for the following system of parabolic equations with the final condition

$$\left.\begin{aligned} \partial_t v_k + \frac{1}{2}\sum_{i,j} a_{ij}\partial^2_{x_i x_j} v_k + \sum_{i=1}^d b_i \partial_{x_i} v_k + (\mathcal{D}v)_k + f_k = 0, \\ v_k(T,x) = \varphi_k(x), \quad x \in \mathbb{R}^d, \end{aligned}\right\} \quad k = 1,\dots,l, \tag{3.1}$$

for $(t,x) \in [0,T]\times\mathbb{R}^d$, or the following system of parabolic equations with initial condition

$$\left.\begin{aligned} \partial_t v_k = \frac{1}{2}\sum_{i,j} a_{ij}\partial^2_{x_i x_j} v_k + \sum_{i=1}^d b_i \partial_{x_i} v_k + (\mathcal{D}v)_k + f_k, \\ v_k(0,x) = \varphi_k(x), \quad x \in \mathbb{R}^d, \end{aligned}\right\} \quad k = 1,\dots,l, \tag{3.2}$$

where $a = \sigma\sigma^*$ and

$$\left.\begin{aligned} \sigma : [0,T]\times\mathbb{R}^d \to \mathbb{R}^{d\times d}, \\ b : [0,T]\times\mathbb{R}^d \to \mathbb{R}^d, \\ \mathcal{D} : [0,T]\times\mathbb{R}^d \to \mathbb{R}^{l\times l}, \\ \varphi : \mathbb{R}^d \to \mathbb{R}^l, \\ f : [0,T]\times\mathbb{R}^d \to \mathbb{R}^l \end{aligned}\right\} \tag{3.3}$$

are Borel-measurable functions. Additional assumptions will be stated below.

At first, for simplicity, assume that $f \equiv 0$ and all the data are regular. If $l = 1$, the equation (3.1), with the final condition, has a unique solution given by the Feynman–Kac formula:

$$v(t,x) = \boldsymbol{E}\left[\varphi(X_T^{t,x}) \exp\left\{\int_t^T \mathcal{D}(r, X_r^{t,x})\,\mathrm{d}r\right\}\right],$$

where $X_s^{t,x}$ is the solution of the stochastic differential equation (SDE)

$$\left.\begin{aligned} \mathrm{d}X_s^{t,x} &= b(s, X_s^{t,x})\,\mathrm{d}s + \sigma(s, X_s^{t,x})\,\mathrm{d}W_s, \quad s \in [t,T], \\ X_t^{t,x} &= x, \end{aligned}\right\} \tag{3.4}$$

where $(W_t)_{t\geqslant 0}$ is a d-dimensional Brownian motion on some filtered probability space. Our aim is to extend such a formula to the case $l > 1$.

Note that, in the case $l = 1$, for each ω, the function

$$u_r^{t,x} = \exp\left\{\int_t^r \mathcal{D}(s, X_s^{t,x})\,\mathrm{d}s\right\}$$

is the solution of the following equation (now $\mathcal{D}$ is a scalar):

$$\left.\begin{aligned} \mathrm{d}u_r^{t,x} &= u_r^{t,x}\mathcal{D}(r, X_r^{t,x})\,\mathrm{d}r, \quad r \in [t,T], \\ u_t^{t,x} &= 1. \end{aligned}\right\} \tag{3.5}$$

So, in the same way, in the case $l > 1$, we will consider the process $U^{t,(x,Y)}$, the solution of the equation

$$\left.\begin{aligned} \mathrm{d}U_r^{t,(x,Y)} &= U_r^{t,(x,Y)}\mathcal{D}(r, X_r^{t,x})\,\mathrm{d}r, \quad r \in [t,T], \\ U_t^{t,(x,Y)} &= Y, \end{aligned}\right\} \tag{3.6}$$

where now both $\mathcal{D}$ and $U^{t,(x,Y)}$ are $l \times l$ matrices. If $Y \equiv I$, we may write $U^{t,x}$ in place of $U^{t,(x,I)}$. Now, the natural conjecture is that, under suitable regularity conditions, the solution of (3.1) is

$$v(t,x) = \boldsymbol{E}[U_T^{t,x}\varphi(X_T^{t,x})]. \tag{3.7}$$

In § 3.1 we will prove (3.7), under suitable regularity conditions on the coefficients. Such a formula needs to be modified in order to handle the case $f \not\equiv 0$, as we show in § 3.2. In § 3.3 we shall provide sufficient conditions for the uniqueness of strong solutions to system (3.1). Finally, in § 3.4 we shall give a Feynman–Kac representation for the solutions of the system (3.2), with an initial condition.

Remark 3.1. When $l = 1$, we can write both $u_r^{t,x}\mathcal{D}$ and $\mathcal{D}u_r^{t,x}$ without distinction in formula (3.5), since they are both scalars. If $l > 1$, the lack of commutativity for the matrix products gives the result that $U_r^{t,x}\mathcal{D}$ and $\mathcal{D}U_r^{t,x}$ are different. The choice in the order of the matrix product in Equation (3.6), and in formula (3.7), derives from the form of the term $\mathcal{D} \cdot v$ in system (3.1). In order to understand this fact, the reader can see the computations in the proof of the uniqueness in Proposition 3.9 (it is convenient to take $f \equiv 0$ for simplicity). However, when one uses backward stochastic equations to represent solutions, the order of U_r and $\mathcal{D}$ in Equation (3.6) changes (see § 3.4).

3.1. The homogeneous case

Throughout this section, we will assume

$$f \equiv 0$$

and that the functions b, σ and $\mathcal{D}$, given in (3.3), are Borel-measurable functions such that

(A_1) b, σ are sublinear with respect to x, uniformly in t,

(A_2) b, σ are locally Lipschitz-continuous in x, uniformly in t,

(A_3) a is differentiable in x and $\partial_{x_i}a$ are locally Lipschitz-continuous in x, uniformly in t,

(A_4) $\mathcal{D}$ is bounded and locally Lipschitz-continuous in x, uniformly in t,

(A_5) φ is bounded and continuous.

In particular, assumptions (A_1), (A_2) and (A_4) ensure the existence of strong solutions, having the property of uniqueness in law, for Equations (3.4) and (3.6). Moreover, from Assumption (A_4), it easily follows that

$$\|U_T^{t,x}\|_{\mathbb{R}^{l\times l}} \leqslant \mathrm{e}^{T\|\mathcal{D}\|_\infty}, \tag{3.8}$$

where $\|\mathcal{D}\|_\infty$ is the sup-norm. Finally, the previous formula and Assumption (A_5) imply that the function v given by (3.7) is well defined and bounded.

We can now state the main result of this section.

Theorem 3.2. *Assume* (A_1)–(A_5) *and* $\varphi \in C_b(\mathbb{R}^d, \mathbb{R}^l)$. *Then the function*

$$v(t,x) = \boldsymbol{E}[U_T^{t,x}\varphi(X_T^{t,x})]$$

is continuous and bounded and solves the Kolmogorov equation (3.1) in the sense of distributions, i.e.

$$\int_0^T \int_{\mathbb{R}^d} vM^*\eta \,\mathrm{d}x\,\mathrm{d}t = 0, \quad \text{for all } \eta \in C_c^\infty((0,T)\times\mathbb{R}^d, \mathbb{R}^l), \tag{3.9}$$

where

$$M^*\eta = -\partial_t\eta + \frac{1}{2}\sum_{ij}\partial^2_{x_ix_j}(a_{ij}\eta) - \sum_i \partial_{x_i}(b_i\eta) + \mathcal{D}^*\eta. \tag{3.10}$$

Remark 3.3. The operator M^* makes sense since, by Assumptions (A_2), (A_3), the functions b_i, $\partial_{x_j}a_{ij}$ are Lipschitz-continuous, and hence by Rademacher's theorem (see, for example, [**14**]), almost everywhere (a.e.) differentiable. Consequently, the functions $\partial_{x_ix_j}a_{ij}$ and $\partial_{x_i}b_i$ are well defined a.e. and essentially bounded on compact sets. Moreover, $M^*\eta$ is bounded in compact sets.

To prove Theorem 3.2, we shall use the *method of new variables* given by Krylov [**24**]. Krylov used such a method in order to transform a parabolic equation on $\mathbb{R}^d\times[0,T]$ with a potential term into a parabolic equation on $\mathbb{R}^{d+2}\times[0,T]$ without a potential term. As observed in § 1, we extend this method to systems of parabolic equations. In our case, the elimination of the potential term has the additional advantage that the coupling between the equations in (3.1) disappears. In other words, we turn the system (3.1) into a system of l *independent* parabolic equations on $\mathbb{R}^{d+l\times l}\times[0,T]$ without the potential term.

We define the new variables $\bar{x} = (x, Y) \in \mathbb{R}^{d+l\times l}$, and, for each function $\psi : \mathbb{R}^d \to \mathbb{R}^l$, we define the function $\bar{\psi} : \mathbb{R}^{d+l\times l} \to \mathbb{R}^l$ as $\bar{\psi}(\bar{x}) = Y\psi(x)$. Finally, if $u(t,x) : [0,T]\times\mathbb{R}^d \to \mathbb{R}^l$, we set $\bar{u}(t,\bar{x}) = Yu(t,x)$.

Prior to the computation of the derivatives of $\bar{v}$, we give some notation. We denote by $\mathbf{0}_{m\times n}$ the $m\times n$ matrix with all entries equal to zero. Given a column vector $\alpha \in \mathbb{R}^d$ and an $l\times l$ matrix A, we define the $(d+l)$ (*exotic*) column vector $\left[\begin{smallmatrix}\alpha\\ A\end{smallmatrix}\right]$, where the first d rows are given by the components of α and the other l rows are the rows of A (the apparent inconsistency is inessential, since we shall only use the scalar product defined below). The scalar product between two such vectors is defined as

$$\left\langle \begin{bmatrix}\alpha\\ A\end{bmatrix}, \begin{bmatrix}\beta\\ B\end{bmatrix} \right\rangle = \alpha\cdot\beta + \langle A : B\rangle,$$

where, as usual, $\langle A : B\rangle = \mathrm{Tr}(A\cdot B) = \sum_{i,j=1}^l A_{ij}Bij$.

Given $u \in C^1([0,T]\times\mathbb{R}^d;\mathbb{R}^l)$, since

$$\frac{\partial\bar{u}_h}{\partial Y_{ij}} = \frac{\partial(Yu)_h}{\partial Y_{ij}} = \frac{\partial}{\partial Y_{ij}}\sum_k Y_{hk}u_k = \delta_{ih}u_j, \quad h = 1,\dots,l, \tag{3.11}$$

it follows that, for each $h = 1, \dots, l$, the gradient $\nabla_{\bar{x}}\bar{u}_h$ of $\bar{u}_h$ with respect to all its variables is given by the following (exotic) column vector:

$$\nabla_{\bar{x}}\bar{u}_h = \begin{bmatrix} \nabla_x (Yu)_h \\ \nabla_Y \bar{u}_h \end{bmatrix} = \begin{bmatrix} \nabla_x (Yu)_h \\ \mathbf{0}_{1\times l} \\ \vdots \\ u \\ \vdots \\ \mathbf{0}_{1\times l} \end{bmatrix},$$

where the d-column vector is the gradient with respect to x and the $l \times l$ matrix has all its rows equal to the l-dimensional vector $\mathbf{0}_{1\times l} = (0, \dots, 0)^{\mathrm{T}}$ except for the hth, which is the vector u.

Next we evaluate the scalar product

$$\left\langle \begin{bmatrix} b \\ Y\mathcal{D} \end{bmatrix}, \nabla_{\bar{x}}\bar{u}_h \right\rangle.$$

Since

$$(Y\mathcal{D})_{ij}\partial_{Y_{ij}}(Yu)_h = (Y\mathcal{D})_{ij}\delta_{ih}u_j = (Y\mathcal{D})_{hj}u_j\delta_{ih},$$

it follows that

$$\left\langle \begin{pmatrix} b \\ Y\mathcal{D} \end{pmatrix}, \nabla_{\bar{x}}\bar{u}_h \right\rangle = b \cdot \nabla_x (Yu)_h + (Y\mathcal{D}u)_h.$$

In particular, if $Y = I$, the above quantity is equal to $b \cdot \nabla_x u_h + (\mathcal{D}u)_h$.

