ON FOURIER TRANSFORMS OF RADIAL FUNCTIONS

JAMES L. GRIFFITH *

(Received 24 January 1964)

1. Introduction

The Fourier transform $F(\boldsymbol{y})$ of a function $f(\boldsymbol{t})$ in $L^1(E_k)$ where E_k is the k-dimensional cartesian space will be defined by

$$F(\boldsymbol{y}) = (2\pi)^{-\frac{1}{2}k} \int_{E_k} e^{i(\boldsymbol{t}\cdot\boldsymbol{y})} f(\boldsymbol{t}) dV_t. \tag{1.1}$$

We consider the inversion formula $f_0(\boldsymbol{x}) = \lim_{R\to\infty} g(\boldsymbol{x}, R)$ where

$$g(\boldsymbol{x}, R) = (2\pi)^{-\frac{1}{2}k} \int_{B_R} (1-s^2/R^2)^n e^{-i(\boldsymbol{x}\cdot\boldsymbol{y})} F(\boldsymbol{y}) dV_y \tag{1.2}$$

in which formula $\boldsymbol{s}$ is the radial vector in $\boldsymbol{y}$-space and B_R is the ball of radius R with centre at the origin. In the cases considered $f_0(\boldsymbol{x}) = f(\boldsymbol{x})$ almost everywhere, but this detail will not concern us at the moment.

Following the method of Bochner [1] we substitute (1.1) in (1.2) and then change the origin to $\boldsymbol{x}$ by writing $\boldsymbol{t} = \boldsymbol{x}+\boldsymbol{z}$. Thus we obtain

$$g(\boldsymbol{x}, R) = (2\pi)^{-k} \int_{E_k(\boldsymbol{z})} f(\boldsymbol{x}+\boldsymbol{z}) dV_z \int_{B_R(\boldsymbol{y})} (1-s^2/R^2)^n e^{i(\boldsymbol{y}\cdot\boldsymbol{z})} dV_y.$$

We now express the y-system in polar co-ordinates and integrate out all of the "angular" co-ordinates. This leaves us with

$$g(\boldsymbol{x}, R) = (2\pi)^{-\frac{1}{2}k} \int_{E_k} f(\boldsymbol{x}+\boldsymbol{z}) dV_z \int_0^R r^{-\frac{1}{2}(k-2)} s^{\frac{1}{2}k} (1-s^2/R^2)^n J_{\frac{1}{2}(k-s)}(rs) ds$$

where now $\boldsymbol{r}$ is the radius vector of the $\boldsymbol{z}$-system. The final simplification is obtained by turning the $\boldsymbol{z}$-system into polar co-ordinates and integrating out all the variables except r. The final result is

$$g(\boldsymbol{x}, R) = \frac{2^{-\frac{1}{2}k+n}\Gamma(n+1)}{[\Gamma(\frac{1}{2})]^k} \int_0^\infty r^{\frac{1}{2}k-n-1} R^{\frac{1}{2}k-n} J_{\frac{1}{2}k+n}(Rr) Q(r) dr$$

where $Q(r) = \int f(\boldsymbol{x}+r\boldsymbol{z}) dA$, the $(k-1)$ dimensional integral (area) over the surface of the unit sphere.

The particular value of $n = \frac{1}{2}(k-1) = \alpha$ is called the critical value of

* This research was supported in part by the National Science Foundation, while the author was at the University of Kansas.

the index (see in particular E.M. Stein [5]). If $n > \alpha$, we may split the integral into two parts and write

$$g(x, R) = \int_0^p + \int_p^\infty \cdots dr$$

and it is obvious that $\lim_{R\to\infty} \int_p \cdots dr = 0$. That is to say $\lim_{R\to\infty} g(x, R)$ depends only on the values of $f(t)$ near $t = x$. The inversion formula will possess a localisation property. When $n < \alpha$ it is easy to construct a function $f(t)$ which is finite near x, but for which the integral will not converge.

The critical value α for the localisation property to hold was obtained on the assumption that $f(t)$ belonged to $L^1(E_k)$. As mentioned by Bochner if we add further conditions on differentiability and integrability on $f(t)$ it is possible to reduce the value of the critical value to zero.

In this paper we will determine what effect symmetry of $f(t)$ will have on the critical value. It will be shown that the critical value is closely related to the singularity (if such exists) of $f(t)$ at the origin.

