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FREE PRODUCTS WITH AMALGAMATION AND p-ADIC LIE GROUPS

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ABSTRACT. Using the theory of p-adic Lie groups we give conditions for a finitely generated group to admit a splitting as a non-trivial free product with amalgamation. This can be viewed as an extension of a theorem of Bass.

1. Introduction.

The existence of a splitting of a finitely generated group as a non-trivial free product with amalgamation, or HNN-extension is an extremely useful tool. The existence of a map to $\mathbf{Z}$ determines an HNN-decomposition, however detecting a free product with amalgamation decomposition is usually harder, but often of more use. One of the main results of this paper is Theorem 1.1 which guarantees a free product with amalgamation under certain conditions.

THEOREM 1.1. *Let Γ be a finitely generated non-elementary subgroup of* $\mathrm{SL}(2,\mathbf{C})$ *whose traces consist of algebraic numbers and which contains an element whose trace is not an algebraic integer. Further suppose that Γ does not contain a free subgroup of finite index.*

Then Γ splits as a non-trivial free product with amalgamation.

The theorem is similar to the GL_2-subgroup theorem of Bass [1], [2] (see Theorem 4.1 below). Indeed, using the result that a finitely generated group is virtually free if and only if the group is a graph of groups where all vertex groups are finite (see [11], Theorem 7.2) one can refine the statement of the theorem. For example, Theorem 1.1 can be viewed as an extension of Theorem 4.1 in the torsion-free case, as it dispenses with the possibility of an HNN-extension. A further discussion of this is given in Section 4.

The methods of the paper are those of p-adic Lie groups, and grew out of our paper [7] with C. Maclachlan. The present paper provides a more elegant proof to the main result of [7]. It also re-proves some well-known results in 3-manifold topology.

2. p-adic Lie groups and Lie algebras.

Here we collect salient points from the theory of p-adic Lie groups and their Lie algebras. Throughout p is a fixed prime and k is a finite extension of $\mathbf{Q}_p$. Our main interest is in the group $\mathrm{SL}(2,k)$. There is considerable overlap with [7] and so we only give a brief summary. See also [5] or [10] for details.

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2.1.

DEFINITION 2.1. Let H be a topological group. Then H is defined to be a p-adic Lie group if H has the structure of an analytic manifold over $\mathbf{Q}_p$ and if the function $H \times H \longrightarrow H$ defined by $(x, y) \longrightarrow xy^{-1}$ is analytic.

By a *Lie algebra* over k we mean a vector space over k with a multiplication which satisfies the conditions

$$x^2 = 0 \quad \text{and} \quad (xy)z + (yz)x + (zx)y = 0.$$

The following theorem summarizes what we need here, a reasonably full account over $\mathbf{Q}_p$ is given in Section 4 of [7]. The arguments for k are identical. We denote by $\mathrm{sl}(2, k)$ the trace-less matrices in $M(2, k)$. This is a 3-dimensional Lie algebra over k when equipped with the obvious Lie bracket.

THEOREM 2.2. *(i)* $\mathrm{SL}(2, k)$ *is a p-adic Lie group whose Lie algebra is* $\mathrm{sl}(2, k)$.
(ii) Let $\mathcal{L}$ *be a* 3*-dimensional non-solvable Lie algebra over k. Then* $\mathcal{L}$ *is isomorphic to* $\mathrm{sl}(2, k)$ *or* $\mathcal{D}_0$*, the pure quaternions in the unique division algebra of quaternions over k. These algebras are non-isomorphic, and any other Lie algebra of dimension at most* 3 *is solvable.*

Part (ii) of Theorem 2.2 follows from the classification theorem for quaternion algebras over local fields, [14], which states there are precisely two isomorphism classes of quaternion algebras over any finite extension k of $\mathbf{Q}_p$; namely $M(2, k)$ or the unique division algebra of quaternions D. The elements of norm 1 in D, which we denote by D^1, is a compact p-adic Lie group whose Lie algebra is $\mathcal{D}_0$.

We require the following result, which is presumably well-known. We fix some notation; we let $\mathcal{L}_k$ denote either of the two Lie algebras in Theorem 2.2(ii).