Let

$$\alpha(t, \bar{x}) = \begin{pmatrix} a(t,x) & \mathbf{0}_{d\times l^2} \\ \mathbf{0}_{l^2\times d} & \mathbf{0}_{l^2\times l^2} \end{pmatrix}, \qquad \beta(t, \bar{x}) = \begin{bmatrix} b(t,x) \\ Y\mathcal{D}(t,x) \end{bmatrix}, \tag{3.12}$$

where we understand that α is defined in blocks, where each entry is a matrix itself (note that $D^2_{\bar{x}}\bar{u}_h$ is also defined in blocks, and the product $\langle \alpha : D^2_{\bar{x}}\bar{u}_h \rangle$ is defined as the sum of the four $\langle \cdot : \cdot \rangle$-products of the corresponding blocks). With this notation, if $u \in C^{1,2}([0,T] \times \mathbb{R}^d; \mathbb{R}^l)$, we have, for each $h = 1, \dots, l$,

$$\begin{aligned} \partial_t \bar{u}_h + \tfrac{1}{2}\langle \alpha : D^2_{\bar{x}}\bar{u}_h \rangle + \langle \beta, \nabla_{\bar{x}}\bar{u}_h \rangle & \\ = (Y\partial_t u)_h + \frac{1}{2}\sum_{i,j} a_{ij}\partial^2_{x_i x_j}(Yu)_h + \sum_i b_i \partial_{x_i}(Yu)_h + (Y\mathcal{D}u)_h & \\ = \left[Y\left(\partial_t u + \frac{1}{2}\sum_{i,j} a_{ij}\partial^2_{x_i x_j}u + \sum_i b_i\partial_{x_i}u + \mathcal{D}u \right) \right]_h. & \end{aligned}$$

From this identity it is straightforward to prove that a field u is a strong solution of (3.1) if and only if $\bar{u}$ is a strong solution of system (3.13), where by ‘strong solution’ we mean a continuous function having continuous first derivatives in time and second derivatives in space, and satisfying the corresponding equation pointwise. In the same way, applying the same ideas used above on the adjoint operator, we have the following equivalence.

Proposition 3.4. *A function u is a weak solution of system (3.1), with a final condition, if and only if $\bar{u}$ is a weak solution of*

$$\partial_t \bar{u}_h + \tfrac{1}{2}\langle \alpha : D^2_{\bar{x}} \bar{u}_h \rangle + \langle \beta, \nabla_{\bar{x}} \bar{u}_h \rangle = 0, \quad h = 1, \dots, l, \tag{3.13}$$

with the final condition $\bar{u}(T, \bar{x}) = Y\varphi(x)$.

Below we prove that, under suitable conditions, the vector field $\bar{v}(t, \bar{x}) = Yv(t, x)$, where v is given by (3.7), is a weak solution of (3.13). In view of Proposition 3.4, this implies that the function given by (3.7) solves system (3.1) in the weak sense.

The main part of the proof that $\bar{v}(t, \bar{x})$ is a weak solution of (3.13) is contained in the following proposition, where we relax some regularity assumptions on the coefficients of a theorem of Krylov [**24**]. Indeed, the drift and the diffusion defined in formulae (3.12) are neither bounded nor globally Lipschitz-continuous, in contrast to the assumptions of [**24**]. The same problem occurs for the final condition. On the other hand, both the drift and the diffusion are locally Lipschitz-continuous and have linear growth (in all variables, including Y).

Proposition 3.5. *Let $m \in \mathbb{N}$ and consider the scalar parabolic equation*

$$\partial_t u + \tfrac{1}{2}\langle \alpha : D^2 u \rangle + \langle \beta, \nabla u \rangle = 0, \quad (t, x) \in [0, T] \times \mathbb{R}^m, \tag{3.14}$$

with final condition $u(T, x) = \psi(x)$, where $\alpha = \gamma\gamma^*$ and

$$\beta : [0, T] \times \mathbb{R}^m \to \mathbb{R}^m, \qquad \gamma : [0, T] \times \mathbb{R}^m \to \mathbb{R}^{m \times m}, \qquad \psi : [0, T] \times \mathbb{R}^m \to \mathbb{R},$$

and assume that

(i) *β, γ are Borel measurable, sublinear and locally Lipschitz-continuous in x, uniformly in t,*

(ii) *ψ is continuous and has polynomial growth,*

(iii) *$\gamma(t, \cdot)$ is continuously differentiable for each t and $\partial_{x_i}\gamma$ are locally Lipschitz-continuous in x, uniformly in t.*

Set $u(t, x) = \boldsymbol{E}[\psi(Z_T^{t,x})]$, where $Z_r^{t,x}$ is the solution of the SDE

$$\left.\begin{aligned} \mathrm{d}Z_r^{t,x} &= \beta(r, Z_r^{t,x})\,\mathrm{d}r + \gamma(r, Z_r^{t,x})\,\mathrm{d}W_r, \quad r \in [t, T], \\ Z_t^{t,x} &= x, \end{aligned}\right\}$$

where $(W_t)_{t \geqslant 0}$ is an m-dimensional standard Brownian motion. Then u is a weak solution of (3.14): for each $\eta \in C_c^\infty((0, T) \times \mathbb{R}^m)$, we have

$$\int_0^T \int_{\mathbb{R}^m} u N^* \eta \,\mathrm{d}x\,\mathrm{d}t = 0,$$

where

$$N^*\eta = -\partial_t \eta + \frac{1}{2}\sum_{i,j} \partial^2_{x_i x_j}(\alpha_{ij}\eta) - \sum_i \partial_{x_i}(\beta_i \eta).$$

Proof. If everywhere in the assumptions of the proposition we have global Lipschitz-continuity (instead of local Lipschitz-continuity), the proposition follows from [**24**, Theorem 5.13]. In the general case, we proceed by truncation. Let $\Psi_n \in C^\infty(\mathbb{R}^m)$ be such that

$$\Psi_n(x) = \begin{cases} 1, & |x| \leqslant n, \\ 0, & |x| \geqslant n+1, \end{cases}$$

and set $\beta^{(n)} = \Psi_n \beta$ and $\gamma^{(n)} = \Psi_n \gamma$. Fix a Brownian motion $(\Omega, \mathcal{F}, \mathcal{F}_t, W_t, \boldsymbol{P})$ and denote by $Z_t^{s,x,n}$ the solutions to the corresponding SDEs. The sequence $Z_t^{s,x,n}$ converges to $Z_t^{s,x}$ in probability uniformly on compact subsets of $[0,T] \times \mathbb{R}^m$.

Suppose first that ψ is bounded. Then $u_n(t,x) = \boldsymbol{E}[\psi(Z_t^{s,x,n})]$ converges to $u(t,x)$ and $\beta_{x_i}^{(n)}$ converges to β_{x_i}, $\partial_{x_i}\alpha^{(n)}$ to $\partial_{x_i}\alpha$ and $\partial_{x_i,x_j}\alpha^{(n)}$ to $\partial_{x_i,x_j}\alpha$ uniformly on compact subsets of $[0,T] \times \mathbb{R}^m$. Let $\eta \in C_c^\infty$, since $N_n^*\eta$ is a bounded sequence (see Remark 3.3), by the dominated convergence theorem, $\int u_n N_n^* \eta$ converges to $\int u N^* \eta$, where N_n^* is the operator corresponding to the approximate coefficients. Since u_n are weak solutions, it follows that u is also a weak solution.

If ψ is not bounded, we take a sequence of bounded continuous functions $\psi_n \to \psi$ such that $|\psi_n(x)| \leqslant |\psi(x)|$. From [**24**, Theorem 4.6], we have $\boldsymbol{E}[|Z_T^{t,x}|^k] \leqslant c(1+|x|^k)$, so that $u_n(t,x) \leqslant c(1+|x|^k)$ by Assumption (ii), and again we conclude by the dominated convergence theorem. □

We are now ready to prove the main theorem.

Proof of Theorem 3.2. First we show that v is bounded and continuous. The boundedness comes from (3.8) and the assumptions on φ. In order to show the continuity, we take a sequence (x_n, t_n) converging to (x,t). From [**24**, Lemma 2.9], the function $(t,x) \to (X^{t,x}, U^{t,(x,I)}) \in C([0,T], \mathbb{R}^{d+l\times l})$ (where by convention $(X_s^{t,x}, U_s^{t,(x,I)}) = (x,I)$ if $s < t$) is continuous in probability. Hence, there is a subsequence such that convergence is almost sure. Finally, the conclusion follows from the bound (3.8), the assumptions on φ and the dominated convergence theorem.

We show then that v is a weak solution. We have the following two ingredients.

(i) The two systems of SDEs (3.4) and (3.6) can be thought of as a unique system where the solution $(X_r^{t,x}, U_r^{t,(x,Y)})$ takes values in $\mathbb{R}^{d+l\times l}$ and drift and diffusion are given by (3.12).

(ii) Since, by uniqueness for Equation (3.6), it follows that $U_T^{t,(x,Y)} = YU_T^{t,x}$, for the function v defined in (3.7), we have

$$\begin{aligned} \bar{v}(t,\bar{x}) = Yv(t,x) &= \boldsymbol{E}[YU_T^{t,x}\varphi(X_T^{t,x})] \\ &= \boldsymbol{E}[U_T^{t,(x,Y)}\varphi(X_T^{t,x})] \\ &= \boldsymbol{E}[\bar{\varphi}(X_T^{t,x}, U_T^{t,(x,Y)})]. \end{aligned}$$

From these two facts, by Proposition 3.5, $\bar{v}$ is a weak solution to system (3.13). By Proposition 3.4, v is a weak solution to system (3.1). □

The regularity assumption (A_4) on the term $\mathcal{D}$ can be relaxed with the following condition:

(A'_4) $\mathcal{D}$ is bounded and uniformly continuous.

In fact, we can deduce the following corollary.

Corollary 3.6. *Assume* (A_1)–(A_3), (A'_4) *and* (A_5). *Then the function*

$$v(t,x) = \boldsymbol{E}[U_T^{t,x}\varphi(X_T^{t,x})]$$

is continuous and bounded and solves the Kolmogorov equation (3.1) in the sense of distributions:

$$\int_0^T \int_{\mathbb{R}^d} vM^*\eta \,\mathrm{d}x\,\mathrm{d}t = 0 \quad \textit{for all } \eta \in C_c^\infty((0,T)\times\mathbb{R}^d,\mathbb{R}^d).$$

Proof. Let ρ_n be a sequence of mollifiers and set $\mathcal{D}_n = \mathcal{D} * \rho_n$ and $v_n(t,x) = \boldsymbol{E}[U_{T,n}^{t,x}\varphi(X_T^{t,x})]$, where $U_{r,n}^{t,x}$ is the solution of (3.6) corresponding to $\mathcal{D}_n$.

Since $\mathcal{D}_n \to \mathcal{D}$ uniformly in $[0,T]\times\mathbb{R}^d$, we have $U_{t,n}^{t,x} \to U_t^{t,x}$ in $L^1(\Omega)$, uniformly in $[0,T]\times\mathbb{R}^d$. Consequently, $v_n(t,x)\to v(t,x)$ and $\mathcal{D}_n v_n \to \mathcal{D}v$ uniformly $[0,T]\times\mathbb{R}^d$. Since v_n are weak solutions of the corresponding approximate problem, in the limit v is a weak solution of $Mv = 0$. □

3.2. The inhomogeneous case

In this section, Theorem 3.2 will be extended to the inhomogeneous case. We will show a Feynman–Kac representation formula for the complete system (3.1), i.e. with $f \not\equiv 0$, with the final condition. Throughout this section we will assume (A_1)–(A_3), (A'_4), (A_5) and the following:

(A_6) $f : [0,T]\times\mathbb{R}^d \to \mathbb{R}^l$ is bounded and uniformly continuous.

Theorem 3.7. *Assume* (A_1)–(A_3), (A'_4), (A_5), (A_6). *Then the function*

$$v(t,x) = \boldsymbol{E}[U_T^{t,x}\varphi(X_T^{t,x})] + \int_t^T \boldsymbol{E}[U_r^{t,x} f(r, X_r^{t,x})]\,\mathrm{d}r \tag{3.15}$$

is a weak solution of (3.2), i.e.

$$\int_0^T \int_{\mathbb{R}^d} (uM^*\eta + f\eta)\,\mathrm{d}t\,\mathrm{d}x = 0, \quad \eta \in C_c^\infty((0,T)\times\mathbb{R}^d,\mathbb{R}^l).$$

The main idea to prove the theorem is to introduce a new component (we apply again the *method of new variables* of Krylov [**24**]) and prove that v is a solution of system (3.1) if and only if $\tilde{v} = (v_1,\ldots,v_l,1)$ solves the system

$$\partial_t\tilde{v} + \frac{1}{2}\sum_{i,j} a_{ij}\partial_{x_i x_j}\tilde{v} + \sum_i b_i\partial_{x_i}\tilde{v} + (\tilde{\mathcal{D}}\tilde{v})_k = 0, \tag{3.16}$$

with final condition

$$\tilde{v}(T,\cdot) = (\varphi_1, \dots, \varphi_l, 1), \quad \text{where } \tilde{\mathcal{D}} = \begin{pmatrix} \mathcal{D} & f \\ 0 & 0 \end{pmatrix}.$$

Note that

$$\tilde{\mathcal{D}}\tilde{v} = \begin{pmatrix} \mathcal{D}v + f \\ 0 \end{pmatrix},$$

so the component $\tilde{v}_{l+1}$ is obviously a solution.

The key lemma follows.