2

In this section we assume that $f(t)$ belongs to $L^1(E_k)$ and is radial, that is $f(t) = g(r)$. We then follow Bochner and Chandrasekharan ([2], p. 67 et seq.) to see that the Fourier transform is also radial and is given by

$$G(s) = s^{-\frac{1}{2}(k-2)} \int_0^\infty r^{\frac{1}{2}k} J_{\frac{1}{2}(k-2)}(sr) g(r) dr. \tag{2.1}$$

The inversion formula we wish to investigate will then be written as $\lim_{R\to\infty} g(r, R)$, where

$$g(r, R) =$$

$$= r^{-\frac{1}{2}(k-2)} \int_0^R (1-s^2/R^2)^n s^{\frac{1}{2}k} J_{\frac{1}{2}(k-2)}(sr) G(s) ds \tag{2.1a}$$

$$= r^{-\frac{1}{2}(k-2)} \int_0^\infty u^{\frac{1}{2}k} g(u) du \int_0^R s(1-s^2/R^2)^n J_{\frac{1}{2}(k-2)}(sr) J_{\frac{1}{2}(k-2)}(su) ds \tag{2.1b}$$

where now $u^{k-1}g(u)$ belongs to $L^1(0, \infty)$. We recall that if $\int_0^{\to\infty} p(r)dr$ exists then $\lim_{R\to\infty} \int_0^R (1-r^2/R^2)^n p(r) dr$ also exists and equals $\int_0^{\to\infty} p(r)dr$ (see Titchmarsh [6], p. 27).

We then divide the integral $\int_0^\infty$ in (2.1b) into

$$\int_0^\infty = \int_0^a + \int_a^{r-b} + \int_{r-b}^{r+b} + \int_{r+b}^\infty .$$

If we take $n = 0$ and use Watson ([7], p. 134, (8)) we obtain the contribution from the last integral to be

https://doi.org/10.1017/S1446788700027725 Published online by Cambridge University Press

$$r^{-\frac{1}{2}(k-2)}\int_{r+b}^{\infty}\frac{Rg(u)u^{\frac{1}{2}k}(rJ_{\frac{1}{2}(k-2)}(uR)J_{\frac{1}{2}k}(rR)-uJ_{\frac{1}{2}(k-2)}(rR)J_{\frac{1}{2}k}(uR))}{r^2-u^2}\,du.$$

The asymptotic expressions for the Bessel Functions and the Riemann Lebesgue Lemma show that this contribution vanishes as $R\to\infty$ (see for example Titchmarsh [6], p. 240 et seq.). A similar remark can be made concerning the contribution $\int_a^{r-b}$. Thus the contributions to $\lim_{R\to\infty}g(r, R)$ from $\int_a^{r-b}$ and $\int_{r+b}^{\infty}$ will both vanish for all $n \geqq 0$.

Now Titchmarsh (l.c.) shows that the contribution from $\int_0^a$ vanishes if $u^{\frac{1}{2}k}g(u)$ belongs to $L^1(0, a)$. This is a heavier condition than we wish to impose. We will assume that

$$\int_0^t u^{k-1}|g(u)|du = P(t) = o(t^c), \text{ for some } c \geqq 0$$

as $t\to 0$.

Writing $\nu=\frac{1}{2}(k-2)$, as is usual, we use the Parseval formula for the Hankel transforms in conjunction with formulae of Erdelyi ([4], p. 26, (33) and p. 52, (31)) to obtain

$$\text{(2.2a)}\quad \begin{aligned}&\int_0^R s(1-s^2/R^2)^n J_\nu(sr)J_\nu(su)ds = I(R)\ \text{(say)}\\ &= \frac{AR^{2-2\nu}}{r^\nu u^\nu}\int_{R|r-u|}^{R|r+u|}\frac{J_{n+\nu+1}(y)}{y^{n+\nu}}\,[y^2-R^2(r-u)^2]^{\nu-\frac{1}{2}}[R^2(r+u)^2-y^2]^{\nu-\frac{1}{2}}dy\end{aligned}$$

where $A = 2^{n-3\nu+1}\Gamma(n+1)/\pi^{\frac{1}{2}}\Gamma(\nu+\frac{1}{2})$. Then after a change of variables

$$\text{(2.2b)}\quad \begin{aligned}&I(R) =\\ &\frac{AR^{\nu-n+1}}{2r^\nu u^\nu}\int_{(r-u)^2}^{(r+u)^2}\frac{J_{n+\nu+1}(Rv^{\frac{1}{2}})}{v^{\frac{1}{2}(n+\nu+1)}}\,[v-(r-u)^2]^{\nu-\frac{1}{2}}[(r+u)^2-v]^{\nu-\frac{1}{2}}dv.\end{aligned}$$

