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# GENERALIZED VARIATION AND FUNCTIONS OF SLOW GROWTH

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**1. Introduction.** Many of the basic results of $H^p$ theory on the disk $\Delta = \{ |z| < 1\}$ are proved using the Cauchy-Stieltjes representation

$$(1.1)\quad f(z) = \frac{1}{2\pi}\int_0^{2\pi} K(z, t)d\mu(t), \quad z \in \Delta,$$

and the Poisson-Stieltjes representation

$$(1.2)\quad u(z) = \frac{1}{2\pi}\int_0^{2\pi} P(z, t)d\mu(t), \quad z \in \Delta.$$

Here, $\mu:\mathbf{R} \rightarrow \mathbf{C}$ is a complex-valued function of a real variable that is of bounded variation on $[0, 2\pi]$ such that $\mu(t + 2\pi) = \mu(t) + \mu(2\pi) - \mu(0)$, $t \in \mathbf{R}$,

$$K(z, t) = \frac{e^{it}}{e^{it} - z}, \quad z \in \Delta, t \in \mathbf{R},$$

is the Cauchy kernel, and

$$P(z, t) = \mathrm{Re}\left(\frac{e^{it} + z}{e^{it} - z}\right) = \frac{1 - |z|^2}{|e^{it} - z|^2}, \quad z \in \Delta, t \in \mathbf{R},$$

is the Poisson kernel. It is therefore natural to generalize these representations in such a way that some of the basic properties and results carry over. Such a generalization occurs when the assumption that $\mu$ is of bounded variation on $[0, 2\pi]$ is replaced by the requirement that it is measurable and bounded on $[0, 2\pi]$ (cf. [**9**]). The integrals in (1.1) and (1.2) are then defined by a formal integration by parts. After some preliminaries in Section 2, we catalogue a variety of results which remain valid in Section 3.

It is classical (see [**7**]) that all functions in $H^1$ and $h^1$ admit representations of the form (1.1) and (1.2), where $\mu$ is of bounded variation on $[0, 2\pi]$. (The latter is sometimes referred to as the Riesz-Herglotz representation.) In Section 4, we define generalizations $H^1_\psi$ and $h^1_\psi$ based

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on $\psi$-bounded variation studied by Musielak and Orlicz [**13**], where $\psi:[0, \infty) \rightarrow [0, \infty)$ is an increasing convex function satisfying $\psi(0) = 0$. When

$$\liminf_{t \to 0} \psi(t)/t > 0,$$

then this reduces to the classical case. We shall be particularly interested in functions $F \in H^\infty$ and $U \in h^\infty$ for which the boundary functions are, up to multiplication by a constant, of $\psi$-bounded variation. In this case

$$iz\frac{dF}{dz} \in H^1_\psi \quad \text{and} \quad \frac{\partial U}{\partial \theta} \in h^1_\psi.$$

In [**6**], Caveny and Novinger showed that when $F$ is analytic on $\Delta$ and $F' \in H^1$, the zero set of the boundary function $F^*$ of $F$ is a BCH (Beurling-Carleson-Hayman) set. The condition that $F' \in H^1$ is equivalent to the assumption that $F$ has a continuous extension to $\bar{\Delta} = \{ |z| \leqq 1\}$ (that is, it is in the disk algebra $A$) and $F^*$ is of bounded variation. In Section 5 we prove the corresponding result for the generalized spaces $H^1_\psi$ in terms of generalized BCH sets (see Section 2). The condition $F' \in H^1$ is replaced by $izF'(z) \in H^1_\psi$, or equivalently, the assumption that $F \in A$ and $cF^*$ is of $\psi$-bounded variation for some $c > 0$.

The subclass $h^\infty$ of $h^1$ consists of functions of the form (1.2) for which the modulus of continuity $\omega_\mu$ of $\mu$ satisfies

$$\omega_\mu(t) = O[t] \quad \text{as } t \rightarrow 0.$$

Let $h^\infty_\psi$ be the subclass of $h^1_\psi$ for which

$$\omega_\mu(t) = O[\psi^{-1}(t)] \quad \text{as } t \rightarrow 0.$$

It is a direct consequence of a result of [**5**] (see Section 3), that when $u \in h^\infty_\psi$, we have

$$(1.3) \qquad |u(z)| \leqq c\frac{\psi^{-1}(1 - |z|)}{1 - |z|}, \quad z \in \Delta,$$

for some $c > 0$. In Section 6 we show that a function $u$ in the larger class $h^1_\psi$ has a similar growth restriction along almost all radii, and for faster growth, the exceptional sets have zero Hausdorff measure. This generalizes a result of Å. Samuelsson [**16**, pp. 489-490] for positive harmonic functions.

A radial Phragmén-Lindelöf theorem proved in joint work of the author with W. Cohn [**2**] asserts that if a subharmonic function $u$ on $\Delta$ has a limit superior of 0 along all except a small exceptional set of radii, then $u \leqq 0$. Here, the size of the exceptional set depends on the rate of growth of $u$. In Section 7 we prove two supplementary results for functions $u$ in $h^1$ for which $u \not\leq 0$. In one, we prove that positive or infinite radial limits must exist on a set of nonzero Hausdorff measure, and in a second, we show that if the exceptional set is a countable union of generalized BCH sets, then the function must grow at least at a given rate along some radius.

In the final section, Section 8, we give some quantitative relationships between the rate at which the maximum modulus $M(r;\ |f|\ )$ of an inner function $f$ tends to 1 as $r \to 1$, and the size of the set of preimages $E_\zeta$ of $\zeta$ under the boundary function $f^*$. For example, we show that if $M(r;\ |f|\ )$ tends to 1 sufficiently slowly as $r \to 1$, then for each $\zeta$ in the unit circle, the set $E_\zeta$ has nonzero Hausdorff measure.

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**2. Preliminaries.** In this section we establish notation and recall some basic facts that will be useful.

We assume throughout that $\omega \not\equiv 0$ is a continuous modulus of continuity on $[0, 2\pi]$. By definition, $\omega = \omega_h$ where $h$ is a continuous complex-valued function on $[0, 2\pi]$ and

$$\omega_h(t) = \sup\{\ |h(x) - h(y)\ |:|x - y| \leqq t\}, \quad t \in [0, 2\pi].$$

When it is convenient, we assume that $\omega$ is extended to $[0, \infty)$ so that it is constant on $[2\pi, \infty)$. It is easy to verify that $\omega(0) = 0$, $\omega$ is monotone nondecreasing, and $\omega$ is subadditive, that is,

$$\omega(t + s) \leqq \omega(t) + \omega(s), \quad t, s \geqq 0.$$

In addition, $\omega$ is its own modulus of continuity,

$$\liminf_{t \to 0} \omega(t)/t > 0,$$

and

$$(2.1) \qquad \frac{\lambda}{\lambda + 1}\omega(t) \leqq \omega(\lambda t) \leqq (\lambda + 1)\omega(t), \quad \lambda, t \geqq 0.$$

A continuous, increasing, concave-downward function on $[0, 2\pi]$ vanishing at 0 is an example of an allowed $\omega$. We say that $\omega$ is *smooth* if it has a continuous derivative on $(0, 2\pi]$ and

$$\lim_{t \to 0} \omega'(t) = \omega'(0) \in (0, +\infty].$$

If $\omega < 1$, then the function

$$(2.2) \qquad \Phi_\omega(t) = \int_0^t \log\left[\frac{1}{\omega(s)}\right] ds, \quad t \in [0, 2\pi],$$

is a smooth concave-downward modulus of continuity.

The following lemmas can be found in [**4**].

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LEMMA 2.1 ([**4**, Theorem 2.1]). *Given* $\epsilon > 0$, *there exists a smooth, concave-downward modulus of continuity* $\bar{\omega}$ *such that* $(1/2 - \epsilon)\omega \leqq \bar{\omega} \leqq \omega$.

LEMMA 2.2 ([**4**, Theorem 3.2]). *If* $\omega$ *is a smooth, concave-downward modulus of continuity such that* $\omega < 1$, *then there exists* $M > 0$ *such that*

$$\Phi_\omega(t) \leqq Mt \log \frac{1}{\omega(t)}, \quad t \in (0, 2\pi].$$

We shall frequently refer to two classes of thin subsets of $C$, those of Hausdorff measure 0 and generalized BCH (Beurling-Carleson-Hayman) sets. These are defined as follows.

For an arbitrary Borel subset $E$ of $C$, the Hausdorff measure $H_\omega(E)$ of $E$ is given by

$$H_\omega(E) = \lim_{r \to 0^+} \left\{ \inf \sum_j \omega(|I_j|) \right\},$$

where the infimum is taken over all countable covers $(I_j)$ of $E$ by open arcs $I_j$ with length $|I_j| \leqq r$. If $\omega(t) = t$, then $H_\omega$ is linear Lebesgue measure $m$.

The class of $\omega$-sets is defined to be the class of closed subsets $K$ of $C$ having $m(K) = 0$, such that

$$\sum \omega(|J_k|) < \infty,$$

where $(J_k)$ is an enumeration of the component arcs of $C - K$. The BCH sets arise when

$$\omega(t) = t \log[(2\pi e)/t].$$

Hausdorff measure and generalized BCH sets are analogously defined on closed intervals in **R**.

The following is a standard result concerning Hausdorff measure (see [**12**, Theorème III, Chapitre II, p. 27]).