Lemma 3.8. *The function $\tilde{v} = (v_1, v_2, \dots, v_l, 1)$ is a weak solution of (3.16) if and only if $v = (v_1, v_2, \dots, v_l)$ is a weak solution of (3.1).*

Proof. A weak solution of (3.1) is a function v such that

$$\iint (vM^*\eta + f\eta) = 0$$

for each test function η, or, equivalently,

$$\iint (vL^*\eta + v\mathcal{D}^*\eta + f\eta) = 0,$$

where the operator M^* has been defined in (3.10) and L^* is defined as

$$L^*\eta = -\partial_t\eta + \frac{1}{2}\sum_{i,j} \partial^2_{x_i x_j}(a_{ij}\eta) - \sum_i \partial_{x_i}(b_i\eta).$$

Let $\tilde{\eta} = (\eta, \eta_{l+1})$ be an $\mathbb{R}^{l+1}$-valued test function. Since

$$\tilde{\mathcal{D}}^* = \begin{pmatrix} \mathcal{D}^* & 0 \\ f^* & 0 \end{pmatrix},$$

we have

$$\tilde{v}\tilde{\mathcal{D}}^*\tilde{\eta} = \begin{pmatrix} v\mathcal{D}^*\eta + f\eta \\ 0 \end{pmatrix}.$$

It comes out that v is a solution of the inhomogeneous equation if and only if $\tilde{v}$ solves

$$\iint (\tilde{v}L^*\tilde{\eta} + f\tilde{\eta}) = 0,$$

that is, if and only if $\tilde{v}$ is a weak solution of system (3.16). □

We can now prove the main theorem of this section.

Proof of Theorem 3.7. Let $\tilde{\varphi}$ be the function $(\varphi_1, \ldots, \varphi_l, 1)$ and $\tilde{U}_s^{t,x}$ be the solution of

$$\left.\begin{aligned} \mathrm{d}\tilde{U}_s^{t,x} &= \tilde{U}_s^{t,x}\tilde{\mathcal{D}}(s, X_s^{t,x})\,\mathrm{d}s, \quad s \in [t,T], \\ \tilde{U}_t^{t,x} &= I_{l+1}. \end{aligned}\right\} \tag{3.17}$$

Since φ, $\mathcal{D}$ and f satisfy Assumptions (A_4')–(A_6), the functions $\tilde{\varphi}$ and $\tilde{\mathcal{D}}$ satisfy Assumptions (A_4') and (A_5). Hence, by Corollary 3.6, the function

$$(x,t) \to \boldsymbol{E}[\tilde{U}_T^{t,x}\tilde{\varphi}(X_T^{t,x})]$$

is a weak solution of system (3.16).

We write $\tilde{U}_s^{t,x}$ in blocks:

$$\tilde{U}_s^{t,x} = \begin{pmatrix} A_s^{t,x} & b_s^{t,x} \\ c_s^{t,x} & d_s^{t,x} \end{pmatrix},$$

where A_s is an $l \times l$ matrix, $b_s \in \mathbb{R}^d$ is a column vector, $c_s \in \mathbb{R}^d$ is a row vector and d_s is a scalar. With this position, the Cauchy problem (3.17) is equivalent to

$$\begin{aligned} \mathrm{d}A_s^{t,x} &= A_s^{t,x}\mathcal{D}(s, X_s^{t,x})\,\mathrm{d}s, & A_t^{t,x} &= I_l, \\ \mathrm{d}b_s^{t,x} &= A_s^{t,x} f(s, X_s^{t,x})\,\mathrm{d}s, & b_t^{t,x} &= 0, \\ \mathrm{d}c_s^{t,x} &= c_s^{t,x}\mathcal{D}(s, X_s^{t,x})\,\mathrm{d}s, & c_t^{t,x} &= 0, \\ \mathrm{d}d_s^{t,x} &= c_s^{t,x} f(s, X_s^{t,x})\,\mathrm{d}s, & d_t^{t,x} &= 1, \end{aligned}$$

and it is easy to see that

$$\begin{aligned} A_s^{t,x} &= U_s^{t,x}, & b_s^{t,x} &= \int_t^s U_r^{t,x} f(r, X_r^{t,x})\,\mathrm{d}r, \\ c_s^{t,x} &= 0, & d_s^{t,x} &= 1. \end{aligned}$$

Consequently,

$$\begin{aligned} \boldsymbol{E}[\tilde{U}_T^{t,x}\tilde{\varphi}(X_T^{t,x})] &= \boldsymbol{E}\left[\begin{pmatrix} U_T^{t,x} & b_T^{t,x} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \varphi(X_T^{t,x}) \\ 1 \end{pmatrix}\right] \\ &= \boldsymbol{E}\left[\begin{pmatrix} U_T^{t,x}\varphi(X_T^{t,x} + b_T^{t,x}) \\ 1 \end{pmatrix}\right] \\ &= \boldsymbol{E}\left[\begin{pmatrix} U_T^{t,x}\varphi(X_T^{t,x} + b_T^{t,x}) + \int_t^s U_r^{t,x} f(r, X_r^{t,x})\,\mathrm{d}r \\ 1 \end{pmatrix}\right]. \end{aligned}$$

□

3.3. A uniqueness result

In the preceding sections, we were concerned with the existence of a weak solution of the parabolic system (3.1) having a nice probabilistic representation. The aim of the present section is to provide sufficient conditions for the uniqueness of solutions. In Proposition 3.9 we shall see that the strong solution, if it exists, is given by our probabilistic representation and, hence, is unique. In Theorem 3.10 we will show, under some special conditions on the coefficients, that weak solutions are also unique and are given by the probabilistic representation. Such special conditions on the coefficients are satisfied in the application of the probabilistic representation to the Navier–Stokes system: if the velocity field is regular enough, the coefficients in the equations for the vorticity satisfy the special conditions. Hence, for each fixed regular velocity, there exists a unique weak solution of the vorticity equation given by the Feynman–Kac formula.

Let $C_b^{1,2}([0,T]\times\mathbb{R}^d,\mathbb{R}^l)$ be the space of continuous functions having first and second derivatives in x and first derivative in t continuous and bounded. We start by showing that, if the solution of the parabolic system is regular, then it is given by (3.15).

Proposition 3.9. *Let $v\in C_b^{1,2}([0,T]\times\mathbb{R}^d,\mathbb{R}^l)$ be a strong solution of system (3.1), with the final condition. Then v is given by (3.15).*

Proof. It is sufficient to show that the process

$$U_r^{t,x}v(r,X_r^{t,x})+\int_t^r U_s^{t,x}f(s,X_s^{t,x})\,\mathrm{d}s,\quad r\in[t,T],$$

is a martingale. Indeed, if $h\in\{1,\dots,l\}$, by the Itô formula (we omit for simplicity $(r,X_r^{t,x})$ from the term $v(r,X_r^{t,x})$ and from the coefficients, and the subscript r from the term $U_r^{t,x}$),

$$\begin{aligned}\mathrm{d}_r(U^{t,x}v)_h&=\sum_k \mathrm{d}(U_{hk}^{t,x}v_k)=\sum_k(U_{hk}^{t,x}v_k+v_k\,\mathrm{d}U_{hk}^{t,x})\\&=\sum_k U_{hk}^{t,x}\biggl[\biggl(\partial_r v_k+\sum_i b_i\partial_{x_i}v_k+\frac12\sum_{i,j}a_{ij}\partial^2_{x_ix_j}v_k\biggr)\mathrm{d}r+\sum_{i,j}\partial_{x_i}v_k\sigma_{ij}\,\mathrm{d}W_r^j\biggr]\\&\quad+\sum_{k,i}v_kU_{hi}^{t,x}\mathcal{D}_{ik}\,\mathrm{d}r\\&=-\sum_k U_{hk}^{t,x}\biggl(f_k+\sum_i\mathcal{D}_{ki}v_i\biggr)\mathrm{d}r+(M_r)_h+\sum_{k,i}v_kU_{hi}^{t,x}\mathcal{D}_{ik}\,\mathrm{d}r\\&=-\mathrm{d}_r\biggl(\int_t^r(U_s^{t,x}f)_h\,\mathrm{d}s\biggr)+(\mathrm{d}M_r)_h,\end{aligned}$$

since v is a solution of system (3.1); $(M_r)_{r\in[t,T]}$ is the d-dimensional martingale, vanishing at $r=t$, given by

$$(\mathrm{d}M_r)_h=\sum_k U_{hk}^{t,x}\sum_{i,j}\partial_{x_i}v_k\sigma_{ij}\,\mathrm{d}W_r^j.$$

Moreover, M_r is square-integrable, since $v\in C_b^{1,2}$, $U_T^{t,x}$ is bounded by (3.8), and

$$\sup_{t\leqslant r\leqslant T}\boldsymbol{E}[|X_r^{t,x}|^2]$$

is bounded. □

Theorem 3.10. *Let φ be bounded and continuous, and let f and $\mathcal{D}$ be bounded and uniformly continuous. Suppose that σ is constant and b is a function that is Borel measurable and Lipschitz-continuous in x such that $\operatorname{div} b = 0$. Then the function*

$$v(t,x) = \boldsymbol{E}[U_T^{t,x}\varphi(X_T^{t,x})] + \int_t^T \boldsymbol{E}[U_r^{t,x} f(r, X_r^{t,x})]\,\mathrm{d}r$$

is the unique weak solution of the parabolic system (3.1).

The proof of the theorem is based on a regularization by convolution, in order to apply the uniqueness result of the previous proposition.

Let $\rho \in C^\infty(\mathbb{R}^d, \mathbb{R})$, $0 \leqslant \rho \leqslant 1$, with support in the ball of radius 1, and set $\rho_n(x) = n^d \rho(nx)$. Let J_n be the convolution operator: $J_n(u) = \rho_n * u$.

Lemma 3.11. *Let $b : \mathbb{R}^d \to \mathbb{R}^d$ be a Lipschitz-continuous function, such that $\operatorname{div} b = 0$ (in the sense of distributions). There is then a constant C such that, for each $u \in C_b(\mathbb{R}^d, \mathbb{R}^l)$,*

$$|([J_n, b\cdot\nabla]u)(x)| \leqslant C \sup_{y\in B_{1/n}(x)} |u(y)| \quad \text{for all } n, \tag{3.18}$$

where $[J_n, b\cdot\nabla]u = J_n((b\cdot\nabla)u) - (b\cdot\nabla)J_n u$ is the commutator. Moreover,

$$[J_n, b\cdot\nabla]u \xrightarrow{n\to\infty} 0 \quad \text{uniformly on compact sets.} \tag{3.19}$$

Proof. Fix $u \in C_b(\mathbb{R}^d, \mathbb{R}^l)$. Since $\operatorname{div} b = 0$, by integration by parts we have

$$\begin{aligned}
([J_n, b\cdot\nabla]u))(x) &= (\rho_n * (b\cdot\nabla)u - (b\cdot\nabla)(\rho_n * u)))(x) \\
&= \int_{\mathbb{R}^d} \rho_n(x-y)(b(y)\cdot\nabla_y)u(y) - (b(x)\cdot\nabla_x)(\rho_n(x-y))u(y)\,\mathrm{d}y \\
&= \int_{\mathbb{R}^d} u(y)(b(x)-b(y))\cdot\nabla_y\rho_n(x-y)\,\mathrm{d}y.
\end{aligned}$$

Taking the norms in $\mathbb{R}^l$ we get

$$\begin{aligned}
|([J_n, b\cdot\nabla]u)(x)| &\leqslant \int_{B_{1/n}(x)} |\nabla\rho_n(x-y)|\cdot|b(y)-b(x)|\cdot|u(y)|\,\mathrm{d}y \\
&\leqslant cL\|\nabla\rho\|_\infty \sup_{y\in B_{1/n}(x)} |u(y)|,
\end{aligned}$$

where L is the Lipschitz constant of b. So far, we have proved (3.18). Concerning (3.19), it is easy to see that the claim is true for $u \in C_b^\infty(\mathbb{R}^d, \mathbb{R}^l)$. If u is only C_b, the claim follows from approximation with C_b^∞ functions (in the sup-norm, on compact sets) and from the bound (3.18). □

We apply now the previous lemma to prove the main theorem.

Proof of Theorem 3.10. Let v be a bounded and continuous weak solution of system (3.1). The sequence $v_n = \rho_n * v$ belongs to $C([0,T], C_b^\infty(\mathbb{R}^d, \mathbb{R}^l))$ and $v_n \to v$ uniformly on compact sets. We want to show that v_n is a weak solution of

$$\partial_t v_n + \frac{1}{2}\sum_{i,j} a_{ij}\partial^2_{x_i x_j} v_n + \sum_i b_i \partial_{x_i} v_n + \mathcal{D} v_n + \rho_n * f + R_n = 0, \tag{3.20}$$

with final condition $v_n(T) = \rho_n * \varphi$, where $R_n = [J_n, b\cdot\nabla]v + [J_n, \mathcal{D}]v$. Indeed, v is a weak solution of (3.1), so that we can use $\zeta_n = \breve{\rho}_n * \eta$ as a test function, where η is again a test function and $\breve{\rho}_n(x) = \rho_n(-x)$, to obtain, by some easy computations,

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^d} (vM^*\zeta + f\zeta) &= \iint v\Big(-\partial_t\zeta_n + \frac{1}{2}\sum_{i,j} a_{x_i x_j}\partial^2_{ij}\zeta_n - \sum_i b_i\partial_{x_i}\zeta_n + \mathcal{D}^*\zeta_n\Big) + f\zeta_n \\ &= \iint \Big[v_n\Big(-\partial_t\eta + \frac{1}{2}\sum_{i,j} a_{ij}\partial^2_{x_i x_j}\eta - \sum_i b_i\partial_{x_i}\eta + \mathcal{D}^*\eta\Big)\Big] \\ &\quad + \iint \eta(J_n f + [J_n, b\cdot\nabla]v + [J_n,\mathcal{D}]v)\end{aligned}$$

(note that, for each u, $\int u(\breve{\rho}_n * \eta) = \int \eta(\rho_n * u)$).