LEMMA 2.3. *Let G be a non-solvable Lie subgroup of* $\mathrm{SL}(2, k)$*. Then the Lie algebra of G is isomorphic to* $\mathcal{L}_\ell$ *for some subfield* ℓ *of k.*

PROOF. Let $L(G)$ denote the Lie algebra of G. Since G is a subgroup of $\mathrm{SL}(2, k)$, as usual, we may identify $L(G)$ as a subalgebra of $\mathrm{sl}(2, k)$. Therefore this subalgebra is defined over a subfield, ℓ say, of k. Now $L(G) \otimes_\ell k$ can be identified with a Lie subalgebra of $\mathrm{sl}(2, k)$ which is defined over k. Since this can have dimension at most 3 over k it follows that $L(G)$ can have dimension at most 3 over ℓ. Since G is non-solvable, Theorem 2.2 implies that $L(G)$ must be $\mathcal{L}_\ell$. ∎

2.2. If G is a group, we let $G^p = \langle g^p \mid g \in G \rangle$. Recall that a *profinite group* is a compact Hausdorff topological group whose open subgroups form a base for the neighbourhoods of the identity and can be characterised as an inverse limit of an inverse system $\{G_i\}$ of finite groups. If the finite groups G_i are all p-groups, we obtain a *pro-p group* and if, furthermore, the maps in the inverse system are all surjective and the quotients G_i/G_i^p

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abelian, then the inverse limit is a *powerful pro-p group*. Finally, a pro-p group is termed *uniform* if it is finitely generated, powerful and satisfies

$$[P_i(G) : P_{i+1}(G)] = [G : P_2(G)] \quad \text{for all } i,$$

where $P_1(G) = G$ and $P_{i+1}(G)$ is defined recursively as $P_i(G)^p[P_i(G), G]$. One should take closures in the previous statement, but the assumption that G is finitely generated makes this unnecessary. ([5] Corollary 1.20). The following fundamental result characterises p-adic Lie groups in terms of these uniform pro-p groups, *cf.* [5] Theorem 9.34.

THEOREM 2.4. *Let G be a topological group. Then G is a p-adic Lie group if and only if G contains an open subgroup which is a uniform pro-p group.* ∎

We now briefly discuss the construction of the Lie algebra of a p-adic Lie group from an open uniform subgroup. See [5] Chapters 7, 8 and 10 for details.

Let U be an open uniform pro-p subgroup of the p-adic Lie group G. Using the discussion in Section 8.2 of [5] $\Lambda = \log U$ can be defined. Then Λ is a $\mathbf{Z}_p$-Lie algebra, and the Lie algebra of G is obtained as $\Lambda \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$. This turns out to be independent of the choice of open uniform subgroup, see [5] Chapter 10.

2.3. We conclude this section with a discussion of $\mathrm{SL}(2,k)$ and the group D^1 (recall Section 2.1). Throughout this section π will denote a local uniformizer for k, ν the valuation on k and $\mathcal{O}$ the valuation ring of k. $\mathcal{O}$ is a compact open subring of k and so $\mathrm{SL}(2,\mathcal{O})$ becomes a compact open subgroup of $\mathrm{SL}(2,k)$, and as such is a p-adic Lie group. One way to view the (unique) p-adic analytic structure on $\mathrm{SL}(2,\mathcal{O})$ is by considering the *principal congruence subgroups* Γ_j obtained as the kernel of the maps

$$\mathrm{SL}(2,\mathcal{O}) \longrightarrow \mathrm{SL}(2,\mathcal{O}/\mathcal{P}^j),$$

where $\mathcal{P}$ is the unique maximal ideal in $\mathcal{O}$. These groups are uniform pro-p groups when p is odd, when p is even, the groups Γ_j are uniform for $j \geq 2$. In either case the groups Γ_j form a basis of open neighbourhoods of the identity in $\mathrm{SL}(2,\mathcal{O})$.

A fact about D^1 that we will make use of is:

LEMMA 2.5. *Traces of elements in* D^1 *lie in* $\mathcal{O}$.

PROOF. The extension of the valuation ν from k to D is simply given by $\omega(a) = \nu\big(n_D(a)\big)$ where n_D is the reduced norm on D.

If a has norm one it follows in particular that $a \in \mathcal{M} = \{d \in D \mid \omega(d) \geq 0\}$, the *valuation ring* of D which is the unique maximal order in D (see [14] Chapter 2). From the definition of an order, elements in $\mathcal{M}$ have traces in $\mathcal{O}$. ∎

If K is a finite extension of k, the uniqueness of a p-adic structure for the Lie group $\mathrm{SL}(2,k)$ (see [5] Theorem 10.6) implies that the induced topology on $\mathrm{SL}(2,k)$ as a subgroup of $\mathrm{SL}(2,K)$ coincides with the one described above.