Integrating equation (2.2b) by parts q times, and using the formula

$$\int v^{-\frac{1}{2}\nu}J_\nu(v^{\frac{1}{2}})dv = 2v^{-\frac{1}{2}(\nu-1)}J_{\nu-1}(v^{\frac{1}{2}})$$

(Watson [7], p. 132, (1), with some change of variable), we write $I(R)$ in the form

$$\text{(2.3)}\quad \begin{aligned}&I(R) =\\ &\frac{R^{\nu-n+1-q}}{u^\nu}\int_{(r-u)^2}^{(r+u)^2}\frac{J_{n+\nu+1-q}(Rv^{\frac{1}{2}})}{v^{\frac{1}{2}(n+\nu+1-q)}}\sum_{p=0}^{q}B_p[v-(r-u)^2]^{\nu-\frac{1}{2}-p}[(r+u)^2-v]^{\nu-\frac{1}{2}-q+p}\,dv\end{aligned}$$

where B_p are constants not containing R or u. Thus

https://doi.org/10.1017/S1446788700027725 Published online by Cambridge University Press

$$
\begin{aligned}
|I(R)| &\leqq \frac{R^{\nu-n+1-q}}{u^{\nu}}\int_{(r-u)^2}^{(r+u)^2}\frac{|J_{n+\nu+1-q}(Rv^{\frac{1}{2}})|}{v^{\frac{1}{2}(n+\nu+1-q)}}\,(4ru)^{2\nu-1-q}\Big(\sum_{p=0}^{q}|B_p|\Big)dv \\
&= O(R^{\nu-n-q+\frac{1}{2}}u^{\nu-q}) \qquad (2.4)\\
&= O\big((Ru)^{\nu-n-q+\frac{1}{2}}u^{n-\frac{1}{2}}\big).
\end{aligned}
$$

If $\nu-\frac{1}{2}$ is an integer we may take $q=\nu+\frac{1}{2}$. It is then easy to show that when $\nu-\frac{1}{2}$ is an integer the estimate (2.4) will hold for all q with $0\leqq q\leqq \nu+\frac{1}{2}$.

If ν is an integer we may only take $q=\nu$, in (2.3). However we may take the integration by parts one step further for each term in the summation in (2.3) except those terms given by $p=0$ and $p=q=\nu$. The first of these terms will be

$$
\begin{aligned}
T(R) &= \\
&\frac{BR^{1-n}}{u^{\nu}}\int_{(r-u)^2}^{(r+u)^2}\frac{J_{n+1}(Rv^{\frac{1}{2}})}{v^{\frac{1}{2}(n+1)}}\,[v-(r-u)^2]^{-\frac{1}{2}}[(r+u)^2-v]^{\nu-\frac{1}{2}}dv \\
&= \frac{CR^{\frac{1}{2}-n}}{u^{\nu}}\int_{(r-u)^2}^{(r+u)^2}[v-(r-u)^2]^{-\frac{1}{2}}[(r+u)^2-v]^{\nu-\frac{1}{2}}\left[\frac{\cos(Rv^{\frac{1}{2}}-w)}{v^{\frac{1}{2}(n+2)}}+O(R^{-1})\right]dv
\end{aligned}
$$

(B, C and w being constants not containing u or R)

$$
\begin{aligned}
&= \frac{2CR^{\frac{1}{2}-n}}{u^{\nu}}\int_{|r-u|}^{r+u}[v^2-(r-u)^2]^{-\frac{1}{2}}[(r+u)^2-v^2]^{\nu-\frac{1}{2}}\left[\frac{\cos(Rv-w)}{v^{n+1}}+O(R^{-1})\right]dv \\
&= \frac{2CR^{\frac{1}{2}-n}}{u^{\nu}}(4ru)^{\nu-\frac{1}{2}}(r-u)^{-n-1}(2r)^{\frac{1}{2}}\int_{r-u}^{p}\cos(Rv-w)(y^2-|r-u|)^{-\frac{1}{2}}dv \\
&\qquad\qquad +O(R^{-\frac{1}{2}-n})
\end{aligned}
$$

with $|r-u|<p<|r+u|$, by a mean value theorem.