THEOREM 2.1. *Let* $E$ *be a Borel subset of* $C$. *The following are equivalent*:

(1) $H_\omega(E) > 0$,

(2) $E$ *supports a finite positive Borel measure* $\mu$ *such that*

$$\omega_\mu(t) = O[\omega(t)] \quad \text{as } t \to 0.$$

Recall that to each complex Borel measure $\mu$ on $C$, there is associated a function $\hat{\mu}:\mathbf{R} \to \mathbf{C}$ defined by

$$\hat{\mu}(t) = \mu([0, t)) \quad \text{for } t \in [0, 2\pi]$$

with the extension to **R** made by the requirement that

$$\hat{\mu}(t + 2\pi) = \hat{\mu}(t) + \hat{\mu}(2\pi), \quad t \in \mathbf{R}.$$

We shall take $\omega_{\mu}(t)$ to be

$$\omega_{\hat{\mu}|[0,2\pi]}, \quad t \in [0, 2\pi].$$

In the sequel, $\mu$ will be identified with $\hat{\mu}$, with the precise interpretation being determined by context.

The relationship between generalized BCH sets and Hausdorff measure 0 sets is studied in [**4**]. One result that is of interest here is that every $\omega$-set $K$ has $H_{\omega}(K) = 0$.

For $\mu:\mathbf{R} \rightarrow \mathbf{R}$ a measurable function, let

$$(2.3) \qquad \bar{D}_{\phi}\mu(t) = \limsup_{|I|\to 0} \frac{\mu(y) - \mu(x)}{\phi(I)}, \quad t \in \mathbf{R},$$

where the 'lim sup' is taken over all nondegenerate closed intervals $I = [x, y]$ with $t \in (x, y)$, and

$$\phi(I) = \omega(\,|I|\,) \quad \text{or} \quad \phi(I) = \nu[\,\{e^{it}:t \in I\}\,]$$

where $\nu$ is a real Borel measure on $C$. The convention $0/0 = 0$ is understood in (2.3). Similarly, define $\underline{D}_{\phi}\mu(t)$ with 'lim sup' replaced by 'lim inf' and write $D_{\phi}\mu$ for the common value when

$$\bar{D}_{\phi}\mu(t) = \underline{D}_{\phi}\mu(t).$$

The corresponding symmetric derivatives $\bar{D}^{s}_{\phi}\mu$, $\underline{D}^{s}_{\phi}\mu$, and $D^{s}_{\phi}\mu$ are defined by assuming that the intervals $I$ are centered at $t$. The natural interpretation of $D_{\phi}\mu$ and $D^{s}_{\phi}\mu$ in terms of real and imaginary parts is made when $\mu$ is complex-valued. If $\mu(y) - \mu(x)$ is replaced by $|\mu(y) - \mu(x)|$ in (2.3), we use the notation $\overline{AD}_{\phi}\mu(t)$, $\underline{AD}_{\phi}\mu(t)$ and $AD_{\phi}\mu(t)$ and add a superscript '$s$' for the corresponding symmetric derivatives. When $\omega(t) = t$ or $\nu$ is equal to linear Lebesgue measure $m$ on $C$, then we omit the subscript $\phi$ from the notation defined above.

If $\mu$ and $\nu$ are complex Borel measures on $C$, then we write $\mu \ll \nu$ if $\mu$ is absolutely continuous with respect to $\nu$ and $\mu \perp \nu$ if $\mu$ and $\nu$ are mutually singular. The following results are standard and can be found in [**15**, p. 166].

THEOREM 2.2. *Let $\mu$ be a complex Borel measure on $C$. The following assertions are valid*:

(a) *$D\mu$ exists* a.e. $[m]$,

(b) $D\mu \in L^1[0, 2\pi]$, *and*

(c) *for every Borel subset $E$ of $C$, we have*

$$\mu(E) = \mu_s(E) + \int_E D\mu(t)dt,$$

*where $\mu_s \perp m$ and $D\mu_s(t) = 0$* a.e. $[m]$.

COROLLARY 2.1. (i) *$\mu \perp m$ if and only if $D\mu(t) = 0$* a.e. $[m]$.

(ii) *$\mu \ll m$ if and only if $\mu(E) = \int_E D\mu(t)dt$ for every Borel set $E$. In this case, the derivative $D\mu$ coincides* a.e. $[m]$ *to the Radon-Nikodym derivative.*

Note that $D\mu$ is the derivative of the function $\mu:\mathbf{R} \to \mathbf{C}$ of bounded variation on $[0, 2\pi]$ associated with the measure $\mu$, and the integrals are understood to be taken over

$$\{t \in [0, 2\pi]:e^{it} \in E\}.$$

The next theorem is a direct generalization of a theorem of de la Vallée Poussin where linear measure $m$ in the statement and proof given in [**15**, Theorem 8.10] is replaced by $\nu$.

THEOREM 2.3. *Let $\mu$ and $\nu$ be finite nonnegative Borel measures on $C$ with $\mu \perp \nu$. Then $D_\nu\mu = +\infty$ a.e. $[\mu]$.*

The following version of Riemann-Stieltjes integration will be used for the generalized Cauchy-Stieltjes and Poisson-Stieltjes integrals.

*Definition* 2.1. Suppose that $-\infty < a < b < +\infty$ and $f$, $g$ are mappings of $\mathbf{R}$ into $\mathbf{C}$. When $f$ is measurable and bounded on $[a, b]$ and $g$ is absolutely continuous, define

$$\int_a^b gdf = g(b)f(b) - g(a)f(a) - \int_a^b fg'dt.$$

Note that $g' \in L^1[a, b]$ so that the last integral is defined as a Lebesgue integral. The usual linearity properties of $\int_a^b gdf$ along with the fact that

$$\int_a^c gdf + \int_c^b gdf = \int_a^b gdf$$

when $a \leqq c \leqq b$ are verified without difficulty. In addition, we note that

$$\left|\int_a^b gdf\right| \leqq |g(b)f(b) - g(a)f(a)| + (b - a)\|f\|_\infty\|g'\|_1.$$

The proofs of the following results are elementary so we omit them.

LEMMA 2.3. *If $g(t + 2\pi) = g(t)$ and $f(t + 2\pi) = f(t) + f(2\pi) - f(0)$, $t \in \mathbf{R}$, then*

$$\int_a^b gdf = \int_{a+2n\pi}^{b+2n\pi} gdf, \quad n = 0, \pm 1, \pm 2, \dots .$$

COROLLARY 2.2. *If $\theta \in \mathbf{R}$, then*

$$\int_0^{2\pi} gdf = \int_{\theta-\pi}^{\theta+\pi} gdf.$$

LEMMA 2.4. *If $\delta > 0$, $\theta \in \mathbf{R}$, and $g(s) = g(-s)$ for $s \in \mathbf{R}$, then*

$$\int_0^\delta g'(s)[f(\theta + s) - f(\theta - s)]ds$$

$$= \int_{\theta-\delta}^{\theta+\delta} [g(\delta) - g(\theta - t)]df(t).$$

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**3. Generalized representations.** Using Definition 2.1, we can write (1.1) and (1.2) more explicitly. Recall that $\mu:\mathbf{R} \to \mathbf{C}$ is assumed to be measurable and bounded on $[0, 2\pi]$ with

$$\mu(t + 2\pi) = \mu(t) + \mu(2\pi) - \mu(0), \quad t \in \mathbf{R},$$

and the Cauchy and Poisson kernels $K(z, t)$ and $P(z, t)$ have continuous partial derivatives with respect to $t$. Thus

$$(3.1) \quad f(z) = \frac{1}{2\pi}\int_0^{2\pi} K(z, t)d\mu(t)$$
$$= \frac{\mu(2\pi) - \mu(0)}{2\pi}\frac{1}{1 - z} + \frac{i}{2\pi}\int_0^{2\pi}\frac{ze^{it}}{(e^{it} - z)^2}\mu(t)dt$$

and

$$(3.2) \quad u(z) = \frac{1}{2\pi}\int_0^{2\pi} P(z, t)d\mu(t)$$
$$= \frac{\mu(2\pi) - \mu(0)}{2\pi}\frac{1 - |z|^2}{|1 - z|^2} + \frac{1}{2\pi}\int_0^{2\pi}\mathrm{Re}\left[\frac{2zie^{it}}{(e^{it} - z)^2}\right]\mu(t)dt.$$

By standard results (see, for example [**11**, Ch. V, Section 6]) these define functions analytic and (complex) harmonic in $\mathbf{C} - C$, where $C = \{\,|z| = 1\}$ is the circumference of the unit disk $\Delta$.

Throughout this section, we assume that $f$ and $u$ are given as in (3.1) and (3.2) and $p(r, \delta) = P(z, t)$ when $z = re^{i\theta}$ and $\delta = \theta - t$. The proofs of the following lemma and theorem are omitted since they closely parallel the proofs of similar results given in [**15**, pp. 256-258].

LEMMA 3.1. *Let $\theta \in \mathbf{R}$ and let $\mu$ be real-valued. If there exists $\delta \in (0, \pi]$ and a real number $k$ such that*

$$\mu(\theta + s) - \mu(\theta - s) < 2sk, \quad s \in (0, \delta],$$

*then*

$$(3.3) \quad u(re^{i\theta}) < k + |k|p(r, \delta) + \frac{3}{\pi}p(r, \delta)M,$$

*where* $M = \sup\{\,|\mu(t)\,|:\theta - \pi \leqq t \leqq \theta + \pi\}$.

To follow the outline of the proof given in [**15**], it would be necessary to use Corollary 2.2 and Lemma 2.4.

THEOREM 3.1. *If $\mu$ is real-valued, then*

$$\underline{D}^s\mu(\theta) \leqq \liminf_{r\to 1} u(re^{i\theta}) \leqq \limsup_{r\to 1} u(re^{i\theta}) \leqq \bar{D}^s\mu(\theta)$$

*for every $\theta \in \mathbf{R}$.*

The next lemma is closely related to one given by Samuelsson [**16**, Lemma 4.2].