Since v_n belongs to $C([0,T], C_b^\infty(\mathbb{R}^d, \mathbb{R}^l))$ and $\rho_n * f + R_n$ is bounded and continuous, we argue that the distributional derivative $\partial_t v_n$ is bounded and continuous and, therefore, a strong derivative. Hence $v_n \in C_b^{1,2}$, and it is a strong solution of (3.20). Proposition 3.9 yields

$$v_n(t,x) = \boldsymbol{E}[U_T^{t,x}\rho_n * \varphi(X_T^{t,x})] + \int_t^T \boldsymbol{E}[U_r^{t,x}(\rho_n * f + R_n)(X_r^{t,x})]\,\mathrm{d}r.$$

It is easy to check that $\|[J_n, D]v\|_\infty \leqslant 2\|D\|_\infty\|v\|_\infty$ and $[J_n, D]v \to 0$, uniformly on compact sets. Hence, by the previous lemma, R_n is bounded, independently of n, and $R_n \to 0$ uniformly on compact sets. Using (3.8) and the dominated convergence theorem, we obtain

$$v(t,x) = \lim_{n\to\infty} v_n(t,x) = \boldsymbol{E}[U_T^{t,x}\varphi(X_T^{t,x})] + \int_t^T \boldsymbol{E}[U_r^{t,x} f(r, X_r^{t,x})]\,\mathrm{d}r.$$

□

3.4. The formula for parabolic systems with an initial condition

In this section we describe the probabilistic representation of weak solutions to the system (3.2), with an initial condition. Indeed, below we will use the results of this section to give a probabilistic representation for the solutions to the Navier–Stokes equations, which is a parabolic equation with an initial condition.

We will obtain the representation formula for the forward parabolic system using the representation for the backward parabolic system and a time inversion of the coefficients.

To this aim, we will consider the following SDEs

$$\left.\begin{aligned} \mathrm{d}X_r^{s,x,t} &= b(t-r, X_r^{s,x,t})\,\mathrm{d}r + \sigma(t-r, X_r^{s,x,t})\,\mathrm{d}W_r, \quad r\in[s,t], \\ X_s^{s,x,t} &= x, \end{aligned}\right\} \tag{3.21}$$

and

$$\left.\begin{aligned} \mathrm{d}U_r^{s,(x,Y),t} &= U_r^{s,(x,Y),t}\mathcal{D}(t-r, X_r^{s,x,t})\,\mathrm{d}r, \quad r\in[s,t], \\ U_s^{s,(x,Y),t} &= Y, \end{aligned}\right\} \tag{3.22}$$

where, as usual, $U^{s,(x,Y),t} = U^{s,x,t}$ when $Y = I$.

Theorem 3.12. *Let the data b, σ, φ, $\mathcal{D}$ and f satisfy Assumptions* (A$_1$)–(A$_3$), (A$_4'$), (A$_5$) *and* (A$_6$). *Then the function*

$$v(t,x) = \boldsymbol{E}[U_t^{0,x,t}\varphi(X_t^{0,x,t})] + \int_0^t \boldsymbol{E}[U_r^{0,x,t} f(t-r, X_r^{0,x,t})]\,\mathrm{d}r \tag{3.23}$$

is a weak solution of (3.2), with an initial condition.

Moreover, if σ is constant and b is globally Lipschitz-continuous in x, then v is the unique weak solution.

Proof. Let $\tilde v(t,x) = v(T-t,x)$. If v is a weak solution of (3.2), by easy computations it follows that $\tilde v$ is a weak solution of

$$\partial_t \tilde v(t,x) + \frac{1}{2}\sum_{i,j} a_{ij}(T-t,x)\partial^2_{x_i x_j}\tilde v(t,x) + \sum_i b_i(T-t,x)\partial_{x_i}\tilde v + \mathcal{D}(T-t,x)\tilde v(t,x) + f(T-t,x) = 0, \tag{3.24}$$

for $t\in[0,T]$, with final condition $\tilde v(T,x) = \varphi(x)$ (and vice versa).

By Theorem 3.7, a solution $\tilde v$ of (3.24) is given by

$$\tilde v(t,x) = \boldsymbol{E}[U_T^{t,x,T}\varphi(X_T^{t,x,T})] + \int_t^T \boldsymbol{E}[U_r^{t,x,T} f(T-r, X_r^{t,x,T})]\,\mathrm{d}r,$$

where $U_r^{t,x,T}$ and $X_r^{t,x,T}$ are given in (3.22) and (3.21), respectively. We can conclude that a solution v of the forward parabolic Equation (3.2) is given by

$$v(t,x) = \boldsymbol{E}[U_T^{T-t,x,T}\varphi(X_T^{T-t,x,T})] + \int_{T-t}^T \boldsymbol{E}[U_r^{T-t,x,T} f(T-r, X_r^{T-t,x,T})]\,\mathrm{d}r.$$

Finally, one can easily check that, for each $r\in[T-t,T]$, the joint law of the random variables $U_r^{T-t,x,T}$ and $X_r^{T-t,x,T}$ is equal to the joint law of the random variables $U_{r+t-T}^{0,x,t}$ and $X_{r+t-T}^{0,x,t}$. In conclusion, formula (3.23) holds. □

The representation formula above appears more complicated than the formula for parabolic systems with final condition (3.15): the stochastic processes X_r in (3.15) are the solutions of a fixed SDE corresponding to different initial conditions, while the stochastic processes $X_r^{0,x,t}$ and $U_r^{0,x,t}$ in (3.23) solve, for each t, a different SDE. A different representation can be given which is more appealing at the heuristic level, even if it is less suitable for stochastic calculus.

Consider the backward SDE

$$Y_r^{t,x} = x + \int_r^t b(s, Y_s^{t,x})\,\mathrm{d}s + \int_r^t \sigma(s, Y_s^{t,x})\,\hat{\mathrm{d}}W_s, \quad r \in [0,t], \tag{3.25}$$

where $\hat{\mathrm{d}}W_s$ denotes the backward stochastic integral with respect to the Brownian motion W_s (see [**25**] for the definition of the backward integral). Note that the final condition $Y_t^{t,x} = x$ has been imposed here. Let $V_r^{s,t,x}$, $0 \leqslant s \leqslant r \leqslant t$, be the solution of

$$\left.\begin{aligned} \mathrm{d}V_r^{s,t,x} &= \mathcal{D}(r, Y_r^{t,x}) V_r^{s,t,x}\,\mathrm{d}r, \quad r \in [s,t], \\ V_s^{s,t,x} &= I. \end{aligned}\right\} \tag{3.26}$$

Theorem 3.13. *Under the same assumptions of the previous theorem, a weak solution of the parabolic system (3.2), with an initial condition, is given by*

$$v(t,x) = \boldsymbol{E}[V_t^{0,t,x} \varphi(Y_0^{t,x})] + \int_0^t \boldsymbol{E}[V_t^{r,t,x} f(r, Y_r^{t,x})]\,\mathrm{d}r, \tag{3.27}$$

where $Y^{t,x}$ and $V^{r,t,x}$ are given by (3.25) and (3.26), respectively.

Remark 3.14. We want to give an interpretation of the representation formula given above. Suppose for clarity that $f \equiv 0$. Consider the trajectory $Y_r^{t,x}(\omega)$ of a virtual particle which is in x at time t, transported by a velocity field and subject to a diffusion, and evaluate $v(0, Y_0^{t,x}(\omega)) = \varphi(Y_0^{t,x}(\omega))$. Then we take into account, through the vector field $V_t^{0,t,x}$, the effects of the tensor $\mathcal{D}$ along the given trajectory in the time interval $[0,t]$. Finally, by taking the expectation, we consider the mean effect of all virtual particles.

Before giving the proof of the theorem, we need the following simple lemma for the time inversion of a stochastic integral.

Lemma 3.15. *Let $(W_s)_{s\geqslant 0}$ be a Brownian motion. Fix $t > 0$ and set*

$$B_s = W_t - W_{t-s}, \quad s \in [0,t].$$

Let

$$\mathcal{F}_s^W = \sigma(W_r \mid r \in [0,s]) \quad \text{and} \quad \mathcal{F}_{s,t}^B = \sigma(B_u - B_v \mid s \leqslant v \leqslant u \leqslant t)$$

and let $g(s)$ be a continuous and bounded process adapted to the filtration F_s^W. Then the process $f(s) = g(t-s)$, $s \in [0,t]$, is $\mathcal{F}_{s,t}^B$-adapted and, for all a, b such that $0 \leqslant a \leqslant b \leqslant t$,

$$\int_a^b g(s)\,\mathrm{d}W_s = \int_{t-b}^{t-a} f(s)\,\hat{\mathrm{d}}B_s.$$

Proof. Since $B_u - B_v = W_{t-v} - W_{t-u}$, we have $\mathcal{F}_{t-s}^W = \mathcal{F}_{s,t}^B$, and this gives the first statement.

Take now a sequence of partitions of the interval $[a,b]$:

$$\pi_n : \{a = s_0^n \leqslant s_1^n \leqslant \cdots \leqslant s_{k_n}^n = b\}$$

such that $|\pi_n| \to 0$. We have

$$\begin{aligned}\int_a^b g(s)\,\mathrm{d}W_s &= \lim_{n\to\infty} \sum g(s_i^n)(W_{s_{i+1}^n} - W_{s_i^n}) \\ &= \lim_{n\to\infty} \sum g(t - r_i^n)(W_{t-r_{i+1}^n} - W_{t-r_i^n}) \\ &= \lim_{n\to\infty} \sum f(r_i^n)(B_{r_i^n} - B_{r_{i+1}^n}) \\ &= \int_{t-b}^{t-a} f(s)\,\hat{\mathrm{d}}B_s,\end{aligned}$$

where $r_i^n = t - s_i^n$, $i = 1, \ldots, k_n$. □

Proof of Theorem 3.13. We need only to show that

$$X_{t-r}^{0,x,t} = Y_r^{t,x} \quad \text{and} \quad U_{t-r}^{0,x,t} = V_t^{r,x,t}, \quad \boldsymbol{P}\text{-almost surely},$$

since such formulae, formula (3.23) and a change of variables give us (3.27).

We prove the first equality. Fix a Brownian motion $(W_r)_{r\geqslant 0}$ and consider the solution $X_r^{0,x,t}$ of Equation (3.21). By Lemma (3.15) above, it follows that $X_{t-r}^{0,x,t}$ satisfies the backward SDE (3.25) with respect to the Brownian motion B_s defined in Lemma 3.15. Since Equation (3.25) has a unique strong solution, we have the first equality.

We proceed to prove the second equality. Fix ω so that $r \to Y_r^{t,x}(\omega)$ is continuous. The key observation is that

$$V_r^{s,t,x}(\omega) = V_r^{0,t,x}(\omega)(V_s^{0,t,x}(\omega))^{-1}, \quad 0 \leqslant s \leqslant r \leqslant t,$$

and it is true since

$$\mathrm{d}(V_r^{0,t,x}(\omega))^{-1} = -(V_r^{0,t,x}(\omega))^{-1}\mathcal{D}(r, Y_r^{t,x}(\omega)),$$

with initial condition $(V_0^{0,t,x}(\omega))^{-1} = I$, so that it easy to check that

$$V_r^{0,t,x}(\omega)(V_s^{0,t,x}(\omega))^{-1}$$

satisfies Equation (3.26). Finally, by evaluating

$$\mathrm{d}_r V_t^{r,t,x}(\omega) = \mathrm{d}_r[V_t^{0,t,x}(\omega)(V_r^{0,t,x}(\omega))^{-1}],$$

we see that both $V_t^{r,t,x}(\omega)$ and $r \to U_{t-r}^{0,x,t}(\omega)$ solve the ordinary differential equation

$$\mathrm{d}U_r = -U_r\mathcal{D}(r, Y_r^{t,x}(\omega))\,\mathrm{d}r, \quad r \in [0,t],$$

with final condition $U_t = I$. □

4. A probabilistic representation for the Newtonian potential and the Biot–Savart law

In the present section we aim to give a probabilistic representation for the velocity field of an incompressible fluid in terms of the vorticity field $\xi = \operatorname{curl} u$.

Under suitable assumptions on ξ, the Poisson equation $-\Delta\psi = \xi$ has a solution, given by

$$\psi(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\xi(y)}{|x-y|} \, \mathrm{d}y$$

(ψ is a vector field and the equation is interpreted componentwise). Let $u(x)$ be defined as $u(x) = \operatorname{curl} \psi(x)$, i.e.

$$u(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\xi(y) \times (x-y)}{|x-y|^3} \, \mathrm{d}y. \tag{4.1}$$

If $\operatorname{div} \xi = 0$, then also $\operatorname{div} \psi = 0$ and $\operatorname{div} u = 0$, and this also implies $\operatorname{curl}\operatorname{curl} \psi = -\Delta\psi$. Therefore, $\operatorname{curl} u = \xi$, i.e. u is the divergence-free velocity field associated to ξ. The equality (4.1) is the *Biot–Savart law*.