If we now put $v=|r-u|+z/R$ in the integral this last expression takes the form

$$
R^{-\frac{1}{2}}\int_{0}^{Rp-R|r-u|}\cos(z+R|r-u|-w)z^{-\frac{1}{2}}dz
$$

in which the integral is bounded uniformly for all R and u. If we then substitute back we see that the contribution from the term in $p=0$ to $I(R)$ will be $O(R^{-n}u^{-\frac{1}{2}})$. A similar treatment will show that the contribution from the term with $p=\nu$ will be of the same order.

We have then shown that the estimate (2.4) holds for $0\leqq q\leqq \nu+\frac{1}{2}$ whether ν is an integer or not.

We now return to equation (2.1b) to examine the contribution form $\int_0^a$. It will be useful to split the range into $\int_0^{1/R}+\int_{1/R}^{a}=K_1+K_2$ (say). K_1 will be dominated by a term of the form

$$S(R) = CR^{\frac{1}{2}k-n-q-\frac{1}{2}} \int_0^{1/R} |g(u)| u^{k-1-q} du$$

where C is independent of R (but is dependent on q). Recalling that $\alpha = \frac{1}{2}(k-1) = \nu+\frac{1}{2}$, we have

$$S(R) = CR^{\alpha-n-q}\{[P(u)u^{-q}]_0^{1/R} + q\int_0^{1/R} P(u)u^{-q-1}du\}.$$

If $c > 0$, then we select $0 < q < c$ and each term is seen to be $o(R^{\alpha-n-c})$ as $R \to \infty$. If $c = 0$, we select $q = 0$ and the second term will vanish so that $S(R) = o(R^{\alpha-n})$.

A similar method shows that K_2 will be dominated by

$$\begin{aligned} V(R) &= CR^{\alpha-n-q}[P(u)u^{-q}]_{1/R}^{a} + q\int_{1/R}^{a} P(u)u^{-q-1}du \\ &= o(R^{\alpha-n-c}) + o(R^{\alpha-n-q}) \end{aligned}$$

the first term being from the upper limits and the second from the lower.

So provided that we choose $n > \alpha - c$ if $c < \alpha$ and $n \geqq 0$ if $c > \alpha$ we can be assured that the contribution from $\int_0^a$ will vanish. That is to say that the inversion integral (1.2) or (2.1a) will be localised if $n > \alpha - c$.

We will now show that in general we cannot improve on this result.

3

Suppose that

$$g(x) = \begin{cases} x^{c-k}, & 0 \leqq x \leqq X \\ 0 & x > X \end{cases} \quad c > 0$$

so that

$$\int_0^t x^{k-1} g(x) dx = t^c/c.$$

We may then write $g(x) = f(x) - h(x)$ with

$$f(x) = x^{c-k} = x^{c-2\nu-2}, \text{ all } x$$

and

$$h(x) = \begin{cases} x^{c-k} = x^{c-2\nu-2}, & x > X \\ 0, & x < X. \end{cases}$$

Thus equation (2.1b) becomes

$$\begin{aligned} g(r, R) &= f(r, R) - h(r, R) \\ &= r^{-\nu}\int_0^\infty u^{c-\nu-1}du \int_0^R s(1-s^2/R^2)^n J_\nu(sr)J_\nu(su)ds \\ &\quad - r^{-\nu}\int_X^\infty u^{c-\nu-1}du \int_0^R s(-1-s^2/R^2)^n J_\nu(sr)J_\nu(su)ds. \end{aligned}$$

Now $\lim_{R\to\infty} h(r, R)$ exists for all $n \geqq 0$ by the proof of theorem 135 of Titchmarsh [6]. We will only need to examine $\lim_{R\to\infty} f(r, R)$.

https://doi.org/10.1017/S1446788700027725 Published online by Cambridge University Press

From Watson ([7], p. 391, (1)),

$$f(r, R) = \frac{r^{-\nu}\Gamma(\frac{1}{2}c)}{2^{\nu-c+1}\Gamma(\nu-\frac{1}{2}c+1)}\int_0^R s^{\nu-c+1}(1-s^2/R^2)^n J_\nu(sr)ds. \tag{3.1}$$

For our purpose we will put $r = 1$ and $n = m-\beta$ where m is an integer and $0 \leqq \beta < 1$. We will assume that $n < \nu+\frac{1}{2}-c$, and will examine

$$I(R) = \int_0^R s^{\nu-c+1}(R^2-s^2)^n J_\nu(s)ds$$

as $R \to \infty$.