LEMMA 3.2. *Let $\theta \in \mathbf{R}$ and let $\mu$ be complex-valued. There exists a constant $c > 0$ (independent of $\theta$, $\mu$, and $\omega$) such that*

$$\limsup_{r \to 1} \frac{1-r}{\omega(1-r)} |u(re^{i\theta})| \leqq c\overline{AD}^s_\omega \mu(\theta).$$

*Proof.* By Lemma 2.1, we can assume without loss of generality that $\omega$ is concave downward so that $\omega(t)/t$ is a monotone nonincreasing function on $(0, \infty)$. If

$$\overline{AD}^s_\omega \mu(\theta) < k < +\infty,$$

then there exists $\delta \in (0, \pi)$ such that

$$|\mu(\theta + s) - \mu(\theta - s)| \leqq k\omega(2s), \quad s \in (0, \delta].$$

From Lemma 2.4, it follows that

$$\begin{aligned}\left|\int_{\theta-\delta}^{\theta+\delta} P(z, t)d\mu(t)\right| &\leqq p(r, \delta)\,|\mu(\theta + \delta) - \mu(\theta - \delta)| \\ &\quad + \int_0^\delta |\mu(\theta + s) - \mu(\theta - s)|\,[-p_s(r, s)]ds \\ &\leqq kp(r, \delta)\omega(2\delta) \\ &\quad + k\int_0^\delta \omega(2s)[-p_s(r, s)]ds.\end{aligned}$$

If $1 - r \in (0, \delta)$, then using (2.1) and the monotonicity of $\omega(t)/t$, we have

$$\begin{aligned}&\frac{1-r}{\omega(1-r)}\int_0^\delta \omega(2s)[-p_s(r, s)]ds \\ &= (1-r)\int_0^{1-r} \frac{\omega(2s)}{\omega(1-r)}[-p_s(r, s)]ds \\ &\quad + 2\int_{1-r}^\delta \frac{\omega(2s)}{2s}\frac{1-r}{\omega(1-r)}s[-p_s(r, s)]ds \\ &\leqq 3(1-r)\int_0^{1-r} [-p_s(r, s)]ds + 2\int_{1-r}^\delta s[-p_s(r, s)]ds \\ &= 3(1-r)[p(r, 0) - p(r, 1-r)] \\ &\quad - 2sp(r, s)\,|_{1-r}^\delta + 2\int_{1-r}^\delta p(r, s)ds \\ &\leqq 3(1-r)\left[\frac{1+r}{1-r} - p(r, 1-r)\right] \\ &\quad - 2\delta p(r, \delta) + 2(1-r)p(r, 1-r) + 2\pi \leqq 6 + 2\pi.\end{aligned}$$

Since $P(z, t) \leqq p(r, \delta)$ and $P_t(z, t) < 0$ for $\delta \leqq \theta - t \leqq \pi$, we have

$$\left|\int_{\theta+\delta}^{\theta+\pi} P(z, t)d\mu(t)\right|$$
$$= \left|\mu(\theta + \pi)p(r, \pi) - \mu(\theta + \delta)p(r, \delta) - \int_{\theta+\delta}^{\theta+\pi} \mu(t)P_t(z, t)dt\right|$$
$$\leqq 3Mp(r, \delta)$$

where

$$M = \sup\{ |\mu(t)| : \theta - \pi \leqq t \leqq \theta + \pi\}.$$

A similar argument also shows that

$$\left|\int_{\theta-\pi}^{\theta-\delta} P(z, t)d\mu(t)\right| \leqq 3Mp(r, \delta).$$

By Corollary 2.2, we have

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_{\theta-\pi}^{\theta+\pi} P(z, t)d\mu(t).$$

Putting together the results obtained above, we see that

$$\frac{1 - r}{\omega(1 - r)}|u(re^{i\theta})| \leqq \frac{k(1 - r)p(r, \delta)\omega(2\delta)}{2\pi\omega(1 - r)} + \frac{k(6 + 2\pi)}{2\pi} + \frac{3(1 - r)Mp(r, \delta)}{\pi\omega(1 - r)}$$

for $1 - r \in (0, \delta)$. Since $p(r, \delta) \to 0$ as $r \to 1$ and

$$\limsup_{r\to 1} \frac{1 - r}{\omega(1 - r)} < +\infty,$$

the first and third term approach 0 and the lemma follows.

The following proposition and theorem are analogues of results appearing in [7, pp. 39-41]. We omit the proofs.

PROPOSITION 3.1. *If* $z \in \mathbf{C} - (C \cup \{0\})$, *then*

$$u(z) = f(z) - f(1/\bar{z}).$$

THEOREM 3.2. *The following statements are equivalent.*
(i) $\int_0^{2\pi} e^{int}d\mu(t) = 0, \quad n = 1, 2, \ldots,$
(ii) $f(z) = 0, \quad |z| > 1,$
(iii) $u(z)$ *is analytic on* $\Delta$.
*If* (i), (ii), *and* (iii) *hold, then* $u(z) = f(z)$, $z \in \Delta$.

Let $u$ be a function of the form (3.2). Then $\tilde{u}(z) = u(z) - u(0)$ is a function admitting a similar representation with $\mu$ replaced by

$\tilde{\mu}(t) = \mu(t) - u(0)t$. If $u$ is not constant and $\mu$ is continuous with $\omega_\mu(t) = O[\omega(t)]$, then $\omega_{\tilde{\mu}}(t) = O[\omega(t)]$ (as $t \to 0$) as well. By (3.2) applied to $\tilde{u}$, we see that

$$\tilde{u}(z) = \frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re}\left[\frac{2zie^{it}}{(e^{it} - z)^2}\right]\tilde{\mu}(t)dt$$
$$= \frac{\partial}{\partial\theta}\tilde{U}(z),$$

where

$$(3.4) \qquad \tilde{U}(z) = \frac{1}{2\pi}\int_0^{2\pi} P(z, t)\tilde{\mu}(t)dt, \quad z = re^{i\theta} \in \Delta.$$

The result of Brudnyi and Gopengauz alluded to in Section 1 asserts that when $\tilde{U}(z)$ is given by (3.4) with $\tilde{\mu}(t)$ continuous, we have

$$\left|\frac{\partial\tilde{U}}{\partial\theta}(z)\right| \leqq c|z|\frac{\omega(1 - |z|)}{1 - |z|}, \quad z \in \Delta,$$

for some $c > 0$. An immediate consequence is the following: If $u$ is given by (3.2) and $\mu$ is continuous, then

$$|u(z)| \leqq c\frac{\omega_\mu(1 - |z|)}{1 - |z|}, \quad z \in \Delta,$$

for some $c > 0$.

If $u(z) = f(z)$ as in Theorem 3.2, then

$$\frac{\partial u}{\partial\theta} = iz\frac{df}{dz}, \quad z \in \Delta.$$

Thus it follows that

$$\left|\frac{df}{dz}\right| \leqq c\frac{\omega_\mu(1 - |z|)}{1 - |z|}, \quad z \in \Delta,$$

for some $c > 0$.

In connection with these results and for use later, we state a result of H. S. Shapiro [**17**] and P. Ahern [**1**, Lemma 1.2] for the case when $\mu$ is monotone nondecreasing.

THEOREM 3.3. *If $\mu$ is a finite nonnegative Borel measure, then there exist constants $c_1, c_2 > 0$ such that*

$$c_1\frac{\omega_\mu(1 - |z|)}{1 - |z|} \leqq M(|z|; u) \leqq c_2\frac{\omega_\mu(1 - |z|)}{1 - |z|}, \quad z \in \Delta,$$

*where*

$$M(r; u) = \max_{\theta\in\mathbf{R}} u(re^{i\theta}), \quad r \in [0, 1), \theta \in \mathbf{R}.$$

**4. Generalized variation.** We start by giving some background concerning generalized variation (see **[13]** ).

Let $\psi:[0, \infty) \rightarrow [0, \infty)$ be an increasing convex function with $\psi(0) = 0$ such that $\psi \circ \omega$ as well as $\omega$ is a modulus of continuity. Recall that $\mu:\mathbf{R} \rightarrow \mathbf{C}$ is measurable and bounded on $[0, 2\pi]$ and satisfies

$$\mu(t + 2\pi) = \mu(t) + \mu(2\pi) - \mu(0), \quad t \in \mathbf{R}.$$

*Definition* 4.1. The function $\mu$ is said to be of *$\psi$-bounded variation* (on $[0, 2\pi]$ ) if there exists $M > 0$ such that

$$(4.1) \qquad \sum \psi[\,|\mu(y_k) - \mu(x_k)|\,] \leqq M$$

whenever $n$ is a positive integer and $0 \leqq x_1 < y_1 \leqq \ldots \leqq x_n < y_n \leqq 2\pi$. Also,

$$\psi\ \mathrm{BV} = \{\mu{:}c\mu \text{ is of } \psi\text{-bounded variation for some } c > 0\}.$$

It is not difficult to show that if $\mu \in \psi$ BV, then it is regulated. This means that $\mu$ has both one-sided limits at each point of $\mathbf{R}$. In particular, $\mu$ is bounded on any closed interval with at most countably many discontinuities, each of these being a jump discontinuity. In this connection we also note that Goffman, Moran, and Waterman **[8]** have shown that each regulated function $\mu$ is contained in $\psi$ BV for some $\psi$.

*Definition* 4.2. Let $h_\psi^1$ denote the class of harmonic functions $u$ defined by (3.2) for which $\mu \in \psi$ BV. The subclass of analytic functions in $h_\psi^1$ is denoted $H_\psi^1$.

Observe that Theorem 3.2 shows that $H_\psi^1$ can be defined equivalently as the functions $f(z)$ admitting a representation (3.1) with $f(z) = 0$ for $|z| > 1$. We note that in case

$$\liminf_{t \to 0} \psi(t)/t > 0,$$

we have $h_\psi^1$, respectively $H_\psi^1$, is the classical Hardy space $h^1$, respectively $H^1$.