In order to give a probabilistic representation of this formula, it is necessary to give a representation of the solution of the Poisson equation and of its derivatives.

4.1. A probabilistic representation for the Newtonian potential

In this section we study a probabilistic representation of the solution of the Poisson equation. The deterministic regularity results are classical (see, for example, [**20**, **33**]), so we will focus on the probabilistic formula.

Let $f : \mathbb{R}^3 \to \mathbb{R}$ be an integrable function. We define the Newtonian potential with density f as

$$Nf(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|x-y|} f(y) \, \mathrm{d}y.$$

If f is regular and with compact support, Nf is a solution of the Poisson equation.

Let $A = \frac{1}{2}\Delta$. It is well known that A generates, on the space $C_0(\mathbb{R}^3)$ of all continuous functions vanishing at infinity, the strongly continuous semigroup

$$P_t f(x) = \boldsymbol{E}[f(x + W_t)], \quad x \in \mathbb{R}^3,\ t \geqslant 0,\ f \in C_0(\mathbb{R}^3),$$

where $(W_t)_{t\geqslant 0}$ is a 3D standard Brownian motion. The resolvent of A can be written as

$$((A - \lambda I)^{-1} f)(x) = \int_0^\infty \mathrm{e}^{-\lambda t} \boldsymbol{E}[f(x + W_t)] \, \mathrm{d}t, \quad f \in C_0(\mathbb{R}^3),$$

so we can argue that the integral

$$\int_0^\infty \boldsymbol{E}[f(x + W_t)] \, \mathrm{d}t. \tag{4.2}$$

converges to $A^{-1} f(x) = 2Nf(x)$ (indeed, at this stage, we do not know if A is invertible).

As a first step, we find some conditions on f in such a way that (4.2) produces a solution of the Poisson equation.

Proposition 4.1. *Let $f \in L^p(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$, with $1 \leqslant p < \frac{3}{2} < q < \infty$. Then the integral in (4.2) is convergent for all $x \in \mathbb{R}^3$ and is equal to $2Nf(x)$. Moreover, $Nf \in C_0(\mathbb{R}^3)$ and*

$$\|Nf\|_\infty \leqslant C_{p,q}(\|f\|_p + \|f\|_q).$$

Proof. For every $r > 1$, by the Hölder inequality,

$$\boldsymbol{E}|f(x+W_t)| = \frac{1}{(2\pi t)^{3/2}} \int_{\mathbb{R}^3} |f(x+y)| \exp\left\{ -\frac{1}{2t}|y|^2 \right\} \mathrm{d}y \leqslant C_r t^{-3/2r} \|f\|_r, \qquad (4.3)$$

so, by using the above inequality with $r = p$ and $r = q$ and by integrating by time,

$$\int_0^\infty \boldsymbol{E}|f(x+W_t)|\,\mathrm{d}t \leqslant \int_0^1 \boldsymbol{E}|f(x+W_t)|\,\mathrm{d}t + \int_1^\infty \boldsymbol{E}|f(x+W_t)|\,\mathrm{d}t \leqslant C(\|f\|_p + \|f\|_q).$$

This will also prove the final inequality, once the other properties are verified. The integral in (4.2) is equal to $2Nf(x)$ since (we can use the Fubini theorem because of the previous inequality)

$$\begin{aligned}\int_0^\infty \boldsymbol{E}[f(x+W_t)]\,\mathrm{d}t &= \int_{\mathbb{R}^3} f(x+y) \int_0^\infty \frac{1}{(2\pi t)^{3/2}} \exp\left\{ -\frac{1}{2t}|y|^2 \right\} \mathrm{d}t\,\mathrm{d}y \\ &= \int_{\mathbb{R}^3} \frac{1}{2\pi|y|} f(x+y)\,\mathrm{d}y \\ &= 2Nf(x).\end{aligned}$$

We know from [**20**] that, by Sobolev embeddings, $f \in L^q(\mathbb{R}^3)$ implies $Nf \in C(\mathbb{R}^3)$. The behaviour at infinity is less standard, so we give a probabilistic proof of it. Thus, let us show that $Nf \in C_0(\mathbb{R}^3)$. Indeed, for each $R > 0$,

$$\int_0^\infty \boldsymbol{E}[f(x+W_t)]\,\mathrm{d}t = \int_0^\infty \boldsymbol{E} f(x+W_t) I_{\{|W_t|>R\}}\,\mathrm{d}t + \int_0^\infty \boldsymbol{E} f(x+W_t) I_{\{|W_t|\leqslant R\}}\,\mathrm{d}t$$

and, in order to show that $Nf(x)$ converges to 0 as $|x| \to \infty$, we will prove that the first term converges to 0, uniformly in x, as $R \to \infty$, and the second term converges to 0 as $|x| \to \infty$ for each $R > 0$.

For the first term the claim is true since, as in (4.3),

$$\sup_{x \in \mathbb{R}^3} \boldsymbol{E}|f(x+W_t)| I_{\{|W_t|>R\}} \leqslant C(\|f\|_p + \|f\|_q)(t^{-3/(2p)} I_{[1,\infty)}(t) + t^{-3/(2q)} I_{[0,1)}(t))$$

and

$$\sup_{x \in \mathbb{R}^3} \boldsymbol{E}|f(x+W_t)| I_{\{|W_t|>R\}} \leqslant C t^{-3/2} \|f\|_p \left(\int_{|y|>R} \exp\left\{ -\frac{1}{2t}|y|^2 \right\} \right)^{1/p'} \to 0$$

as $R \to \infty$. As regards the second term, we can proceed as in (4.3) and bound the term $\boldsymbol{E}|f(x+W_t)| I_{\{|W_t|\leqslant R\}}$ with

$$C(t^{-3/(2p)} \|f(y) I_{\{|y-x|\leqslant R\}}\|_p I_{[1,\infty)}(t) + t^{-3/(2q)} \|f(y) I_{\{|y-x|\leqslant R\}}\|_q I_{[0,1)}(t)),$$

so that, after the integration in time, the above term converges to 0, since $f \in L^p(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$. □

In the second step, we study the derivatives of Nf. Note that, for a regular f, the Bismut–Elworthy–Li formula (see [**5, 11**]) gives

$$D_{x_i}\boldsymbol{E}[f(x+W_t)] = \frac{1}{t}\boldsymbol{E}[f(x+W_t)(W_t)_i].$$

In this simple case, with the Brownian motion, such a formula can be easily checked by means of the Gaussian density.

As in the previous proposition, one could expect that, under suitable conditions, it is possible to write the derivatives of Nf with the probabilistic representation suggested by the formula above. Indeed, this is the case, as the following proposition shows.

Proposition 4.2. *Let $f \in L^p(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$ for some $1 \leqslant p < \frac{3}{2} < 3 < q < +\infty$. Then $\nabla Nf \in C_0(\mathbb{R}^3)$ and, for each $x \in \mathbb{R}^3$,*

$$2D_{x_i}Nf(x) = \int_0^\infty \frac{1}{t}\boldsymbol{E}[f(x+W_t)(W_t)_i]\,\mathrm{d}t, \quad i = 1, 2, 3. \tag{4.4}$$

Moreover,

$$\|\nabla Nf\|_\infty \leqslant C_{p,q}(\|f\|_p + \|f\|_q). \tag{4.5}$$

Proof. By the Hölder inequality,

$$\begin{aligned}\frac{1}{t}\boldsymbol{E}|f(x+W_t)(W_t)_i| &= \frac{C}{t^{5/2}}\int_{\mathbb{R}^3} f(x+y)y_i \exp\left\{-\frac{1}{2t}|y|^2\right\}\mathrm{d}y \\ &\leqslant \frac{C}{t^{5/2}}\|f\|_p\sqrt{t}t^{3/(2p')} \\ &\leqslant C\|f\|_p t^{-(1/2)-3/(2p)} \end{aligned} \tag{4.6}$$

and, as in the proof of the previous proposition, the time integral is finite and bounded with respect to x, by the assumptions on p and q. Moreover, it can be easily seen, by the same arguments as used in the previous proposition, that the formula (4.4) and inequality (4.5) hold and that $\nabla Nf \in C_0(\mathbb{R}^3)$. □

In the last step, we study the second derivatives of the Newtonian potential. The regularity of the following theorem is based on the classical Schauder estimates.

Proposition 4.3. *Let $f \in L^p(\mathbb{R}^3) \cap C_b^\alpha(\mathbb{R}^3)$, with $1 \leqslant p < \frac{3}{2}$. Then $Nf \in C_b^{2,\alpha}(\mathbb{R}^3) \cap C_0(\mathbb{R}^3)$,*

$$\|Nf\|_{C_b^{2,\alpha}(\mathbb{R}^3)} \leqslant \tilde{C}(\|f\|_{L^p(\mathbb{R}^3)} + \|f\|_{C_b^\alpha(\mathbb{R}^3)})$$

and Nf is the unique solution of the Poisson equation in $C_0(\mathbb{R}^3) \cap C^2(\mathbb{R}^3)$.

Proof. From the previous proposition, we know that $Nf \in C_b^1(\mathbb{R}^3)$. The Bismut–Elworthy–Li formula gives us

$$D_{x_ix_j}\boldsymbol{E}f(x+W_t) = \frac{2}{t}\boldsymbol{E}[(D_{x_i}\psi)(x+W_{t/2})(W_{t/2})_j], \tag{4.7}$$

where $\psi(x) = \boldsymbol{E} f(x + W_{t/2})$. Hence, in order to show that

$$D_{x_i x_j} N f(x) = \int_0^\infty \frac{1}{t} \boldsymbol{E}[(D_{x_i}\psi)(x + W_{t/2})(W_{t/2})_j] \,\mathrm{d}t$$

holds, it is sufficient to show that (4.7) is integrable in time in the interval $[0, \infty)$.

First, by the Bismut–Elworthy–Li formula, we see that

$$D_{x_i}\psi(x) = \frac{2}{t} \boldsymbol{E}[f(x + W_{t/2})(W_{t/2})_i]$$

and, by (4.6), that

$$\|D_{x_i}\psi\|_\infty \leqslant C t^{-(1/2)-3/(2p)} \|f\|_p. \tag{4.8}$$

Moreover, since $f \in C_b^\alpha(\mathbb{R}^3)$,

$$\begin{aligned} |D_{x_i}\psi(y) - D_{x_i}\psi(x)| &= \frac{2}{t} \boldsymbol{E}|f(y + W_{t/2}) - f(x + W_{t/2})|\,|(W_{t/2})_i| \\ &\leqslant C t^{-1/2} [f]_\alpha |x - y|^\alpha. \end{aligned} \tag{4.9}$$

Now we show that (4.7) is integrable in time. By (4.8)

$$\frac{2}{t} \boldsymbol{E}|(D_{x_i}\psi)(x + W_{t/2})(W_{t/2})_j| \leqslant C t^{-(3/2)-3/(2p)} \|f\|_p \boldsymbol{E}|(W_{t/2})_j| \leqslant C t^{-1-(3/(2p))} \|f\|_p$$

and (4.7) is integrable in $[1, \infty)$. By (4.9) it follows that

$$\begin{aligned} \frac{2}{t} |\boldsymbol{E}(D_{x_i}\psi)(x + W_{t/2})(W_{t/2})_j| &= \frac{2}{t} |\boldsymbol{E}[(D_{x_i}\psi)(x + W_{t/2}) - (D_{x_i}\psi)(x)](W_{t/2})_j| \\ &\leqslant \frac{2}{t} \boldsymbol{E}|[(D_{x_i}\psi)(x + W_{t/2}) - (D_{x_i}\psi)(x)](W_{t/2})_j| \\ &\leqslant C t^{-3/2} \boldsymbol{E}|W_{t/2}|^\alpha |(W_{t/2})_i| \\ &\leqslant C t^{-1+\alpha/2} [f]_\alpha \end{aligned}$$

and (4.7) is integrable in $[0, 1)$.

In conclusion, the probabilistic representation formula for the second derivatives holds and

$$\|D_{x_i x_j} N f\|_\infty \leqslant C(\|f\|_p + [f]_\alpha).$$

By Schauder's theory, since $Nf \in C_b^2(\mathbb{R}^3)$ and $f \in C_b^\alpha(\mathbb{R}^3)$, it follows that $Nf \in C_b^{2,\alpha}(\mathbb{R}^3)$ and

$$\|Nf\|_{C_b^{2,\alpha}(\mathbb{R}^3)} \leqslant C(\|f\|_p + \|f\|_{C_b^\alpha})$$

(see, for example, [**27**]). Moreover, Nf solves the Poisson equation [**20**, Lemma 4.2], and the solution is unique by the maximum principle. □

4.2. A probabilistic representation for the Biot–Savart law

We now apply the theory developed in the previous section. The following theorem, which is actually a mere corollary of the above results, is the well-known *Biot–Savart law.*

Theorem 4.4. *Let $\xi \in L^p(\mathbb{R}^3,\mathbb{R}^3)\cap C_b^\alpha(\mathbb{R}^3,\mathbb{R}^3)$, with $1 \leqslant p < \frac{3}{2}$ and $0<\alpha<1$. There is a unique $u \in C_b^{1,\alpha}(\mathbb{R}^3,\mathbb{R}^3)\cap C_0(\mathbb{R}^3,\mathbb{R}^3)$ such that*

$$\operatorname{curl} u = \xi, \qquad \operatorname{div} u = 0,$$

and such a solution is given by

$$u(x) = \frac{1}{2}\int_0^\infty \frac{1}{t}\boldsymbol{E}[\xi(x+W_t)\times W_t]\,\mathrm{d}t, \quad x \in \mathbb{R}^3,$$

where $(W_t)_{t\geqslant 0}$ is a standard 3D Brownian motion.