The main technical lemma in this article is the following.

LEMMA 2.6. *Let G be a non-compact, non-solvable p-adic Lie subgroup of* $\mathrm{SL}(2,k)$*. Then the Lie algebra of G is isomorphic to* $\mathrm{sl}(2,\ell)$ *for some subfield* ℓ *of k.*

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PROOF. Let ℓ be the field generated over $\mathbf{Q}_p$ by all the traces of elements in G.

Since G is a p-adic Lie group, Theorem 2.4 implies that G contains an open uniform subgroup O. The ℓ-algebra, A, generated by finite ℓ-combinations of elements of O is a quaternion algebra over ℓ. Therefore A is isomorphic to $M(2, \ell)$ or the unique division algebra of quaternions D over ℓ. Let A_0 denote the pure quaternions in A. Since ℓ is a complete field, A is a complete algebra.

We claim that $G \subset A$. To see this note that O contains a basis $\{1, e_1, e_2, e_3\}$ for A over ℓ. Furthermore, this extends to a k-basis of $M(2, k)$. Thus any element of g is a k-combination of the basis elements. We will show g is an ℓ-combination.

Let $g = a + \Sigma x_i e_i$, then by definition $\mathrm{tr}(g) = 2a \in \ell$. Now consider ge_i^{-1} for each $i = 1, 2, 3$. Since $ge_i^{-1} \in G$ it is easily seen that each $x_i \in \ell$ as required.

The proof now proceeds as follows. By Lemma 2.3 the Lie algebra of G is one of $\mathrm{sl}(2, \ell)$ or $\mathcal{D}_0$ (defined over ℓ). We shall eliminate the latter as a possibility. To do this we use the construction of the Lie algebra $\log O$ discussed above in Section 2.2.

Using the action of the tree of $\mathrm{SL}(2, k)$ (*cf.* [9]), O is conjugate into $\mathrm{SL}(2, \mathcal{O})$, and on passing to a subgroup of finite index if necessary (which will not change the Lie algebra), we can assume that this conjugate, which we will continue to call O, is a subgroup of some Γ_j. We assume that A is conjugated also and continue to call it A. Thus, every element in O has the form $x = 1 + \pi^j a$ for some $a \in M(2, \mathcal{O})$. Note that since $x \in O \subset A$, $x - 1 \in A$ and so for any integer m, $(\pi^j a)^m \in A$.

Now consider $\log(x)$. By definition

$$\log(1 + \pi^j a) = \sum (-1)^{n+1} \frac{(\pi^j a)^n}{n}.$$

By our remark above, each term in the summation is an element of A, and as A is a complete algebra we deduce that the sum above converges to an element of A. We next claim that $\log(1 + \pi^j a) \in A_0$. It suffices to show that the reduced trace of $\log(1 + \pi^j a)$ is zero. First note that if $\bar{x}$ denotes the usual canonical involution on $M(2, k)$, then its restriction to A coincides with the canonical involution on A. Thus, with this the following is easy to establish;

$$\begin{aligned}\mathrm{tr}\big(\log(1 + \pi^j a)\big) &= \log(1 + \pi^j a) + \overline{\log(1 + \pi^j a)} = \log(1 + \pi^j a) + \log(1 + \pi^j \bar{a}) \\ &= \log[(1 + \pi^j a)(1 + \pi^j \bar{a})].\end{aligned}$$

Now as $1 + \pi^j a \in O$, it has reduced norm equal to one (as the determinant is the reduced norm), and so we deduce that $\mathrm{tr}\big(\log(1 + \pi^j a)\big) = 0$ as was claimed. Hence we conclude from the discussion in Section 2.2 that the Lie algebra of G is A_0.

Let us assume that $A_0 \cong \mathcal{D}_0$ (recall Theorem 2.2), so that $A \cong D$. Now $A \subset M(2, k)$, and by standard results in quaternion algebras we can embed D in $M(2, K)$ for some quadratic extension K of k. By the Skölem-Noether theorem (see [14]) this isomorphism is achieved by an an inner automorphism of $M(2, K)$ and so trace is preserved. From above $G \subset A$, and indeed as elements in G have determinant 1, G is contained in the norm 1 elements of A. As G is non-compact it contains an element whose trace is not in $\mathcal{O}$. However, these comments together with Lemma 2.5 means G cannot be conjugate into D^1. Hence A cannot be isomorphic to D, and this contradiction means $A_0 \cong \mathrm{sl}(2, \ell)$ as was required. ∎

3**. Main result.** Our main result is a classification theorem for Lie subgroups of $\mathrm{SL}(2, k)$, where throughout this section k will always be a finite extension of $\mathbf{Q}_p$ for some prime p.