In the sequel, $A$ denotes the class of analytic functions $F$ on $\Delta$ admitting a continuous extension to $\bar{\Delta}$. For any analytic function $f$ on $\Delta$ of bounded characteristic, let $f^*$ denote the radial limit function of $f$. The next lemma and corollary are classical so we omit their proofs.

LEMMA 4.1. *If $F \in H^\infty$ and $\mu:\mathbf{R} \rightarrow \mathbf{C}$ has period $2\pi$ with*

$$F^*(e^{it}) = \mu(t) \text{ a.e.},$$

*then $\mu$ cannot have a jump discontinuity.*

COROLLARY 4.1. *Let $\mu$ be a regulated function. If $F \in H^\infty$ and*

$$F^*(e^{it}) = \mu(t) \text{ a.e.},$$

*then $F \in A$ and*

$$F^*(e^{it}) = \mu(t), \quad t \in \mathbf{R}.$$

The following proposition parallels a well-known classical result (for $H^1_\psi = H^1$).

PROPOSITION 4.1. *An analytic function $F$ on $\Delta$ satisfies $izF'(z) \in H^1_\psi$ if and only if $F \in A$ and $F^*(e^{it}) \in \psi$ BV.*

*Proof.* If $F \in A$ and $F^*(e^{it}) \in \psi$ BV, then a direct calculation shows that $f(z) = izF'(z)$ admits a representation (3.1) with

$$\mu(t) = F^*(e^{it}), \quad t \in \mathbf{R}.$$

Conversely, if $izF'(z) \in H^1_\psi$, then $f(z) = izF'(z)$ has a representation (3.1) with $\mu(0) = \mu(2\pi) = 0$ (since $f(0) = 0$) and $\mu \in \psi$ BV. Hence

$$F(z) = \frac{1}{2\pi}\int_0^{2\pi} K(z, t)\mu(t)dt + c, \quad z \in \Delta,$$

where $c$ is a complex constant. In particular, we have $F \in H^\infty$ and

$$F^*(e^{it}) - c = \mu(t) \text{ a.e.}$$

Corollary 4.1 now implies that $F \in A$ and $F^*(e^{it}) \in \psi$ BV as required.

**5. A generalization of a theorem of Caveny and Novinger.** We start with a lemma.

LEMMA 5.1. *Let $\alpha \in (0, 1)$ and suppose that $\omega = \psi^{-1}|_{[0,2\pi]}$ is a smooth concave-downward modulus of continuity such that $0 \not\equiv \omega < 1$. Then*

$$\lim_{t\to 0} \frac{\Phi_\omega(t)}{\psi[\omega(t)^\alpha]} = 0.$$

*Proof.* By l'Hopitâl's rule

$$\lim_{t\to 0} \frac{\Phi_\omega(t)}{\psi[\omega(t)^\alpha]} = \lim_{t\to 0} \frac{\log[1/\omega(t)]\,\omega'[\psi(\omega(t)^\alpha)]}{\alpha\omega(t)^{\alpha-1}\omega'(t)}$$

$$= \frac{1}{(1-\alpha)\alpha} \lim_{t\to 0} \omega^{1-\alpha}(t)\log[1/\omega^{1-\alpha}(t)] \times \frac{\omega'[\psi(\omega(t)^\alpha)]}{\omega'(t)}$$

$$\leqq \frac{1}{(1-\alpha)\alpha} \lim_{t\to 0} \omega^{1-\alpha}(t) \log[1/\omega^{1-\alpha}(t)]$$

$$= 0.$$

Here we used the fact that $\psi[\omega(t)^\alpha] \geqq t$, $\omega'$ is a monotone nonincreasing function, and

$$\lim_{s\to 0^+} s \log(1/s) = 0.$$

This completes the proof.

The following generalizes a theorem and proof of Caveny and Novinger which corresponds to the case when $\psi(t) = t$.

THEOREM 5.1. *Let $F \not\equiv 0$ be an analytic function on $\Delta$ such that*

$$izF'(z) \in H^1_\psi.$$

*Then $F \in A$ and $Z(F^*) \equiv \{F^* = 0\}$ is a $\Phi_\omega$-set, where*

$$\omega = \psi^{-1}|_{[0,2\pi]}.$$

*Proof.* By Lemma 2.1 we can assume without loss of generality that $\omega$ is a smooth concave-downward modulus of continuity with $0 \not\equiv \omega < 1$.

Proposition 4.1 implies that $F \in A$ and $F^*(e^{it}) \in \psi$ BV. We suppose, as we may, that $|F| \leqq 1$ and $F^*(e^{it})$ is of $\psi$-bounded variation. Since $F \not\equiv 0$ and $F \in A$, the set $Z(F^*)$ is a closed set of measure 0. Let $\alpha \in (0, 1)$. Divide the component arcs of $C - Z(F^*)$ into two subsequences $(I_k)$ and $(J_l)$ such that

$$(5.1) \qquad |F^*(\eta)| \leqq \omega(|I_k|)^\alpha, \quad \eta \in I_k,$$

and

$$(5.2) \qquad |F^*(\xi_l)| > \omega(|J_l|)^\alpha$$

for some $\xi_l \in J_l$. Since $F$ is of bounded characteristic (see [7, Theorem 2.2]), we have

$$\infty > \int_C \log\frac{1}{|F^*(\eta)|}\,|d\eta| \geqq \sum \alpha|I_k| \log\frac{1}{\omega(|I_k|)}$$

$$\geqq \frac{\alpha}{M} \sum \Phi_\omega(|I_k|)$$

for some $M > 0$ using (5.1) and Lemma 2.2.

It remains to prove that $\sum \Phi_\omega(|J_l|) < \infty$. For each $l$, let $\xi_l$ be as in (5.2) and let $J_l^*$ be the closed subarc of $J_l$ having one common endpoint $\eta_l$, and one endpoint $\xi_l$. Since $F^*(e^{it})$ is of $\psi$-bounded variation and $F^*(\eta_l) = 0$ for each $l$, there exists a constant $M > 0$ such that for every positive integer $n$, we have

$$M \geqq \psi[\,|F^*(\xi_1)|\,] + \ldots + \psi[\,|F^*(\xi_n)|\,]$$
$$\geqq \psi[\omega(\,|J_1|\,)^\alpha] + \ldots + \psi[\omega(\,|J_n|\,)^\alpha]$$

using (5.2) and the monotonicity of $\psi$. By Lemma 5.1 and the fact that

$$\lim_{l\to\infty} |J_l| = 0,$$

it follows that $\sum \Phi_\omega(\,|J_l|\,) < \infty$ as required.

Theorem 5.1 is established.

We note that only the fact that $|F^*(e^{it})\,| \in \psi$ BV has been used in the second part of the proof. Also, the sharpness of Theorem 5.1 is established by a theorem of Shirokov [**18**] which insures that for any $\Phi_\omega$-set $K$, there exists $F \in A$ with modulus of continuity $\omega_F \leqq \omega$ such that $Z(F^*) = K$.

**6. Radial growth of functions in $h^1_\psi$.** Recall that $\omega \not\equiv 0$ is a continuous modulus of continuity and $\psi$:[0, $\infty$) $\to$ [0, $\infty$) is an increasing convex function that vanishes at 0. With $\mu$ as before, let

$$E^R_\omega = \{t \in (0, 2\pi):\overline{AD}^s_\omega\mu(t) \geqq R\}, \quad R \in [0, \infty].$$

The following is a generalization of [**16**, Lemma 4.3].

THEOREM 6.1. *If $\mu$ is of $\psi$-bounded variation, then*

$$H_{\psi\circ\omega}(E^R_\omega) < \infty \quad \textit{for all } R \in (1, \infty) \quad \textit{and}$$
$$\lim_{R\to\infty} H_{\psi\circ\omega}(E^R_\omega) = 0.$$

*In particular*, $H_{\psi\circ\omega}(E^\infty_\omega) = 0$.

Note that in the case when $\psi(t) = t^\beta$, $\beta \in [1, \infty)$, the condition $R \in (1, \infty)$ can be replaced by $R \in (0, \infty)$. In fact, the theorem for $R \in (1, \infty)$ can be applied to $\epsilon\omega$ for any $\epsilon \in (0, 1]$, and the more general result follows from the observation that

$$E^R_{\epsilon\omega} = E^{\epsilon R}_\omega \quad \text{and} \quad H_{\psi\circ(\epsilon\omega)} = \epsilon^\beta H_{\psi\circ\omega}.$$

*Proof.* By assumption, there exists a constant $M > 0$ such that whenever $n$ is a positive integer and $0 \leqq x_1 < y_1 \leqq \ldots \leqq x_n < y_n \leqq 2\pi$, we have

$$\text{(6.1)} \qquad \sum \psi[\,|\mu(y_k) - \mu(x_k)\,|\,] \leqq M.$$

Suppose that $\delta$, $x \in (0, \infty)$, $R > 1$, and $H_{\psi\circ\omega}(E^R_\omega) = \infty$. Then there exists a compact set $K \subseteq E^R_\omega$ such that $H_{\psi\circ\omega}(K) > x$ (see [**14**, p. 50] ). By the definition of $E^R_\omega$, there exists for each $t \in K$ a closed interval

$$[t - \delta_t, t + \delta_t] \subseteq (0, 2\pi)$$

with $\delta_t \in (0, \delta)$ such that

$$\frac{|\mu(t + \delta_t) - \mu(t - \delta_t)|}{\omega(2\delta_t)} \geqq \frac{1 + R}{2}. \tag{6.2}$$