Proof. The probabilistic formula derives from Proposition 4.2 and the regularity of u from Propositions 4.2 and 4.3. We prove the uniqueness of the representation: since $\operatorname{div} u = 0$, we have $u = \operatorname{curl}\psi$, where ψ is the stream function. Now, by the maximum principle, the unique solution of the problem

$$\Delta u = 0, \quad u \to 0 \text{ as } |x|\to\infty,$$

is $u \equiv 0$. □

Since we are interested in the time evolution of the vector fields, it is appropriate to give a time-dependent version of the previous theorem. We recall that the spaces $\mathcal{U}^\alpha(T)$ and $\mathcal{U}^\alpha_M(T)$ have been defined in (2.1) and (2.3), and the spaces $\mathcal{V}^{\alpha,p}(T)$ and $\mathcal{V}^{\alpha,p}_L(T)$ have been defined in (2.2) and (2.4).

Corollary 4.5. *Let $\alpha \in (0,1)$ and $1 \leqslant p < \frac{3}{2}$. The map $\mathcal{B}s : \mathcal{V}^{\alpha,p}(T) \to \mathcal{U}^\alpha(T)$, defined as*

$$\mathcal{B}s(\xi)(t,x) = \frac{1}{2}\int_0^\infty \frac{1}{s}\boldsymbol{E}[\xi(t,x+W_s)\times W_s]\,\mathrm{d}s,$$

is linear bounded and $\|\mathcal{B}s\| \leqslant \tilde{C}$, where $\tilde{C}$ is the constant, independent of T, appearing in Proposition 4.3.

Moreover, if $L, M > 0$ are constant such that $M \geqslant \tilde{C}L$, then the map $\mathcal{B}s : \mathcal{V}^{\alpha,p}_L(T) \to \mathcal{U}^\alpha_M(T)$ is linear bounded.

5. The representation map

The section is devoted to the study of the properties of the representation map $\mathcal{N}s$, defined as

$$\mathcal{N}s(u)(t,x) = \boldsymbol{E}[U_t^{x,t}\psi(X_t^{x,t})] + \int_0^t \boldsymbol{E}[U_t^{x,t} g(t-s, X_s^{x,t})]\,\mathrm{d}s,$$

where $\psi = \psi(x)$, $g = g(t,x)$ and $X_s^{x,t}$ are the Lagrangian paths, defined in (5.5), and $U_s^{x,t}$ are the deformation matrices, defined in (5.6).

In the first part, some regularity properties of the Lagrangian paths and of the deformation matrices are obtained. In the second part we show that $\mathcal{NS}$ maps the space $\mathcal{U}^\alpha(T)$ in $\mathcal{V}^{\alpha,p}(T)$ (for the definition of the spaces, see (2.1) and (2.2)). Finally, in the third part, we prove that $\mathcal{NS}$ is Lipschitz-continuous from $\mathcal{U}^\alpha(T)$ to $\mathcal{V}^{\alpha,p}(T)$.

5.1. Regularity of the Lagrangian paths

In this section we study some regularity properties of the Lagrangian paths

$$\left.\begin{aligned} \mathrm{d}X_s^x &= u(s,X_s^x)\,\mathrm{d}s + \sqrt{2\nu}\,\mathrm{d}W_s, \quad s\in[0,T],\\ X_0^x &= x, \end{aligned}\right\}$$

and of the deformation matrices

$$\left.\begin{aligned} \mathrm{d}U_s^x &= U_s^x\mathcal{D}(s,X_s^x)\,\mathrm{d}s, \quad s\in[0,T],\\ U_0^x &= I, \end{aligned}\right\}$$

where $u\in C([0,T];C_b^1(\mathbb{R}^3,\mathbb{R}^3))$ and $\mathcal{D}\in C([0,T];C_b^\alpha(\mathbb{R}^3,\mathbb{R}^{3\times3}))$ are given. Note that both equations have unique strong solutions. Hence, for a fixed 3D Brownian motion $((W_s)_{s\geqslant0},(\mathcal{F}_s)_{s\geqslant0})$ on the probability space $(\Omega,\mathcal{F},\boldsymbol{P})$, for each $x\in\mathbb{R}^3$ there is a process $(X_s^x,U_s^x)_{s\geqslant0}$ that solves the corresponding equations, and the solution is unique up to indistinguishability. The equations can be solved pathwise, by choosing the $\omega\in\Omega$ for which $s\to W_s(\omega)$ is a continuous function. Hence, the statements of this section are true for all such ω, independently of x and s. First define

$$\|v\|_{\infty,s} = \int_0^s \|v(r)\|_\infty\,\mathrm{d}r;$$

note that $\|v\|_{\infty,s}\leqslant s\sup_{0\leqslant r\leqslant s}\|v\|_\infty$.

Lemma 5.1. *Assume $u\in C([0,T];C_b^1(\mathbb{R}^3,\mathbb{R}^3))$. Then*

$$|X_s^x - X_s^y| \leqslant |x-y|\mathrm{e}^{\|\nabla u\|_{\infty,s}}, \quad s\geqslant0,\ x,y\in\mathbb{R}^3. \tag{5.1}$$

Moreover, if $\operatorname{div} u = 0$, then, for all $s\geqslant0$ and $\omega\in\Omega$, the map

$$x\in\mathbb{R}^3\mapsto X_s^x(\omega)\in\mathbb{R}^3$$

is a diffeomorphism, the determinant of its Jacobian is everywhere equal to 1 and

$$\int_{\mathbb{R}^3}\varphi(X_s^x(\omega))\,\mathrm{d}x = \int_{\mathbb{R}^3}\varphi(x)\,\mathrm{d}x, \quad \varphi\in L^1(\mathbb{R}^3). \tag{5.2}$$

Proof. First we prove (5.1). By easy computations,

$$|X_s^x - X_s^y| \leqslant |x-y| + \int_0^s \|\nabla u(r)\|_\infty |X_r^x - X_r^y|\,\mathrm{d}r,$$

and by applying Gronwall's lemma, we can complete the proof of (5.1).

Using [**25**, Theorem 4.6.5] (actually the assumption of the Hölder continuity on u is useless for our aim, since we deal with an additive noise; see also [**7**, Theorem 4.1.1]), one can deduce that $x \mapsto X_t^x$ is a diffeomorphism. Moreover, the determinant $J(t,x)$ of the Jacobian matrix of $x \mapsto X_t^x$ solves the following problem with random coefficients:

$$\left.\begin{aligned} \dot{J}(t,x) &= \operatorname{div}(u)(X_t^x)J(t,x), \\ J(0,x) &= 1. \end{aligned}\right\}$$

Since $\operatorname{div} u = 0$ by the assumptions of the lemma, the determinant J is equal to 1 for all times. Hence, by a change of variables and a density argument, (5.2) can also be deduced. □

Lemma 5.2. *Assume*

$$u \in C([0,T]; C_b^1(\mathbb{R}^3,\mathbb{R}^3)) \quad \textit{and} \quad \mathcal{D} \in C([0,T]; C_b^\alpha(\mathbb{R}^3,\mathbb{R}^{3\times 3})).$$

Then

$$|U_s^x| \leqslant \mathrm{e}^{\|\mathcal{D}\|_{\infty,s}}, \quad x \in \mathbb{R}^3,\ s \in [0,T], \tag{5.3}$$

and, for $x, y \in \mathbb{R}^3$ *and* $s \in [0,T]$,

$$|U_s^x - U_s^y| \leqslant |x-y|^\alpha \mathrm{e}^{\|\mathcal{D}\|_{\infty,s} + \alpha\|\nabla u\|_{\infty,s}} \int_0^s [\mathcal{D}(r)]_\alpha \,\mathrm{d}r. \tag{5.4}$$

Proof. Property (5.3) follows from Gronwall's lemma, since

$$|U_s^x| = \left| I + \int_0^s U_r^x \mathcal{D}(r, X_r^x) \right| \leqslant 1 + \int_0^s |U_r^x| \, \|\mathcal{D}(r)\|_{L^\infty} \,\mathrm{d}r,$$

while property (5.4) follows from (5.1), (5.3) and

$$|U_s^x - U_s^y| \leqslant \int_0^s (\|\mathcal{D}(r)\|_{L^\infty} |U_r^x - U_r^y| + \mathrm{e}^{\|\mathcal{D}\|_{\infty,r}} [\mathcal{D}(r)]_\alpha |X_r^x - X_r^y|^\alpha) \,\mathrm{d}r,$$

and again from Gronwall's lemma. □

Let $\mathcal{B}_b(\mathbb{R}^3,\mathbb{R}^3)$ be the space of all bounded Borel-measurable functions and define the operator

$$Q_s\varphi(x) = \boldsymbol{E}[U_s^x \varphi(X_s^x)], \quad x \in \mathbb{R}^3.$$

Lemma 5.3. *Let* $s \geqslant 0$, *then*

(1) $Q_s \in \mathcal{L}(\mathcal{B}_b(\mathbb{R}^3,\mathbb{R}^3))$ *and* $\|Q_s\|_{\mathcal{L}(\mathcal{B}_b)} \leqslant \mathrm{e}^{\|\mathcal{D}\|_{\infty,s}}$;

(2) $Q_s \in \mathcal{L}(C_b^\alpha(\mathbb{R}^3,\mathbb{R}^3))$ *and* $\|Q_s\|_{\mathcal{L}(C_b^\alpha)} \leqslant \mathrm{e}^{\|\mathcal{D}\|_{\infty,s}+\alpha\|\nabla u\|_{\infty,s}}(1 + \int_0^s [\mathcal{D}(r)]_\alpha \,\mathrm{d}r)$.

Moreover, if $\operatorname{div} u = 0$, *then*

(3) $Q_s \in \mathcal{L}(L^p(\mathbb{R}^3,\mathbb{R}^3))$ *and* $\|Q_s\|_{\mathcal{L}(L^p)} \leqslant \mathrm{e}^{\|\mathcal{D}\|_{\infty,s}}$.

Proof. The first property is an obvious consequence of the previous lemma. As to the second property, using the two lemmas above,

$$|\boldsymbol{E}[U^x_s\varphi(X^x_s) - U^y_s\varphi(X^y_s)]| \leqslant \boldsymbol{E}|U^x_s - U^y_s| \cdot |\varphi(X^x_s)| + \boldsymbol{E}|U^y_s| \cdot |\varphi(X^x_s) - \varphi(X^y_s)|$$
$$\leqslant \left(1 + \int_0^s [\mathcal{D}(r)]_\alpha \,\mathrm{d}r\right) \mathrm{e}^{\|\mathcal{D}\|_{\infty,s} + \alpha\|\nabla u\|_{\infty,s}} \|\varphi\|_{C^\alpha_b} |x-y|^\alpha.$$

Finally, assume $\operatorname{div} u = 0$. Using (5.2), the Hölder inequality and the previous lemma, we get

$$\int_{\mathbb{R}^3} |Q_s\varphi(x)|^p \leqslant \mathrm{e}^{p\|\mathcal{D}\|_{\infty,s}} \boldsymbol{E} \int_{\mathbb{R}^3} |\varphi(X^x_s)|^p \leqslant \mathrm{e}^{p\|\mathcal{D}\|_{\infty,s}} \|\varphi\|^p_p.$$

□

5.2. Definition of the representation map

Here we prove that $\mathcal{NS}$ maps $\mathcal{U}^\alpha(T)$ in $\mathcal{V}^{\alpha,p}(T)$. Before proving such a claim, we need some preliminary definitions and results. For each $u \in \mathcal{U}^\alpha(T)$, consider for all $x \in \mathbb{R}^3$ and $t \in [0,T]$ the Lagrangian paths

$$\left.\begin{aligned} \mathrm{d}X^{x,t}_s &= -u(t-s, X^{x,t}_s)\,\mathrm{d}s + \sqrt{2\nu}\,\mathrm{d}W_s, \quad s \in [0,t], \\ X^{x,t}_0 &= x, \end{aligned}\right\} \tag{5.5}$$

and the deformation matrices

$$\left.\begin{aligned} \mathrm{d}U^{x,t}_s &= U^{x,t}_s \mathcal{D}_u(t-s, X^{x,t}_s)\,\mathrm{d}s, \quad s \in [0,T], \\ U^{x,t}_0 &= I, \end{aligned}\right\} \tag{5.6}$$

where $\mathcal{D}_u = \nabla u$ or $\mathcal{D}_u = \frac{1}{2}(\nabla u + \nabla u^{\mathrm{T}})$.

Lemma 5.4. *Let $u \in \mathcal{U}^\alpha(T)$ and $\psi \in C_b(\mathbb{R}^3,\mathbb{R}^3) \cap L^p(\mathbb{R}^3,\mathbb{R}^3)$. The function*

$$(s,t) \in \{0 \leqslant s \leqslant t \leqslant T\} \mapsto \boldsymbol{E}[U^{\cdot,t}_s \psi(X^{\cdot,t}_s)] \in L^p(\mathbb{R}^3,\mathbb{R}^3) \cap C_b(\mathbb{R}^3,\mathbb{R}^3)$$

is continuous with respect to both variables.