We will require the two formulae

$$\int_0^z t^{\frac{1}{2}\nu} J_\nu(at^{\frac{1}{2}})dt = 2a^{-1}z^{\frac{1}{2}(\nu+1)}J_{\nu+1}(az^{\frac{1}{2}}) \tag{3.2a}$$

(Watson [7], p. 133, (1)) and

$$\int_0^z (z-t)^b t^{\frac{1}{2}\nu} J_\nu(at^{\frac{1}{2}})dt = 2^{b+1}\Gamma(b+1)a^{-b-1}z^{\frac{1}{2}(\nu+b+1)}J_{\nu+b+1}(az^{\frac{1}{2}}), \tag{3.2b}$$

which is found by expanding the Bessel function in a series form and integrating.

Now

$$2I(R) = \int_0^{R^2} x^{-\frac{1}{2}c}(R^2-x)^n x^{\frac{1}{2}\nu} J_\nu(x^{\frac{1}{2}})dx. \tag{3.3}$$

We will show that as $R \to \infty$ the dominating part of $I(R)$ can be expressed in the form $AR^{\nu+n-c}J_{\nu+n+1}(R)$. More exactly we shall show that as $R \to \infty$

$$R^{c-n-\nu+\frac{1}{2}}I(R) = AR^{\frac{1}{2}}J_{\nu+n+1}(R)+o(1). \tag{3.4}$$

We now expand (3.3) using integration by parts m times. Then $I(R)$ will be expressed as a linear combination of terms of the type

$$S_{a,b} = \int_0^{R^2} x^{-\frac{1}{2}c-a}(R^2-x)^{n-b}x^{\frac{1}{2}(\nu+a+b)}J_{\nu+a+b}(x^{\frac{1}{2}})dx. \tag{3.5}$$

The expansion will contain only one term involving $b = m$. We leave this term unaltered but carry out one integration by parts step on all the other terms. We then split the formula for $S_{a,b}$ into $\int_0^1 + \int_1^{R^2} = S_1+S_2$. From which we see that as $R \to \infty$

$$S_1 = O(R^{2n-2b}) \quad \text{and} \quad S_2 = O(R^{\nu-a-b+2n-c-1\frac{1}{2}}).$$

The only terms in (3.5) which will possibly contribute a term of sufficiently great order will be that in which $a = 0$ and $b = m$.

So

$$2I(R) = \frac{2^m\Gamma(n+1)}{\Gamma(1-\beta)}\int_0^{R^2} x^{-\frac{1}{2}c}(R^2-x)^{-\beta}x^{\frac{1}{2}(\nu+m)}J_{\nu+m}(x^{\frac{1}{2}})dx$$

$$+ \text{ terms of lower order.}$$

Further

$$(\Gamma(1-\beta)/2^m\Gamma(n+1))2I(R)$$
$$= \left[x^{-\frac{1}{2}c}\int_0^x (R^2-u)^{-\beta}u^{\frac{1}{2}(\nu+m)}J_{\nu+m}(u^{\frac{1}{2}})du\right]_0^{R^2}$$
$$+\tfrac{1}{2}c\int_0^{R^2} x^{-\frac{1}{2}c-1}dx\int_0^x (R^2-u)^{-\beta}u^{\frac{1}{2}(\nu+m)}J_{\nu+m}(u^{\frac{1}{2}})du$$
$$+ \text{ terms of lower order.}$$

The first term is $2^{1-\beta}\Gamma(1-\beta)R^{\nu+n-c+1}J_{\nu+n+1}(R)$. To make an estimate of the second term we divide the range of integration. (In the next few lines A will denote a constant but not necessarily the same constant).

$$I_1(R) = \int_1^{R^2} x^{-\frac{1}{2}c-1}dx\int_1^x (R^2-u)^{-\beta}u^{\frac{1}{2}(\nu+m)}J_{\nu+m}(u^{\frac{1}{2}})du$$
$$= \int_1^{R^2} x^{-\frac{1}{2}c-1}(R^2-x)^{-\beta}dx\int_p^x u^{\frac{1}{2}(\nu+m)}J_{\nu+m}(u^{\frac{1}{2}})du,\ 1\leqq p\leqq x$$
$$= 2\int_1^{R^2} 2x^{-\frac{1}{2}c-1}(R^2-x)^{-\beta}[u^{\frac{1}{2}(\nu+m+1)}J_{\nu+m+1}(u^{\frac{1}{2}})]_p^x dx.$$