Since $K$ is compact, it can be covered by finitely many of the open intervals $(t - \delta_t, t + \delta_t)$. Consider the corresponding finite sequence of closed intervals $([t_k - \delta_k, t_k + \delta_k])$ (where the notation $\delta_k$ is used in place of $\delta_{t_k}$). If there is a point of $(0, 2\pi)$ that lies in three of these intervals, then one of the three is contained in the other two and it can be removed without altering the union. In this fashion, some of the intervals can be removed if necessary so that no point of $(0, 2\pi)$ lies in more than two of them and the union of the closed intervals contains $K$. Since $H_{\psi\circ\omega}(K) > x$ (and finite sets have zero $H_{\psi\circ\omega}$ measure), it follows that for $\delta$ sufficiently small,

$$\sum \psi \circ \omega(2\delta_k) \geqq x.$$

Since each point in $(0, 2\pi)$ is contained in at most two intervals, there exists a finite, mutually disjoint subcollection

$$([t_{k_j} - \delta_{k_j}, t_{k_j} + \delta_{k_j}])_{j=1}^{n}$$

such that

$$\sum \psi \circ \omega(2\delta_{k_j}) \geqq \frac{x}{4}. \tag{6.3}$$

By (6.1)-(6.3) and the fact that $\psi(kx) \geqq k\psi(x)$ when $k \in [1, \infty)$, we have

$$\begin{aligned} M &\geqq \sum \psi[\,|\mu(t_{k_j} + \delta_{k_j}) - \mu(t_{k_j} - \delta_{k_j})|\,] \\ &\geqq \sum \psi\left[\frac{1 + R}{2}\omega(2\delta_{k_j})\right] \\ &\geqq \frac{1 + R}{2} \sum \psi \circ \omega(2\delta_{k_j}) \\ &\geqq \frac{1 + R}{8} x. \end{aligned} \tag{6.4}$$

Since $x \in (0, \infty)$ was arbitrary, we get a contradiction.

For the second assertion, assume (to get a contradiction) that there exists $x \in (0, \infty)$ and a sequence $(R_j)_1^\infty$ in $(1, \infty)$ such that

$$\lim_{j \to \infty} R_j = \infty \quad \text{and}$$

$$H_{\psi\circ\omega}(E_\omega^{R_j}) > x, \quad j = 1, 2, \ldots .$$

Proceeding as before, we arrive at the inequalities

$$M \geqq \frac{1 + R_j}{8} x, \quad j = 1, 2, \ldots .$$

Since

$$\lim_{j\to\infty} R_j = \infty,$$

we have a contradiction.

The last assertion of the theorem follows from the preceding one, and the proof is complete.

COROLLARY 6.1. *Let* $u \in h^1_\psi$. *Then for* $R \in (0, \infty)$ *sufficiently large,*

$$H_{\psi\circ\omega}(W^R_\omega) < \infty \quad \text{and} \quad \lim_{R\to\infty} H_{\psi\circ\omega}(W^R_\omega) = 0,$$

*where*

$$W^R_\omega = \{\eta \in C : \limsup_{r\to 1} \frac{1 - r}{\omega(1 - r)} |u(r\eta)| \geqq R\}, \quad R \in [0, \infty].$$

*Proof.* This follows directly from Theorem 6.1 and Lemma 3.2.

COROLLARY 6.2. *If* $u \in h^1_\psi$, *then*

$$m(E^\infty_\omega) = m(W^\infty_\omega) = 0 \quad \text{when } \omega = \psi^{-1}|_{[0,2\pi]}.$$

*In particular, there exists a constant* $c_\eta > 0$ *such that*

$$|u(r\eta)| \leqq c_\eta \frac{\psi^{-1}(1 - r)}{1 - r}, \quad r \in [0, 1),$$

*for almost all* $\eta$ *in* $C$.

We note that Corollaries 6.1 and 6.2 apply to $\partial U/\partial\theta$ and $dF/dz$ when $U \in h^\infty$ and $F \in H^\infty$ with $U^*$ and $F^*$ equal almost everywhere to a function in $\psi$ BV.

Though the next theorem does not have a direct application to the radial growth of functions in $h^1_\psi$, it is of interest in relation to Theorem 6.1. Let

$$L^R_\omega = \{t \in [0, 2\pi] : \underline{AD}_\omega \mu(t) \geqq R\}, \quad R \in [0, \infty],$$

and recall that $\psi \circ \omega$ is assumed to be a modulus of continuity.

THEOREM 6.2. *If* $\mu$ *is a continuous function of* $\psi$-*bounded variation and*

$$\lim_{t\to 0} \psi \circ \omega(t)/t = \infty,$$

*then there exists a sequence* $(K_n)^\infty_1$ *of* $\psi \circ \omega$-*sets* (*in* $[0, 2\pi]$ ) *such that*

$$L^R_\omega \subseteq \bigcup_1^\infty K_n \quad \text{for } R \in (1, \infty).$$

*Proof.* For each $R \in (1, \infty)$ we have

$$L_\omega^R \subseteq E_\omega^R \cup \{0, 2\pi\}$$

so that Theorem 6.1 implies $H_{\psi\circ\omega}(L_\omega^R) < \infty$. Since

$$\lim_{t\to 0} \psi \circ \omega(t)/t = \infty,$$

it follows that $m(L_\omega^R) = 0$. For each positive integer $n$, let

$$K_n = \left\{t \in [0, 2\pi]: \frac{|\mu(y) - \mu(x)|}{\omega(y - x)} \geqq 1 + \frac{1}{n} \text{ whenever } x < t < y \quad \text{and} \quad y - x < \frac{1}{n}\right\}.$$

It is evident that

$$L_\omega^R \subseteq \bigcup_1^\infty K_n \quad \text{for each } R \in (1, \infty).$$

Since

$$K_n \subseteq L_\omega^{1+(1/n)},$$

we have $m(K_n) = 0$ for all $n$. Also by the continuity of $\mu$, we see that each $K_n$ is closed. Let $n$ be a positive integer and let $\{(x_j, y_j)\}$ be an enumeration of the open intervals of $(0, 2\pi) - K_n$. Suppose that

$$\sum \psi \circ \omega(y_j - x_j) = \infty.$$

Then there exists a positive integer $N$ such that

$$y_j - x_j \leqq 1/(2n) \quad \text{for } j \geqq N.$$

Let $P$ be a positive integer greater than $N$ and

$$x_p = \sum_{j=N}^{P} \psi \circ \omega(y_j - x_j).$$

Since $\mu$ is of $\psi$-bounded variation, there exists a constant $M > 0$ such that

$$\sum_{j=N}^{P} \psi[|\mu(y_j) - \mu(x_j)|] \leqq M.$$

From the definition of $K_n$, we have

$$M \geqq \sum_{j=N}^{P} \psi\left[\left(1 + \frac{1}{n}\right)\omega(y_j - x_j)\right]$$

$$\geqq \left(1 + \frac{1}{n}\right) \sum_{j=N}^{P} \psi \circ \omega(y_j - x_j)$$

$$\geqq \left(1 + \frac{1}{n}\right) x_P.$$

Since these inequalities hold for all $P > N$ and

$$\lim_{P \to \infty} x_P = \infty,$$

we have a contradiction. Thus each $K_n$ is a $\psi \circ \omega$-set and the proof is complete.

We note that Theorem 6.2 remains valid with essentially the same proof if in the definition of $\underline{AD}_w$, we consider only intervals with one endpoint equal to $t$.

**7. A subclass of $h^1$.** Let $S_\omega$ denote the class of all subharmonic functions $u$ on $\Delta$ satisfying the growth condition

$$(7.1) \qquad u(z) \leqq c\frac{\omega(1 - |z|)}{1 - |z|}, \quad z \in \Delta,$$

where $c$ is a positive constant (depending on $u$). In [**2**], the following Phragmén-Lindelöf theorem was proved.

THEOREM 7.1. *Let $E \subseteq C$ with $H_\omega(E) = 0$. If $u \in S_\omega$ satisfies*

$$(7.2) \qquad \limsup_{r \to 1} u(r\eta) \leqq 0, \quad \eta \in C - E,$$

*then* $u \leqq 0$.

This result was demonstrated to be sharp in [**2**] by showing that when $E$ is a Borel set with $H_\omega(E) > 0$, there exists a positive harmonic function $u \in S_\omega$ for which (7.2) holds.

In this section we prove two results concerning a subclass of $h^1 \cap S_\omega$ that relate to Theorem 7.1. Define $\mathcal{RH}_\omega$ to be the class of functions $h$ that can be written in the form $h = u - v$ where $u$ and $v$ are nonnegative harmonic functions on $\Delta$ and $u$ satisfies (7.1).

Theorem 7.1 asserts that if $u$ is any subharmonic function in $S_\omega$ such that $u(z) > 0$ for some $z \in \Delta$, then there exists a Borel subset $E$ of $C$ with $H_\omega(E) > 0$ such that

$$\limsup_{r \to 1} u(r\eta) > 0, \quad \eta \in E.$$

The next theorem shows that more can be said for the functions in $\mathcal{RH}_\omega$.

THEOREM 7.2. *If $h \in \mathcal{RH}_\omega$ and $h(z) > 0$ for some $z \in \Delta$, then there exists a Borel subset $E$ of $C$ with $H_\omega(E) > 0$ such that $h$ has radial limits*

$$h^*(\eta) \in (0, +\infty], \quad \eta \in E.$$

The proof relies on the following result concerning measures.

THEOREM 7.3. *Let $\sigma$ be a complex Borel measure on $C$ and $\lambda$ a finite nonnegative Borel measure on $C$. Suppose that $\sigma \perp \lambda$ and $\mu = \sigma + \lambda$. Then*

(7.3) $\quad 0 < D\lambda \leqq +\infty \quad$ a.e. $[\lambda]$

*and*

(7.4) $\quad D\mu = D\lambda \quad$ a.e. $[\lambda]$.