Proof. First we show the continuity in C_b. If $0 \leqslant s \leqslant t \leqslant T$ and $0 \leqslant r \leqslant v \leqslant T$, with $t \leqslant v$, then, for each $x \in \mathbb{R}^3$,

$$\begin{aligned} &|\boldsymbol{E}[U^{x,t}_s\psi(X^{x,t}_s)] - \boldsymbol{E}[U^{x,v}_r\psi(X^{x,v}_r)]| \\ &\qquad \leqslant \boldsymbol{E}|U^{x,t}_s - U^{x,v}_s|\,|\psi(X^{x,t}_s)| + \boldsymbol{E}|U^{x,v}_s|\,|\psi(X^{x,t}_s) - \psi(X^{x,v}_s)| \\ &\qquad\quad + \boldsymbol{E}|U^{x,v}_s - U^{x,v}_r|\,|\psi(X^{x,v}_s)| + \boldsymbol{E}|U^{x,v}_r|\,|\psi(X^{x,v}_s) - \psi(X^{x,v}_r)|. \end{aligned} \tag{5.7}$$

In order to estimate the different terms of the above inequality, we see that, from equations (5.5) and (5.6),

$$|U^{x,v}_s - U^{x,v}_r| = \left|\int_s^r U^{x,v}_\sigma \mathcal{D}_u(v-\sigma, X^{x,v}_\sigma)\,\mathrm{d}\sigma\right| \leqslant \mathrm{e}^{\|\mathcal{D}_u\|_{\infty,v}} |r-s| \sup_{(0,v)} \|\mathcal{D}_u\|_\infty$$

and

$$|X_r^{x,v} - X_s^{x,v}| \leqslant \|u\|_\infty |s-r| + \sqrt{2\nu}|W_r - W_s|. \tag{5.8}$$

Moreover,

$$\begin{aligned}|X_s^{x,t} - X_s^{x,v}| &\leqslant \int_0^s |u(t-\sigma, X_\sigma^{x,t}) - u(v-\sigma, X_\sigma^{x,v})| \,\mathrm{d}\sigma \\ &\leqslant \int_0^s \|u(t-\sigma) - u(v-\sigma)\|_\infty + \int_0^s \|\nabla u(v-\sigma)\|_\infty |X_\sigma^{x,t} - X_\sigma^{x,v}|\end{aligned}$$

and, by Gronwall's lemma,

$$|X_s^{x,t} - X_s^{x,v}| \leqslant \mathrm{e}^{\|\nabla u\|_{\infty,v}} \int_0^s \|u(t-\sigma) - u(v-\sigma)\|_\infty \,\mathrm{d}\sigma.$$

Finally,

$$\begin{aligned}|U_s^{x,t} - U_s^{x,v}| \leqslant &\int_0^s |U_\sigma^{x,t}|\, |\mathcal{D}_u(t-\sigma, X_\sigma^{x,t}) - \mathcal{D}_u(t-\sigma, X_\sigma^{x,v})| \,\mathrm{d}\sigma \\ &+ \int_0^s |U_\sigma^{x,t}|\, |\mathcal{D}_u(t-\sigma, X_\sigma^{x,v}) - \mathcal{D}_u(v-\sigma, X_\sigma^{x,v})| \,\mathrm{d}\sigma \\ &+ \int_0^s |\mathcal{D}_u(v-\sigma, X_\sigma^{x,v})|\, |U_\sigma^{x,t} - U_\sigma^{x,v}| \,\mathrm{d}\sigma \\ \leqslant &\ \mathrm{e}^{\|\nabla u\|_{\infty,v}} \Big(\sup_{(0,v)} \|\mathcal{D}_u\|_\infty\Big) \sup_{[0,v]} |X_\sigma^{x,t} - X_\sigma^{x,v}| \\ &+ \mathrm{e}^{\|\nabla u\|_{\infty,v}} \sup_{[0,v]} \|\mathcal{D}_u(t-\sigma) - \mathcal{D}_u(v-\sigma)\|_\infty \\ &+ \sup_{(0,v)} \|\nabla u\|_\infty \int_0^s |U_\sigma^{x,t} - U_\sigma^{x,v}| \,\mathrm{d}\sigma \\ \leqslant &\ A(t,v) + C\int_0^s |U_\sigma^{x,t} - U_\sigma^{x,v}| \,\mathrm{d}\sigma,\end{aligned}$$

where $\boldsymbol{E}A(t,v) \to 0$ as $|t-v| \to 0$ and, by Gronwall's lemma,

$$|U_s^{x,t} - U_s^{x,v}| \leqslant A(t,v)\mathrm{e}^{Cv}.$$

Using the above estimates in (5.7), it is easy to show continuity with values in C_b. In order to show continuity in L^p, we remark that the above estimates ensure convergence for all $x \in \mathbb{R}^3$, so that we need only to show uniform integrability. To this end, note that, by Lemma 5.2 and by the change of variables $x \to y = X_s^{x,t}$,

$$\begin{aligned}&\int_{|x|\geqslant K} |\boldsymbol{E}[U_s^{x,t}\psi(X_s^{x,t})]|^p \\ &\qquad \leqslant C\boldsymbol{E}\int_{X_s^{\cdot,t}(B_K{}^c)} |\psi(y)|^p \,\mathrm{d}y \\ &\qquad \leqslant C\boldsymbol{E}\int_{X_s^{\cdot,t}(B_K{}^c)} |\psi(y)|^p (\mathbf{1}_{\{|X_s^{x,t}-x|\leqslant K/2\}} + \mathbf{1}_{\{|X_s^{x,t}-x|\geqslant K/2\}}) \,\mathrm{d}y,\end{aligned}$$

and, since the set of $y = X_s^{x,t}$ such that $|y - x| \leqslant K/2$ and $|x| \geqslant K$, is contained in the set of y such that $|y| \geqslant K/2$,

$$\begin{aligned}\int_{|x|\geqslant K} |\boldsymbol{E}[U_s^{x,t}\psi(X_s^{x,t})]|^p &\leqslant C\int_{|y|\geqslant K/2} |\psi(y)|^p \boldsymbol{P}[|X_s^{x,t} - x| \leqslant K/2]\,\mathrm{d}y \\ &\qquad\qquad + C\|\psi\|_p^p \boldsymbol{P}[|X_s^{x,t} - x| \geqslant K/2], \\ &\leqslant C\int_{|y|\geqslant K/2} |\psi(y)|^p\,\mathrm{d}y + C\|\psi\|_p^p \boldsymbol{P}[|X_s^{x,t} - x| \geqslant K/2],\end{aligned}$$

where $C = T\mathrm{e}^{\|\nabla u\|_{\infty,T}}$ and, because of (5.8), for $K \to \infty$, the above quantity converges to 0 independently of s, t. □

Now it is possible to prove the above-mentioned result on the map $\mathcal{NS}$.

Proposition 5.5. *Given* $1 \leqslant p < \frac{3}{2}$ *and* $0 < \alpha < 1$, *let* $\psi \in C_b^\alpha(\mathbb{R}^3, \mathbb{R}^3) \cap L^p(\mathbb{R}^3, \mathbb{R}^3)$ *and* $g \in \mathcal{V}^{\alpha,p}(T)$. *Then* $\mathcal{NS}$ *maps* $\mathcal{U}^\alpha(T)$ *in* $\mathcal{V}^{\alpha,p}(T)$ *and*

$$\begin{aligned}&\|\mathcal{NS}(u)(t)\|_{C_b^\alpha\cap L^p} \\ &\qquad \leqslant \exp\Big\{3t\sup_{(0,t)}\|\nabla u\|_\infty\Big\}(1 + t\|\nabla u\|_{C_b^\alpha})\bigg(\|\psi\|_{C_b^\alpha\cap L^p} + \int_0^t \|g(s)\|_{C_b^\alpha\cap L^p}\,\mathrm{d}s\bigg). \quad (5.9)\end{aligned}$$

Proof. First, $\mathcal{NS}(u) \in C_b^\alpha \cap L^p$ follows by Lemma 5.3. Moreover, estimate (5.9) can also be easily deduced. Finally, from the previous lemma it follows that

$$t \mapsto \mathcal{NS}(u)(t) \in C_b^\alpha \cap L^p$$

is continuous. □

5.3. Lipschitz-continuity of the representation map

Let $g \in \mathcal{V}^{\alpha,p}(T)$ and $\psi \in C_b^\alpha(\mathbb{R}^3, \mathbb{R}^3) \cap L^p(\mathbb{R}^3, \mathbb{R}^3)$, and consider the map

$$\mathcal{NS} : \mathcal{U}^\alpha(T) \to \mathcal{V}^{\alpha,p}(T)$$

defined in the previous section. The aim of the present section is to show that such a map is locally Lipschitz-continuous. In order to do this, we will use the Girsanov formula. First we rewrite $\mathcal{NS}$ in a more appropriate form, namely

$$\mathcal{NS}(u)(t,x) = \boldsymbol{E}[F_{t,u}(X^{x,t,u})],$$

where, for each trajectory $w \in C([0,T];\mathbb{R}^3)$,

$$F_{t,u}(w) = V_t^{t,u}(w)\psi(w_t) + \int_0^t V_s^{t,u}(w)g(t-s,w_s)\,\mathrm{d}s$$

and $V^{t,u}(w)$ is the solution of the following differential equation:

$$\begin{aligned}\dot{V}_s^{t,u} &= V_s^{t,u}\mathcal{D}_u(t-s,w_s), \quad s \leqslant t, \\ V_0^{t,u}(w) &= I.\end{aligned}$$

Note that $U_s^{x,t,u}(\omega) = V_s^{t,u}(X^{x,t,u}(\omega))$, for each $\omega \in \Omega$, and we have made an explicit reference to the dependence on u in the Lagrangian paths $X^{x,t,u}$ and in the deformation matrices $U^{x,t,u}$.

By the Girsanov formula, we have

$$\boldsymbol{E}[F_{t,u}(X^{x,t,u})] = \boldsymbol{E}[Z_t^{x,t,u} F_{t,u}(x + \sqrt{2\nu} W)],$$

where

$$Z_s^{x,t,u} = \exp\biggl[\frac{1}{\sqrt{2\nu}} \int_0^s \langle u(t-r, x + \sqrt{2\nu} W_r), \mathrm{d}W_r\rangle - \frac{1}{4\nu} \int_0^s |u(t-r, x + \sqrt{2\nu} W_r)|^2 \, \mathrm{d}r\biggr],$$

with $s \leqslant t$, so that, for each u,

$$\begin{aligned}\mathcal{N\!S}(u)(t,x) = \boldsymbol{E}[Z_t^{x,t,u} V_t^{t,u}(x + \sqrt{2\nu} W) \psi(x + \sqrt{2\nu} W_t)] \\ + \int_0^t \boldsymbol{E}[Z_t^{x,t,u} V_s^{t,u}(x + \sqrt{2\nu} W) g(t-s, x + \sqrt{2\nu} W_t)] \, \mathrm{d}s.\end{aligned}$$

Using this representation, we will prove the following proposition.

Proposition 5.6. *Given $1 \leqslant p < \frac{3}{2}$ and $0 < \alpha < 1$, let $\psi \in C_b^\alpha(\mathbb{R}^3, \mathbb{R}^3) \cap L^p(\mathbb{R}^3, \mathbb{R}^3)$ and $g \in \mathcal{V}^{\alpha,p}(T)$, and set*

$$\varepsilon_0 = \|\psi\|_{C_b^\alpha \cap L^p} + \int_0^T \|g(s)\|_{C_b^\alpha \cap L^p} \, \mathrm{d}s.$$

For each u, $v \in \mathcal{U}_M^\alpha(T)$,

$$\sup_{0 \leqslant t \leqslant T} \|\mathcal{N\!S}(u)(t,\cdot) - \mathcal{N\!S}(v)(t,\cdot)\|_{C_b^\alpha \cap L^p} \leqslant C(\nu,p) C_M(T) \varepsilon_0 \sup_{0 \leqslant t \leqslant T} \|u(t,\cdot) - v(t,\cdot)\|_{C_b^{1,\alpha}},$$

where $C(\nu,p)$ is a constant depending only on p and ν and $C_M(T) \to 0$ as $T \to 0$.