Then since $p\geqq 1$ and $\frac{1}{2}(\nu+m+1)>\frac{1}{2}$,

$$|I_1(R)| \leqq A\int_1^{R^2} x^{-\frac{1}{2}(c-\nu-m+1\frac{1}{2})}(R^2-x)^{-\beta}dx$$
$$= O(R^{\nu+m-c+\frac{1}{2}-2\beta}).$$
$$I_2(R) = \int_1^{R^2} x^{-\frac{1}{2}c-1}dx\int_0^1 (R^2-u)^{-\beta}u^{\frac{1}{2}(\nu+m)}J_{\nu+m}(u^{\frac{1}{2}})du,$$
$$I_2(R) < A\int_1^{R^2} x^{-\frac{1}{2}c-1}(R^2-1)^{-\beta}dx = O(R^{-2\beta}).$$
$$I_3(R) = \int_0^1 x^{-\frac{1}{2}c-1}dx\int_0^x (R^2-u)^{-\beta}u^{\frac{1}{2}(\nu+m)}J_{\nu+m}(u^{\frac{1}{2}})du$$
$$= O(R^{-2\beta}).$$

Thus examining I_1, I_2 and I_3, the second term in (3.6) is seen to be of lower order than $R^{\nu+n-c+\frac{1}{2}}$ provided that $\beta>0$. If $\beta=0$ then

$$I_1(R) = \int_1^{R^2} x^{-\frac{1}{2}c-1}\,2[u^{\frac{1}{2}(\nu+m+1)}J_{\nu+m+1}(u^{\frac{1}{2}})]_1^x dx$$
$$= O(R^{\nu+m-c-\frac{1}{2}})$$

after one step of integration by parts.

We have thus shown that if $n<\alpha-c$,

$$I(R) = AR^{\nu+n-c+1}J_{\nu+n+1}(R) + \text{ terms of lower order.}$$

This result confirms the assertion that we cannot in general take $n<\alpha-c$ for all $c>0$. Putting this result in another way we can say that for each

$n < \alpha - c$ we can find a $g(t)$ so that $\int_0^t x^{k-1}|g(x)|dx = o(t^c)$ for which the inversion theorem will not be localised.

Further noting that if $\int_0^t x^{k-1}|g(x)|dx = o(t^c)$ for $c > 0$, then $\int_0^t x^{k-1}|g(x)| dx = o(1)$, we can extend our result to say that if $n < \alpha$ we can find a $g(t)$ so that $\int_0^t x^{k-1}|g(x)|dx = o(1)$ and for which the inversion theorem will not be localised.

4

Up to this point no comment has been made concerning the contribution in $\lim_{R\to\infty} g(r, R)$ from the part of the integral $\int_{r-b}^{r+b}$ in equation (2.1b).

If $g(u)$ is of bounded variation in $[r-b, r+b]$, then the limit in (2.1b)

$$\lim_{R\to\infty} r^{-\frac{1}{2}(k-2)} \int_{r-b}^{r+b} \cdots du \int_0^R \cdots ds = \tfrac{1}{2}(f(r+)+f(r-)) \tag{4.1}$$

for $n = 0$, (Titchmarsh [6], Th. 135). Equation (4.1) confirms the assumption made in the previous section that the contribution from $\int_{r-b}^{r+b}$ did not effect the convergence or otherwise of the integral treated there.

Keeping equation (4.1) in mind we will consider

$$F(r, R) = r^{-\frac{1}{2}(k-2)} \int_{r-b}^{r+b} u^{\frac{1}{2}k} f(u)du \int_0^R s(1-s^2/R^2)^n J_{\frac{1}{2}(k-2)}(sr) J_{\frac{1}{2}(k-2)}(su)ds, \tag{4.2}$$

where $f(u) = g(u) - C$, and we will assume that as $t \to 0$,

$$\int_{r-t}^{r+t} |f(u)|du = o(t)$$

(a condition corresponding to that in Chandrasekharan and Minakshisundaram [3], p. 117). It will be profitable to use formula (2.2a). However the estimate in (2.4) will fail when $|Rv^{\frac{1}{2}}| < 1$. We examine first the integrals in which $|Rv^{\frac{1}{2}}| > 1$.