Note that a consistent interpretation of an infinite derivative in (7.4) must be given in case $\mu$ is not real.

*Proof.* By Theorem 2.2 we have $\lambda = \lambda_a + \lambda_s$, where $\lambda_s \perp m$ and $\lambda_a \ll m$ with

$$\lambda_a(E) = \int_E D\lambda(t)dt$$

for all Borel subsets $E$ of $C$. Also

$$D\lambda = D\lambda_a \text{ a.e. } [m] \quad \text{and} \quad D\lambda_a \in L^1([0, 2\pi]).$$

Clearly

$$D\lambda_a > 0 \text{ a.e. } [\lambda_a].$$

By Corollary 2.1 we have

$$D\lambda_s = 0 \text{ a.e. } [m]$$

so that

$$D\lambda_s = 0 \text{ a.e. } [\lambda_a].$$

Hence

(7.5) $\quad 0 < D\lambda < +\infty$ a.e. $[\lambda_a]$.

From Theorem 2.3 it follows that

$$D\lambda_s = \infty \text{ a.e. } [\lambda_s]$$

and hence

(7.6) $\quad D\lambda = +\infty$ a.e. $[\lambda_s]$.

We conclude from (7.5) and (7.6) that (7.3) holds.
It also follows from Theorem 2.3 that

(7.7) $\quad D_\lambda\sigma = 0$ a.e. $[\lambda]$.

Now for any nondegenerate closed arc $I$ such that $\lambda(I) > 0$, we have

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$$(7.8) \quad \frac{\mu(I)}{|I|} = \frac{\lambda(I) + \sigma(I)}{|I|} = \frac{\lambda(I)}{|I|}\left[1 + \frac{\sigma(I)}{\lambda(I)}\right].$$

Equality (7.4) now follows from (7.3), (7.7), and (7.8). The theorem is thereby established.

*Proof of Theorem* 7.2. Suppose that $u \in \mathcal{RH}_\omega$, $E$ is a Borel subset of $C$ with $H_\omega(E) = 0$, and (7.2) holds. Let $\mu$ be the real Borel measure in the Riesz-Herglotz representation of $h$, that is,

$$h(z) = \frac{1}{2\pi}\int_0^{2\pi} P(z, t)d\mu(t), \quad z \in \Delta.$$

Let $\mu = \lambda - \sigma$ be the Jordan decomposition of $\mu$ where $\lambda$ and $\sigma$ are nonnegative measures such that $\sigma \perp \lambda$ (see [**15**, p. 128] ). By Theorem 7.3 we have

$$0 < D\mu \leqq +\infty \text{ a.e. } [\lambda].$$

By the assumption on $u$ and Theorem 3.3, it follows that $\omega_\lambda(t) \leqq c\omega(t)$ for some $c > 0$. We conclude from Theorem 2.1 that $0 < D\mu \leqq +\infty$ on a Borel set $W$ with $H_\omega(W) > 0$. By the Fatou radial limit theorem (Theorem 3.1), we have $h^*(\eta) > 0$ for $\eta \in W$ and this completes the proof.

Let $\mathcal{N}_\omega$ be the class of analytic functions $f$ on $\Delta$ of bounded characteristic which can be represented in the form $f = Bg$ where $B$ is a Blaschke product whose zeros $(a_k)$ in $\Delta - \{0\}$ (counted according to multiplicity) satisfy

$$(7.9) \quad \sum \omega(1 - |a_k|) < \infty,$$

and $g$ is a nonvanishing analytic function for which

$$u = \log|g| \in \mathcal{RH}_\omega.$$

Recall that a Blaschke product $B$ is a function of the form

$$z^m \prod_1^n (\bar{a}_k/|a_k|)L_{a_k} \quad \text{or} \quad z^m \prod_1^\infty (\bar{a}_k/|a_k|)L_{a_k}$$

where $m$ and $n$ are nonnegative integers, $(a_k)_1^\infty$ is a sequence taking values in $\Delta - \{0\}$ satisfying the Blaschke condition

$$\sum_1^\infty (1 - |a_k|) < +\infty, \quad \text{and}$$

$$L_a(z) = (a - z)/(1 - \bar{a}z), \quad a \in \Delta, z \in \mathbf{C}.$$

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(The convention

$$\prod_1^0 L_{a_k} = 1$$

is used.) For background concerning functions of bounded characteristic, see [**7**, Chapter 2].

COROLLARY 7.1. *Suppose the modulus of continuity* $\omega$ *has a continuous derivative on* $(0, 2\pi]$ *and satisfies*

$$(7.10) \quad \liminf_{t \to 0} \frac{\omega(t)}{t\omega'(t)} > 1.$$

*If* $f \in \mathcal{N}_\omega$ *and there exists* $z \in \Delta$ *such that* $|f(z)| > 1$, *then there is a Borel subset* $E$ *of* $C$ *with* $H_\omega(E) > 0$ *such that* $f^*$ *exists and has modulus greater than* 1 (*and possibly infinite*) *at each point of* $E$.

The proof of Corollary 7.1 is easily given using the proof of Theorem 7.2 and the following result appearing in [**3**] that is a generalized form of a result of Frostman and Carleson.

THEOREM 7.4. *Suppose that* $\omega$ *is a modulus of continuity that has a continuous derivative in* $(0, 2\pi]$ *and* (7.10) *holds. If* $B$ *is a Blaschke product with zeros* $(a_k)$ (*enumerated according to multiplicity*) *satisfying* (7.9), *then there exists a subset* $E$ *of* $C$ *with* $H_\omega(E) = 0$ *such that* $B$ *and all of its subproducts have radial limits of modulus* 1 *at each point of* $C - E$.

Condition (7.10) is unnecessary if we restrict to the subclass of nonvanishing functions in $\mathcal{N}_\omega$.

We consider next the minimum $m(r; h) = -M(r; -h)$, $r \in [0, 1)$, for $h \in \mathcal{RH}_\nu$, where $\nu \not\equiv 0$ is a continuous modulus of continuity with $\nu = o[\omega]$. Theorem 7.1 insures that if $h(z) < 0$ for some $z \in \Delta$ and

$$(7.11) \quad \liminf_{r \to 1} h(r\eta) \geqq 0, \quad \eta \in C - E,$$

with $H_\omega(E) = 0$, then

$$-m(r; h) \neq O\left[\frac{\omega(1 - r)}{1 - r}\right], \quad r \to 1.$$

A stronger conclusion is obtained in the following result.

THEOREM 7.5. *Let* $\omega, \nu \not\equiv 0$ *be continuous moduli of continuity with*

$$\nu(t) = o[\omega(t)] \quad \text{as } t \to 0.$$

*Let* $E$ *be a countable union of* $\omega$*-sets and let* $h \in \mathcal{RH}_\nu$ *such that* (7.11) *holds. Then there exists* $\eta \in C$ *and a constant* $c > 0$ *such that*

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(7.12) $$-h(r\eta) \geqq c\frac{\omega(1-r)}{1-r}$$

*and*

(7.13) $$-m(r; h) \geqq c\frac{\omega(1-r)}{1-r}$$

*for* $r \in [0, 1)$ *sufficiently close to* 1.

By setting $\nu(t) = t$, we arrive at the next corollary.

COROLLARY 7.2. *Suppose* $\omega'(0) = \infty$ *and* $E$ *is a countable union of* $\omega$*-sets. If* $u$ *is a harmonic function bounded below such that* $u(z) > 0$ *for some* $z \in \Delta$ *and* (7.2) *holds, then there exists* $\eta \in C$ *and a constant* $c > 0$ *such that*

$$u(r\eta) \geqq c\frac{\omega(1-r)}{1-r}$$

*and*

$$M(r; u) \geqq c\frac{\omega(1-r)}{1-r}$$

*for* $r \in [0, 1)$ *sufficiently close to* 1.

Note that by Theorem 7.1, the corollary is vacuous for the case $\omega(t) = t$. The proof of Theorem 7.5 depends on Theorem 3.3 and two other results, the first a lemma that was given by Samuelsson (see [**16**, Lemma 4.2] ).

LEMMA 7.1. *Let* $\mu$ *be a finite nonnegative Borel measure in* $C$ *and let*

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} P(z, t)d\mu(t), \quad z \in \Delta.$$

*Then there exists a constant* $c > 0$ *such that*

$$\underline{D}_\omega\mu(t) \leqq c \liminf_{r\to 1} \frac{1-r}{\omega(1-r)}u(re^{it}), \quad t \in \mathbf{R}.$$

The next result is central to the proof of Theorem 7.5 [cf. Theorem 6.2 with $\psi(t) = t$].

THEOREM 7.6. *Let* $\mu$ *be a nonnegative finite Borel measure on* $C$ *and let* $K$ *be an* $\omega$*-set. Then there exists a constant* $\delta > 0$ *such that for every Borel subset* $W$ *of* $K$ *we have*

$$\underline{D}_\omega\mu \geqq \delta \text{ a.e. } [\mu] \text{ on } W.$$

*Proof.* Putting aside the trivial case, assume that $\mu(W) > 0$. For each $\delta \in (0, \infty)$, let

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$$W(\delta) = \{e^{it} \in W:\underline{D}_\omega\mu(t) < \delta\}.$$

We shall show that $\mu[W(\delta)] = 0$ for $\delta$ sufficiently small. If this is not the case, then there exists $\epsilon > 0$ such that $\mu[W(\delta)] > \epsilon$ for all $\delta$.

