also $$d I_1^2 - d A^2 = D I_1^2 - D A^2 = r^2,$$
therefore $$DB^2 \sim DA^2 = r^2 \sim r_1^2 \qquad (1)$$
$A a$ and $B b$ meet on the radical axis, and
$$A X^2 \sim B X^2 = r^2 \sim r_1^2 \qquad (2)$$
if M be the mid-point of $A B$ from (1) and (2), we get
$$2 A B \,.\, M D = r^2 \sim r_1^2 \text{ and } 2 A B \,.\, M X = r^2 \sim r_1^2,$$
so that $M D = M X$, and the two axes are equidistant from M.

Again, at any point between I and I_1 a circle can be drawn which all the coaxials d_1, d_2, etc., cut at ends of diameters. When the point is outwith $I I_1$ on $A B$ the circles become the orthogonals to d_1, d_2, etc.

The question arises here, which are the real circles, A, B, etc., or the orthogonal circles, of which $D d$ is the radical axis.

Townsend, Art. 152, says: "All the circles whose centres are between I and I_1 are imaginary"; still, by foregoing they seem real enough.

WILLIAM FINLAYSON

The Limits of $\left(\cos \frac{x}{n}\right)^n$ and $\left(\sin \frac{x}{n} \Big/ \frac{x}{n}\right)^n$ when n tends to infinity.

These limits may be proved very simply by applying the following theorem in inequalities:—

If n is a positive integer and $r a$ a positive proper fraction for the values 1, 2, 3, ... n of r, then
$$1 - n a < (1 - a)^n < \frac{1}{1 + n a} \qquad (1)$$

These particular cases of the well-known inequalities generally used in connection with infinite products are easily established. Thus
$$(1 - a)^2 = 1 - 2 a + a^2 > 1 - 2 a;$$
$$(1 - a)^3 = (1 - a)(1 - a)^2 > (1 - a)(1 - 2 a)$$
$$[(1 - a)^3 = (1 - a)(1 - a)^2, \text{ etc.}]$$
since $1 - a$ and $1 - 2 a$ are both positive; therefore
$$(1 - a)^3 > 1 - 3 a + 2 a^2 > 1 - 3a,$$
and so on. The general result is easily proved by induction, though it is really obvious; thus we have the first inequality
$$1 - n a < (1 - a)^n.$$

In the same way we find

$$(1+a)^n > 1 + na. \quad \dots\dots\dots\dots\dots\dots\dots \quad (2)$$

Now $$(1-a)^n(1+a)^n = (1-a^2)^n < 1,$$

and therefore $(1-a)^n < \dfrac{1}{(1+a)^n} < \dfrac{1}{1+na}$ by (2).

The only restriction on a is that na must be a positive proper fraction, n being any positive integer.

To apply these results to the trigonometric limits we take the inequalities

$$1 > \cos\frac{x}{n} > 1 - \frac{x^2}{2n^2}. \quad \dots\dots\dots\dots\dots\dots\dots\dots \quad (3)$$

Since n is to tend to infinity, we may suppose it to be such that $x^2/2n$ is a proper fraction, and then let $a = x^2/2n^2$. The inequalities (1) give

$$1 - \frac{x^2}{2n} < \left(1 - \frac{x^2}{2n^2}\right)^n < \frac{1}{1 + \dfrac{x^2}{2n}}.$$

Both $1 - \dfrac{x^2}{2n}$ and $1 + \dfrac{x^2}{2n}$ tend to unity when n tends to infinity, provided x is fixed, or, if variable, is such that $x^2/2n$ tends to zero; therefore $\left(1 - \dfrac{x^2}{2n^2}\right)^n$ also tends to unity. But by (3)

$$1 > \left(\cos\frac{x}{n}\right)^n > \left(1 - \frac{x^2}{2n^2}\right)^n.$$

Thus $\left(\cos\dfrac{x}{n}\right)^n$ lies between unity and a number which tends to unity when n tends to infinity; the limit of $\left(\cos\dfrac{x}{n}\right)^n$ is therefore unity.

Again the inequalities

$$1 > \sin\frac{x}{n}\Big/\frac{x}{n} > \cos\frac{x}{n}$$

give $$1 > \left(\sin\frac{x}{n}\Big/\frac{x}{n}\right)^n > \left(\cos\frac{x}{n}\right)^n;$$

thus $\left(\sin\dfrac{x}{n}\Big/\dfrac{x}{n}\right)^n$ also tends to unity when n tends to infinity.

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We might dispense with the inequalities (3) by writing

$$\left(\cos\frac{x}{n}\right)^n = \left(1 - 2\sin^2\frac{x}{2n}\right)^n$$

and putting a equal to $2\sin^2\frac{x}{2n}$; it is obvious that for large values of n the function $2\sin^2\frac{x}{2n}$ differs but little from $\frac{x^2}{2n^2}$, and therefore na but little from $\frac{x^2}{2n}$.

So far as the limit of $\left(\cos\frac{x}{n}\right)^n$ is concerned, we may, in place of the inequalities (1), put the single theorem

$$\underset{n \to \infty}{\mathrm{L}} (1-a)^n = 0$$

if na is a positive proper fraction. The restriction that n should tend to infinity by integral values is easily removed, for n will lie between two integers, m and $m+1$ say. Then

$$(1-a)^{m+1} < (1-a)^n < (1-a)^m,$$

and therefore, by applying the inequalities (1),

$$1-(m+1)a < (1-a)^n < \frac{1}{1+ma}.$$

G. A. GIBSON.

Centre of Curvature.

Professor Bryan's article on Curvature, etc., in your last issue (p. 219) ends with a challenge. I wonder whether he would be satisfied with the following reasoning to prove that the intersection of "consecutive" normals and the centre of the circle through these "consecutive" points of a curve have the same point as limiting position.

The circumcentre of the triangle formed by three points on a curve is the intersection of the mid-normals of the sides PQ, QR, and these are distant by infinitesimal amounts of higher order than PQ, QR from the normals to the curve at the points of the arcs PQ and QR where they are touched by tangents parallel to the chords PQ and QR respectively.

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