If we put $q = \nu + \frac{1}{2}$ in (2.4) we see that

$$\int_{r+1/R}^{r+b} u^{\nu+1} f(u)du \int_{(r-u)^2}^{(r+u)^2} \cdots dv$$

is dominated by a term of the type

$$AR^{-n} \int_{r+1/R}^{r+b} u^{\nu+\frac{1}{2}} |f(u)|du,$$

which $\to 0$ for any $n > 0$. A similar conclusion may be drawn concerning the integrals

$$\int_{r-b}^{r-1/R} \cdots du \int_{(r-u)^2}^{(r+u)^2} \cdots dv, \int_r^{r+1/R} \cdots du \int_{1/R^2}^{(r+u)^2} \cdots dv \text{ and } \int_{r-1/R}^{r} \cdots du \int_{1/R^2}^{(r+u)^2} \cdots dv.$$

We are finally left with one part

$$\int_{r-1/R}^{r+1/R} \cdots du \int_{(r-u)^2}^{1/R} \cdots dv = K(R), \text{ say.}$$

Now

$$K(R) = BR^{\nu-n+1}\int_{r-1/R}^{r+1/R} uf(u)du\int_{(r-u)^2}^{1/R^2} \frac{J_{n+\nu+1}(Rv^{\frac{1}{2}})}{v^{\frac{1}{2}(n+\nu+1)}}\,(v-(r-u)^2)^{\nu-\frac{1}{2}}((r+u)^2-v)^{\nu-\frac{1}{2}}dv$$

(with B constant, by equation (2.2a)).

Thus using the estimate for the Bessel function when $|Rv^{\frac{1}{2}}| < 1$, we see that

$$K(R) \leqq CR^{2\nu+2}\int_{r-1/R}^{r+1/R} u|f(u)|du\int_{(r-u)^2}^{1/R^2} (v-(r-u)^2)^{\nu-\frac{1}{2}}((r+u)^2-v)^{\nu-\frac{1}{2}}dv \qquad (C \text{ constant})$$

$$\leqq CR^{2\nu+2}\int_{r-1/R}^{r+1/R} u|f(u)|R^{-2\nu-1}(4ru)^{\nu-\frac{1}{2}}du$$

$$\leqq DR\int_{r-1/R}^{r+1/R} |f(u)|du = o(1) \text{ as } R\to\infty \qquad (D \text{ constant}).$$

We are then assured that if

$$G_1(r, R) = r^{-\frac{1}{2}(k-2)}\int_{r-b}^{r+b} u^{\frac{1}{2}k}g(u)du\int_0^R s(1-s^2/R^2)^n J_{\frac{1}{2}(k-2)}(sr)J_{\frac{1}{2}(k-2)}(su)ds$$

and there exists a C so that

$$\int_{r-t}^{r+t} |f(u)|du = o(t),\ f(u) = g(u)-C,$$

as $t\to 0+$, then

$$\lim_{R\to\infty} G_1(r, R) = C$$

for all $n > 0$.

We have thus shown that if $f(t)$ in (1.1) is radially symmetric, and is written $f(t) = g(u)$, and

$$\int_0^t u^{k-1}g(u)du = o(t^c) \text{ as } t\to 0+,$$

then $\lim_{R\to\infty} g(x, R)$ in (1.2) is localised if $n > \frac{1}{2}(k-1)-c$, $n > 0$. Also if there exists a C so that

$$\int_{r-t}^{r+t} |g(u)-C|du = o(t) \text{ as } t\to 0+,$$

then $\lim_{R\to\infty} g(x, R) = C$.

References

[1] Bochner, S., Lectures on Fourier Integrals (Princeton, 1959).
[2] Bochner, S. and Chandrasekharan, K., Fourier Transforms (Princeton, 1949).
[3] Chandrasekharan, K. and Minakshisundaram, S., Typical Means (Tata Institute, Bombay).

[4] Erdelyi and others, Tables of Integral Transforms (McGraw Hill, New York, 1954).
[5] Stein, E. M. Localisation and summability of Multiple Fourier Series, Acta Math. 100 (1958), 93–147.
[6] Titchmarsh, E. C., Fourier Integrals (Oxford, 1948).
[7] Watson, G. N., Theory of Bessel Functions (Cambridge, 1952).

University of New South Wales
and University of Kansas