Let $(I_j)$ be an enumeration of the component arcs of $C - K$ and fix $\delta$. By the regularity of $\mu$, there is a compact subset $K(\delta)$ of $W(\delta)$ such that

$$\mu[K(\delta)] > \epsilon.$$

From the definition of $W(\delta)$, there exists for each $\eta \in K(\delta)$ an open arc $J_\eta$ containing $\eta$ such that

$$\mu(J_\eta)/\omega(|J_\eta|) < \delta.$$

By the compactness of $K(\delta)$, the open cover $(J_\eta:\eta \in K(\delta))$ has a finite subcover $\{J_1, \ldots, J_n\}$ (where the subscript $k$ is used in place of $\eta_k$ for $k = 1, \ldots, n$) for which each point of $C$ is contained in at most two of these arcs and each arc $I_j$ has nonempty intersection with at most two of them. Then

$$\epsilon < \sum \mu(J_k) \leqq \delta \sum \omega(|J_k|) \leqq \delta \sum_k \omega\left(\sum_{I_j \cap J_k \neq \emptyset} |I_j|\right)$$

$$\leqq \delta \sum_k \sum_{I_j \cap J_k \neq \emptyset} \omega(|I_j|) \leqq 2\delta \sum \omega(|I_j|)$$

using the subadditivity of $\omega$. Since $K$ is an $\omega$-set, we have

$$\sum \omega(|I_j|) < \infty.$$

Therefore, because $\epsilon$ is independent of $\delta$ and the inequalities hold for all $\delta \in (0, \infty)$, we get a contradiction. This completes the proof.

*Proof of Theorem* 7.5. Let $\mu$ be the real Borel measure in the Riesz-Herglotz representation of $h$. Let $\mu = \lambda - \sigma$ be the Jordan decomposition of $\mu$ into a difference of mutually singular nonnegative Borel measures and let $u$ and $v$ be the nonnegative harmonic functions which are the Poisson integrals with respect to $\lambda$ and $\sigma$. Then $h = u - v$ and it follows from the assumption $h \in \mathscr{R}\mathscr{H}_\nu$ that

$$(7.14) \quad M(r; u) \leqq c\frac{\nu(1 - r)}{1 - r}, \quad r \in [0, 1),$$

for some constant $c > 0$.

By Theorem 7.3 and the Fatou radial limit theorem, we have

$$D\mu \in [-\infty, 0) \quad \text{and} \quad h^*(\eta) \in [-\infty, 0) \text{ a.e. } [\sigma].$$

From (7.11), it follows that $\sigma$ is concentrated on $E$. Because

$$E = \bigcup_{i=1}^{\infty} K_i$$

where each $K_i$ is an $\omega$-set, we must have $\sigma(K_n) > 0$ for some $n$. Applying Theorem 7.6 and Lemma 7.1, we obtain (7.12) with $v$ replacing $h$. However, by (7.14) and the fact that

$$\nu(t) = o[\omega(t)] \quad \text{as } t \to 0,$$

we conclude that (7.12) holds for $h$. Inequality (7.13) is an immediate consequence of (7.12).

**8. Covering properties of the boundary function of an inner function.** A classical result (see [**19**, pp. 283-284]) asserts that an inner function $f$ is either a finite Blaschke product or else its radial limit function $f^*$ covers the circumference $C$ infinitely often. In this section we obtain some quantitative relationships between the covering properties of $f^*$ and the maximum and minimum modulus

$$M(r; |f|) = \max\{ |f(z)|: \quad |z| = r\}$$

and

$$m(r; |f|) = \min\{ |f(z)|: \quad |z| = r\}$$

for $r \in [0, 1)$.

Recall that an inner function $f$ is, by definition, a bounded analytic function on $\Delta$ for which $|f^*| = 1$ a.e. (with respect to linear measure) on $C$. The two primary subclasses of inner functions consist of the Blaschke products (the definitions of which were given in Section 7) and singular inner functions. The latter are functions of the form

$$(8.1) \qquad S_\mu(z) = \exp\left\{-\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}d\mu(t)\right\}, \quad z \in \mathbf{C} - C,$$

where $\mu$ is a real-valued monotone nondecreasing function on $\mathbf{R}$ that is singular, i.e.,

$$D\mu = 0 \text{ a.e.} \quad \text{and} \quad \mu(t + 2\pi) = \mu(t) + \mu(2\pi), \quad t \in \mathbf{R}.$$

In general, every inner function $f$ has a canonical factorization $f = \eta B S_\mu$ where $B$ is a Blaschke product, $S_\mu$ is a singular inner function, and $\eta$ is a constant of modulus 1. For these and other facts concerning inner functions, see [**7**, Chapter 2].

In the sequel,

$$E_\zeta = \{\eta \in C: \quad f^*(\eta) = \zeta\}, \quad \zeta \in \bar{\Delta},$$

when $f$ is an inner function.

THEOREM 8.1. *Let $f$ be a nonconstant inner function and suppose $\zeta \in \Delta$ such that $f(z) = \zeta$ for only finitely many $z$ in $\Delta$. If*

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$$(8.2) \qquad m(r; |f|) \geqq \exp\left[-c\frac{\omega(1-r)}{1-r}\right], \quad r \to 1,$$

*for some constant* $c > 0$, *then* $H_\omega(E_\zeta) > 0$.

*Proof.* Since $\zeta$ is taken on only finitely often by $f$, we have

$$L_\zeta \circ f = \eta B S_\mu$$

where

$$L_\zeta(z) = (\zeta - z)/(1 - \bar{\zeta}z),$$

$B$ is a finite Blaschke product, $|\eta| = 1$, and $S_\mu$ is a singular inner function given by (8.1). Furthermore, $L_\zeta \circ f/(\eta B)$ also satisfies a condition of the type (8.2) so we can assume without loss of generality that $\zeta = 0$ and $f = S_\mu$. From (8.2), we have

$$M(r; u) \leqq c\frac{\omega(1-r)}{1-r}, \quad r \in [0, 1),$$

where $u = \log(1/|S_\mu|)$, and Theorem 3.3 implies that

$$\omega_\mu(t) \leqq c\omega(t), \quad t \in [0, 2\pi],$$

for some constant $c > 0$. By Theorem 2.3, we have

$$D\mu = +\infty \text{ a.e. } [\mu]$$

and it follows from Theorem 2.1 that $H_\omega(I) > 0$ where

$$I = \{e^{it}: \ D\mu(t) = +\infty\}.$$

We conclude from the Fatou radial limit theorem that $H_\omega(E_0) > 0$ as required.

Next we consider $\zeta \in C$ and give a sufficient condition for $H_\omega(E_\zeta) > 0$ depending on the descent rate of $m(r; f - \zeta)$ to 0 as $r \to 1$.

THEOREM 8.2. *If $f$ is a nonconstant inner function and $\zeta \in C$ such that*

$$(8.3) \qquad m(r; |f - \zeta|) \leqq c\frac{1-r}{\omega(1-r)}, \quad r \in (0, 1],$$

*for some constant* $c > 0$, *then* $H_\omega(E_\zeta) > 0$.

*Proof.* Consider the function

$$g(z) = \exp\left[-\frac{\zeta + f(z)}{\zeta - f(z)}\right], \quad z \in \Delta.$$

It is straightforward to verify that $g$ is a singular inner function and that $g^*(\eta) = 0$ implies $f^*(\eta) = \zeta$ for $\eta \in C$. By (8.3) there exists a constant $c > 0$ such that

$$|g(z)| = \exp\left[-\frac{1 - |f(z)|^2}{|\zeta - f(z)|^2}\right] \geqq \exp\left[-\frac{2}{|\zeta - f(z)|}\right]$$
$$\geqq \exp\left[\frac{-2}{m(r; |f - \zeta|)}\right] \geqq \exp\left[-c\frac{\omega(1 - r)}{1 - r}\right],$$

where $r = |z|$. Applying Theorem 8.1, we arrive at the desired conclusion that $H_\omega(E_\zeta) > 0$.

An immediate consequence of Theorem 8.2 and the inequality

$$m(r; |f - \zeta|) \geqq 1 - M(r; |f|), \quad r \in [0, 1),$$

is the following.

COROLLARY 8.1. *If $f$ is a nonconstant inner function and $c > 0$ such that*

$$1 - M(r; |f|) \geqq c\frac{1 - r}{\omega(1 - r)}, \quad r \in [0, 1),$$

*then $H_\omega(E_\zeta) > 0$ for every $\zeta \in C$.*

The next two results show that for any nonconstant inner function $f$, there exists a modulus of continuity $\omega(\not\equiv 0)$ such that $1 - M(r; |f|)$ is equivalent to $(1 - r)/\omega(1 - r)$. By definition, two functions $\nu_1$, $\nu_2$ defined on a subset of $\mathbf{R}$ are said to be *equivalent* (denoted $\nu_1 \approx \nu_2$) if there exist constants $c_1, c_2 > 0$ such that

$$c_1\nu_2 \leqq \nu_1 \leqq c_2\nu_2.$$

LEMMA 8.1. *Let $\nu(t)$, $t \in [0, 2\pi]$, be a real-valued function that vanishes only at 0 and let $t/\nu(t)$, $t \in (0, 2\pi]$, be extended to $[0, 2\pi]$ by defining it to be 0 when $t = 0$. Then $\nu$ is equivalent to a nontrivial modulus of continuity if and only if the same is true of $t/\nu$. Furthermore, $\nu$ (respectively $t/\nu$) is equivalent to $t$ if and only if $t/\nu$ (respectively $\nu$) is equivalent to a discontinuous modulus of continuity.*

In connection with Lemma 8.1 and its proof, we recall that if $\nu \not\equiv 0$ is a modulus of continuity, then either $\nu \approx t$, $\nu$ is equivalent to a discontinuous modulus of continuity, or else $\nu$ is continuous with $\nu'(0) = \infty$; see, for example [**4**].