The proof of the above proposition will be carried on using the subsequent lemmas. In order to make the explanations easier, we introduce the following notation. We define $\Delta_{xy} f = f(x) - f(y)$ for any function f. Note that

$$\Delta_{xy}(fg) = (\Delta_{xy} f) g(x) + f(y)(\Delta_{xy} g). \tag{5.10}$$

If the functions depend on two variables, we define Δ_{uvxy} as $\Delta_{uv}\Delta_{xy}$ and, by applying the above formula twice,

$$\begin{aligned}\Delta_{uvxy}(fg) &= \Delta_{uv}[(\Delta_{xy} f) g(\cdot, x) + f(\cdot, y)(\Delta_{xy} g)] \\ &= (\Delta_{uvxy} f) g(u,x) + [\Delta_{xy} f(v)][\Delta_{uv} g(x)] \\ &\quad + [\Delta_{uv} f(y)][\Delta_{xy} g(u)] + f(v,y)(\Delta_{uvxy} g).\end{aligned} \tag{5.11}$$

Lemma 5.7. *Let $u,\ v \in \mathcal{U}_M^\alpha(T)$. Then, for each $w, w' \in C([0,T];\mathbb{R}^3)$ and for all $s \leqslant t \leqslant T$,*

$$\begin{aligned} |V_s^{t,u}(w)| &\leqslant \mathrm{e}^{tM}, \\ |\Delta_{uv} V_s^{t,\cdot}(w)| &\leqslant t\mathrm{e}^{2tM} \|u-v\|_{C_b^1}, \\ |\Delta_{ww'} V_s^{t,u}(\cdot)| &\leqslant 2Mt\mathrm{e}^{2tM} \|w-w'\|_\infty^\alpha, \\ |\Delta_{uvww'} V_s^{t,\cdot}(\cdot)| &\leqslant (1+3tM)t\mathrm{e}^{3tM} \|w-w'\|_\infty^\alpha \|u-v\|_{C_b^{1,\alpha}}. \end{aligned}$$

Proof. The proofs of these properties are similar, we just give the proof of the last one. Indeed, using formula (5.11),

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s}(\Delta_{uvww'} V_s^{t,\cdot}(\cdot)) &= \Delta_{uvww'} \left(\frac{\mathrm{d}}{\mathrm{d}s} V_s^{t,\cdot}(\cdot) \right) \\ &= \Delta_{uvww'} (V_s^{t,\cdot}(\cdot) \mathcal{D}(t-s,\cdot)) \\ &= [\Delta_{uvww'} V_s^{t,\cdot}(\cdot)] \mathcal{D}_u(t-s,w_s) + V_s^{t,v}(w')[\Delta_{ww'} \mathcal{D}_{u-v}(t-s,\cdot)] \\ &\quad + [\Delta_{ww'} V_s^{t,v}(\cdot)] \mathcal{D}_{u-v}(t-s,w_s) + [\Delta_{uv} V_s^{t,\cdot}(w')][\Delta_{ww'} \mathcal{D}_u(t-s,\cdot)], \end{aligned}$$

so that, by using the other inequalities of this lemma,

$$\begin{aligned} &|\Delta_{uvww'} V_s^{t,\cdot}(\cdot)| \\ &\quad \leqslant M \int_0^s |\Delta_{uvww'} V_r^{t,\cdot}(\cdot)| \,\mathrm{d}r + \|w-w'\|_\infty^\alpha \|u-v\|_{C_b^{1,\alpha}} \int_0^s |V_r^{t,v}(w')| \,\mathrm{d}r \\ &\qquad + \|u-v\|_{C_b^{1,\alpha}} \int_0^s |\Delta_{ww'} V_r^{t,v}(\cdot)| \,\mathrm{d}r + M \|w-w'\|_\infty^\alpha \int_0^s |\Delta_{uv} V_r^{t,\cdot}(w')| \,\mathrm{d}r \\ &\quad \leqslant M \int_0^s |\Delta_{uvww'} V_r^{t,\cdot}(\cdot)| \,\mathrm{d}r + (1+3tM)s\mathrm{e}^{2tM} \|w-w'\|_\infty^\alpha \|u-v\|_{C_b^{1,\alpha}} \end{aligned}$$

and, by Gronwall's lemma, the inequality follows. □

Using the previous lemma and (5.10) and (5.11), we can easily deduce similar properties for the functional F.

Lemma 5.8. *Let $u,\ v \in \mathcal{U}_M^\alpha(T)$. Then for each $w, w' \in C([0,T];\mathbb{R}^3)$ and, for all $t \in [0,T]$,*

$$\begin{aligned} |F_{t,u}(w)| &\leqslant \mathrm{e}^{tM} \left[|\psi(w_t)| + \int_0^t |g(t-s,w_s)| \,\mathrm{d}s \right], \\ |\Delta_{uv} F_{t,\cdot}(w)| &\leqslant t\mathrm{e}^{2tM} \|u-v\|_{C_b^1} \left[|\psi(w_t)| + \int_0^t |g(t-s,w_s)| \,\mathrm{d}s \right], \\ |\Delta_{ww'} F_{t,u}(\cdot)| &\leqslant (1+2tM)\mathrm{e}^{2tM} \varepsilon_0 \|w-w'\|_\infty^\alpha, \\ |\Delta_{uvww'} F_{t,\cdot}(\cdot)| &\leqslant (2+3tM)t\mathrm{e}^{3tM} \varepsilon_0 \|w-w'\|_\infty \|u-v\|_{C_b^{1,\alpha}}, \end{aligned}$$

where

$$\varepsilon_0 = \|\psi\|_{C_b^\alpha} + \int_0^t \|g(s)\|_{C_b^\alpha} \,\mathrm{d}s.$$

Finally, we estimate the same quantities on the process Z.

Lemma 5.9. *Let u, $v \in \mathcal{U}_M^\alpha(T)$ and $q \geqslant 2$. Then, for all $s \leqslant t$,*

$$\begin{aligned}
\boldsymbol{E}|Z_s^{x,t,u}|^q &\leqslant C\mathrm{e}^{Ct^{q/2}M^q}, \\
\boldsymbol{E}|\Delta_{uv} Z_s^{x,t,\cdot}|^q &\leqslant Ct^{q/2}\mathrm{e}^{CM^q t^{q/2}}\|u-v\|_{C_b}^q, \\
\boldsymbol{E}|\Delta_{xy} Z_s^{\cdot,t,u}|^q &\leqslant Ct^{q/2}M^q\mathrm{e}^{CM^q t^{q/2}}|x-y|^{\alpha q}, \\
\boldsymbol{E}|\Delta_{uvxy} Z_s^{\cdot,t,\cdot}|^q &\leqslant Ct^{3q/2}M^{2q}\mathrm{e}^{CM^q t^{q/2}}|x-y|^{\alpha q}\|u-v\|_{C_b^{1,\alpha}}^q,
\end{aligned}$$

where $C = C(q,\nu)$ is a constant depending only on q and ν.

Proof. From the definition, we see that $Z_s^{x,t,u}$ solves

$$\left.\begin{aligned}
\mathrm{d}Z_s^{x,t,u} &= \frac{1}{\sqrt{2\nu}} Z_s^{x,t,u} u(t-s, x+\sqrt{2\nu}W_s)\,\mathrm{d}W_s, \quad s \leqslant t, \\
Z_0^{x,t,u} &= 1.
\end{aligned}\right\}$$

Again, the proofs of the four inequalities are similar and we prove only the last one. By applying (5.11), we get

$$\begin{aligned}
&\mathrm{d}_s(\Delta_{uvxy} Z_s^{\cdot,t,\cdot}) \\
&\quad = \Delta_{uvxy}(\mathrm{d}_s Z_s^{\cdot,t,\cdot}) \\
&\quad = \frac{1}{\sqrt{2\nu}}[(\Delta_{uvxy} Z_s^{\cdot,t,\cdot})u(t-s,Y_s^x)\,\mathrm{d}W_s + Z_s^{y,t,v}\Delta_{xy}[u(t-s,Y_s^\cdot) - v(t-s,Y_s^\cdot)]\,\mathrm{d}W_s \\
&\qquad + (\Delta_{xy} Z_s^{\cdot,t,v})[u(t-s,Y_s^x) - v(t-s,Y_s^x)]\,\mathrm{d}W_s + (\Delta_{uv} Z_s^{y,t,\cdot})[\Delta_{xy} u(t-s,Y_s^\cdot)]\,\mathrm{d}W_s],
\end{aligned}$$

where, for the sake of brevity, we have set $Y_s^x = x + \sqrt{2\nu}W_s$. By the Burkholder–Davis–Gundy inequality,

$$\begin{aligned}
\boldsymbol{E}|\Delta_{uvxy} Z_s^{\cdot,t,\cdot}|^q \leqslant C\bigg[M^q \boldsymbol{E}\bigg[\int_0^s |\Delta_{uvxy} Z_r^{\cdot,t,\cdot}|^2\,\mathrm{d}r\bigg]^{q/2} \\
+ \|u-v\|_{C_b^\alpha}^q |x-y|^{\alpha q}\boldsymbol{E}\bigg[\int_0^s |Z_r^{y,t,v}|^2\,\mathrm{d}r\bigg]^{q/2} \\
+ \|u-v\|_{C_b}^q \boldsymbol{E}\bigg[\int_0^s |\Delta_{xy} Z_r^{\cdot,t,v}|^2\,\mathrm{d}r\bigg]^{q/2} \\
+ M^q|x-y|^{\alpha q}\boldsymbol{E}\bigg[\int_0^s |\Delta_{uv} Z_r^{y,t\cdot}|^2\,\mathrm{d}r\bigg]^{q/2}\bigg],
\end{aligned}$$

so that, by using the Hölder inequality and the other inequalities of this lemma, we get

$$\begin{aligned}
\boldsymbol{E}|\Delta_{uvxy} Z_s^{\cdot,t,\cdot}|^q \leqslant CM^q s^{q/2-1}\int_0^s \boldsymbol{E}|\Delta_{uvxy} Z_r^{\cdot,t,\cdot}|^q\,\mathrm{d}r \\
+ CM^{2q}t^q s^{q/2}\exp\{CM^q t^{q/2}\}|x-y|^{\alpha q}\|u-v\|_{C_b^{1,\alpha}}^q.
\end{aligned}$$

Finally, using Gronwall's lemma, we obtain the required inequality. □

We are now able to prove the main result of this section.

Proof of Proposition 5.6. Let $u, v \in \mathcal{U}_M^\alpha(T)$. We start with the estimates in C_b and L^p. Using (5.10) and the Hölder inequality, we get, for each $x \in \mathbb{R}^3$ and $t > 0$,

$$\begin{aligned}|[\Delta_{uv}\mathcal{NS}(\cdot)](t,x)| &= |\boldsymbol{E}[\Delta_{uv}(Z_t^{x,t,\cdot}F_{t,\cdot}(Y^x))]| \\ &\leqslant C(q)[(\boldsymbol{E}|\Delta_{uv}Z_t^{x,t,\cdot}|^{q'})^{1/q'}(\boldsymbol{E}|F_{t,u}(Y^x)|^q)^{1/q} \\ &\qquad + (\boldsymbol{E}|Z_t^{x,t,v}|^{q'})^{1/q'}(\boldsymbol{E}|\Delta_{uv}F_{t,\cdot}(Y^x)|^q)^{1/q}], \end{aligned} \tag{5.12}$$

where $q \geqslant 1$, q' is the Hölder conjugate exponent of q, and we have set $Y_s^x = x + \sqrt{2\nu}W_s$. Using the estimates in Lemmas 5.8 and 5.9, and the inequality above with $q = 2$, we obtain the estimate in the C_b norm,

$$\sup_{t\leqslant T}\|\Delta_{uv}\mathcal{NS}(\cdot)\|_{C_b} \leqslant C\varepsilon_0(T+\sqrt{T})\mathrm{e}^{(CM^2+2M)T}\|u-v\|_{C_b^{1,\alpha}}.$$

Using again Lemmas 5.8 and 5.9 and the inequality (5.12) above, with $q = p$, we can obtain the estimate in the L^p norm,

$$\sup_{t\leqslant T}\|\Delta_{uv}\mathcal{NS}(\cdot)\|_{L^p}^p \leqslant C\varepsilon_0^p(T^p+T^{p/2})\exp\{2TM+CM^{p'}t^{p'/2}\}\|u-v\|_{C_b^{1,\alpha}}.$$

To conclude the proof, we need the estimate in the C_b^α norm. For all $x, y \in \mathbb{R}^3$ and $t > 0$, by applying formula (5.11) we get

$$\begin{aligned}|\Delta_{uvxy}\mathcal{NS}(\cdot)(t,\cdot)| &\leqslant \boldsymbol{E}[\Delta_{uvxy}(Z^{\cdot,t,\cdot}F_{t,\cdot}(Y^\cdot))] \\ &\leqslant \boldsymbol{E}[(\Delta_{uvxy}Z_t^{\cdot,t,\cdot})F_{t,u}(Y^x) + Z_t^{y,t,v}[\Delta_{uvxy}F_{t,\cdot}(Y^\cdot)] \\ &\qquad + (\Delta_{uv}Z_t^{y,t,\cdot})[\Delta_{xy}F_{t,u}(Y^\cdot)] + (\Delta_{xy}Z_t^{\cdot,t,v})[\Delta_{uv}F_{t,\cdot}(Y^x)]]\end{aligned}$$

Using the inequalities in Lemmas 5.8 and 5.9, it follows that

$$|\Delta_{uvxy}\mathcal{NS}(\cdot)(t,\cdot)| \leqslant C\varepsilon_0(\sqrt{T}+T+MT^{3/2}+MT^2)\mathrm{e}^{3TM+CTM^2}|x-y|^\alpha\|u-v\|_{C_b^{1,\alpha}}.$$

□

Acknowledgements. The authors thank M. Cannone for the bibliographical suggestions regarding the vorticity approach to the Navier–Stokes equations. The authors also thank the referee, whose suggestions improved the paper.

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