*Proof.* The last assertion is verified without difficulty. We shall assume for the remainder of the proof that $\omega$ is a continuous modulus of continuity with $\omega'(0) = \infty$.

Suppose $\nu \approx \omega$. By Lemma 2.1 we can assume without loss of generality that $\omega$ is concave downward. Then $t/\nu \approx t/\omega$ and the latter is monotone nondecreasing by the concavity of $\omega$. Also $[t/\omega(t)]/t = 1/\omega(t)$ is a monotone nonincreasing function on $(0, 2\pi]$ and it is easy to verify that this implies the subadditivity of $t/\omega$. Hence $t/\omega$ is a modulus of continuity with which $t/\nu$ is equivalent.

Conversely, if $t/\nu \approx \omega$, then $t/\omega \approx \nu$. By what was just proved, $t/\omega$ is equivalent to a modulus of continuity and the same conclusion follows for $\nu$. The proof is thereby completed.

THEOREM 8.3. *If $f$ is a nonconstant inner function, then there exists a continuous modulus of continuity $\omega \not\equiv 0$ such that*

$$1 - M(r; |f|) \approx \omega(1 - r), \quad r \in [0, 1).$$

*Proof.* It suffices to prove $\nu(t) \equiv \log[1/M(1 - t; |f|)]$ is equivalent to a modulus of continuity $\omega$ for $t > 0$ near 0. By Hardy's convexity theorem (see [7, Theorem 1.5]), the function

$$\omega(t) = \log[1/M(e^{-t}; |f|)]$$

is a monotone nondecreasing concave-downward function on $(0, \infty)$. Therefore,

$$\omega(2t) \leqq 2\omega(t) \quad \text{for } t \in (0, \infty).$$

Using the inequalities

$$\exp(-2t) < 1 - t < \exp(-t)$$

for $t > 0$ near 0 and the monotonicity of the functions involved, we see that $\nu(t) \leqq \omega(2t) \leqq 2\omega(t)$ and $\nu(t) \geqq \omega(t)$ for $t > 0$ near 0 as required.

COROLLARY 8.2. *If $f$ is a nonconstant inner function such that*

$$1 - r = o[1 - M(r; |f|)], \quad r \to 1,$$

*then there exists a continuous modulus of continuity $\omega$ such that $H_\omega(E_\zeta) > 0$ for every $\zeta \in C$.*

This corollary follows immediately from Corollary 8.1, Lemma 8.1, and Theorem 8.3. In our next theorem, we show that if $\omega \not\equiv 0$ is any continuous modulus of continuity, there exists a nonconstant singular inner function $S_\mu$ such that

$$1 - M(r; |f|) \geqq \omega(1 - r), \quad r \in [0, 1).$$

The following lemma is elementary. We omit the proof since it involves only a trivial estimate of the Poisson kernel.

LEMMA 8.2. *There exists a constant $c > 0$ such that for every nonnegative Borel measure $\mu$ on $C$ and any nondegenerate closed arc $A$ of $C$ having length $|A|$, we have*

$$P[d\mu](z) \geqq c\frac{\mu(A)}{|A|}$$

*where $|z| = 1 - |A|/2\pi$ and $z/|z| \in A$.*

THEOREM 8.4. *Let $\omega \not\equiv 0$ be a continuous modulus of continuity such that $\omega < 1$. Then there exists a singular inner function $f = S_\mu$ such that*

$$1 - M(r; |f|) \geqq \omega(1 - r) \quad \text{for } r \in [0, 1).$$

*Proof.* If $f = S_\mu$ is a singular inner function, then

$$\log[1/M(r; |f|)] \approx 1 - M(r; |f|) \quad \text{for } r \in [0, 1).$$

Observing that

$$\log[1/M(r; |f|)] = m(r; P[d\mu]) \quad \text{for } r \in [0, 1),$$

we see that it suffices to construct a singular measure $\mu$ on $C$ such that

$$(8.4) \qquad P[d\mu](z) \geqq c\omega(1 - |z|), \quad z \in \Delta,$$

for some constant $c > 0$. (We can then multiply $\mu$ by a suitably large positive number to insure that the constant $c$ can be omitted.) By Theorem 8.3 and the observations made above, we have $m(P[d\mu]; 1 - r)$ is equivalent to a modulus of continuity, so by (2.1) it is enough to insure that (8.4) holds for $|z| = 1 - 2^{-n}$, $n = 1, 2, \ldots$.

We shall construct $\mu$ in the form of a monotone nondecreasing 'jump' function satisfying

$$(8.5) \qquad \mu(t + 2\pi) = \mu(t) + \mu(2\pi), \quad t \in \mathbf{R}.$$

We start by defining $\mu$ on $[0, 2\pi]$ as a sum of monotone nondecreasing functions $\mu_n$ having $2^n$ jumps for $n = 1, 2, \ldots$. For each positive integer $n$, let

$$\mu_n(t) = \begin{cases} 0 & t \in [0, \pi 2^{-n}], \\ k2^{-n}[\omega(2^{-n}) - \omega(2^{-n-1})] & t \in (\pi(2k - 1)2^{-n}, \pi(2k + 1)2^{-n}] \\ & \quad k = 1, \ldots, 2^n - 1, \\ \omega(2^{-n}) - \omega(2^{-n-1}) & t \in (\pi(2^{n+1} - 1)2^{-n}, 2\pi]. \end{cases}$$

Clearly each $\mu_n$ is a monotone nondecreasing singular function with

$$\mu_n(0) = 0 \quad \text{and} \quad \mu_n(2\pi) = \omega(2^{-n}) - \omega(2^{-n-1}).$$

Thus

$$\sum_{n=1}^{\infty} \mu_n(2\pi) = \omega(2^{-1})$$

and

$$\mu = \sum_{n=1}^{\infty} \mu_n$$

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is a singular monotone nondecreasing function on $[0, 2\pi]$ which we can extend to $\mathbf{R}$ by the requirement that it satisfy (8.5).

A direct calculation shows that for each positive integer $n$, we have

$$\begin{aligned}&\mu[\pi(2k)2^{-n}] - \mu[\pi(2k-2)2^{-n}]\\&\geqq \sum_{j=0}^{\infty} 2^j 2^{-n-j}[\omega(2^{-n-j}) - \omega(2^{-n-j-1})]\\&= 2^{-n}\omega(2^{-n}), \quad k = 1, \dots, 2^n.\end{aligned}$$

Taken together with Lemma 8.2, this yields (8.4) for $|z| = 1 - 2^{-n}$, $n = 1, 2, \dots$ . The required conclusion follows and the proof is complete.

From Corollary 8.1, Lemma 8.1, and Theorem 8.4, we obtain the next result.

COROLLARY 8.3. *If $\omega$ is a continuous modulus of continuity with $\omega'(0) = \infty$, then there exists an inner function $f$ such that*

$$H_\omega(E_\zeta) > 0 \quad \textit{for every } \zeta \in C.$$

In the remainder of this section, we give some consequences of the assumption that $E_\zeta$ is contained in a countable union of $\omega$-sets.

THEOREM 8.5. *Suppose that $f$ is a nonconstant inner function, $\zeta \in \Delta$, and $f(z) = \zeta$ for at most finitely many $z$ in $\Delta$. If*

$$E_\zeta \subseteq \bigcup_1^\infty K_n$$

*where each $K_n$ is an $\omega$-set, then*

$$m(r; |f|) \leqq \exp\left[-c\frac{\omega(1-r)}{1-r}\right], \quad r \in [0, 1),$$

*for some constant $c > 0$.*

*Proof.* By considering $L_\zeta \circ f/(\eta B)$ in place of $f$ if necessary (where $B$ is a suitable finite Blaschke product and $|\eta| = 1$), we can assume without loss of generality that $\zeta = 0$ and $f$ is a singular inner function. Since

$$\{e^{it}: D\mu(t) = +\infty\} \subseteq E_\zeta \subseteq \bigcup_1^\infty K_n$$

and $D\mu = +\infty$ a.e. $[\mu]$, we see that $\mu(K_n) > 0$ for some $n$. Using Theorem 7.6 and Lemma 7.1 we obtain the desired result.

THEOREM 8.6. *If $f$ is a nonconstant inner function, $\zeta \in C$, and*

$$E_\zeta \subseteq \bigcup_1^\infty K_n$$

*where each* $K_n$ *is an* $\omega$*-set, then*

$$m(r;\ |f - \zeta|) \leqq c\frac{1 - r}{\omega(1 - r)}, \quad r \in [0, 1),$$

*for some constant* $c > 0$.

*Proof.* As before, the function

$$g(z) = \exp\left[-\frac{\zeta + f(z)}{\zeta - f(z)}\right], \quad z \in \Delta,$$

is a singular inner function for which

$$E_0(g) \subseteq \bigcup_1^\infty K_n.$$

Therefore by Theorem 8.5 we have

$$\exp\left[-c\frac{\omega(1 - r)}{1 - r}\right] \geqq m(r;\ |g|)$$

$$= \min_{|z|=r} \exp\left[-\frac{1 - |f(z)|^2}{|\zeta - f(z)|^2}\right]$$

$$\geqq \min_{|z|=r} \exp\left[-\frac{2}{|\zeta - f(z)|}\right]$$

$$= \exp\left[-\frac{2}{m(r;\ |f - \zeta|)}\right], \quad r \in [0, 1).$$

This leads immediately to the inequality we set out to prove.

COROLLARY 8.4. *If* $f$ *is a nonconstant inner function such that for some* $\zeta \in C$ *we have*

$$E_\zeta \subseteq \bigcup_1^\infty K_n,$$

*where each* $K_n$ *is an* $\omega$*-set, then*

$$1 - M(r;\ |f|) \leqq c\frac{1 - r}{\omega(1 - r)}, \quad r \in [0, 1),$$

*for some constant* $c > 0$.

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