THEOREM 3.1. *Let G be a non-compact, non-solvable Lie subgroup of* $\mathrm{SL}(2, k)$*. Then there is a subfield ℓ of k, containing $\mathbf{Q}_p$, such that G is conjugate to* $\mathrm{SL}(2, \ell)$ *over a finite extension of k.*

The proof is essentially the argument of [7] Theorem 5.4. We will make use of some results in [10] which we state for convenience (see [10], p. 130–131).

THEOREM 3.2. *Let G be a p-adic Lie group and H_1 and H_2 Lie subgroups. Then $H_1 \cap H_2$ is a Lie subgroup, and if $L(H_1)$, $L(H_2)$ and $L(H_1 \cap H_2)$ denote the Lie algebras of H_1, H_2 and $H_1 \cap H_2$ respectively (identified with subalgebras of the Lie algebra of G), then $L(H_1 \cap H_2) = L(H_1) \cap L(H_2)$.* ∎

THEOREM 3.3. *With the hypothesis as above, if $L(H_1) = L(H_2)$, then in a neighbourhood of the identity $H_1 = H_2$.* ∎

PROOF OF THEOREM 3.1. As in the proof of Lemma 2.6 let ℓ be the subfield of k generated over $\mathbf{Q}_p$ by the traces of elements of G. Denote the valuation ring of ℓ by R. Since G is non-compact it follows from the action on the tree of $\mathrm{SL}(2, k)$ (see [9] or [7]) that G contains an element g whose trace does not lie in R. By conjugating we may assume that g is diagonal and $G \subset \mathrm{SL}(2, K)$ where K is at most a quadratic extension of k.

By Lemma 2.6, G has Lie algebra $\mathrm{sl}(2, \ell)$. By definition $\mathrm{SL}(2, R)$ is a Lie subgroup of $\mathrm{SL}(2, K)$, whose Lie algebra is $\mathrm{sl}(2, \ell)$. Hence by Theorem 3.2, $G \cap \mathrm{SL}(2, R)$ is p-adic Lie group whose Lie algebra is $\mathrm{sl}(2, \ell)$. By Theorem 3.3, G and $\mathrm{SL}(2, R)$ agree on a neighbourhood V of the identity in $\mathrm{SL}(2, K)$. As discussed in Section 2.3, the topology on $\mathrm{SL}(2, R)$ coincides with induced topology from $\mathrm{SL}(2, K)$ and so it follows that V must contain an open subgroup of $\mathrm{SL}(2, R)$, and hence one of the principal congruence subgroups Γ_j of $\mathrm{SL}(2, R)$ (recall Section 2.3). Hence G contains a group Γ_j.

Now it is well-known that $\mathrm{SL}(2, \ell)$ is generated by the subgroups ([8]),

$$U = \left\{ \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix} \,\middle|\, q \in \ell \right\} \quad \text{and} \quad L = \left\{ \begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix} \,\middle|\, q \in \ell \right\}.$$

It follows using the diagonal element g that the group $\langle g, \Gamma_j \rangle$ contains both U and L and so $G = \mathrm{SL}(2, \ell)$ (the details are completely analogous to that given in [7] Theorem 5.4). This completes the proof. ∎

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4**. An extension of a theorem of Bass.** We now discuss some applications of Theorem 3.1. We begin with a discussion of Bass's theorem ([1] and [2]). We will work in SL(2) rather than GL(2), and in this setting Bass's theorem is:

THEOREM 4.1. *Let* Γ *be a finitely generated subgroup of* $\mathrm{SL}(2,\mathbf{C})$*. Then one of the following cases occurs:*

1. *There is an epimorphism* $f\colon\Gamma\longrightarrow\mathbf{Z}$ *such that* $f(u)=0$ *for all unipotent elements* $u\in\Gamma$*.*
2. Γ *is a non-trivial free product with amalgamation.*
3. Γ *is conjugate to a group of upper triangular matrices* $\begin{pmatrix} a & * \\ 0 & 1/a\end{pmatrix}$ *with a a root of unity.*
4. Γ *is conjugate to subgroup of* $\mathrm{SL}(2,\mathbf{A})$ *where* $\mathbf{A}$ *is a ring of algebraic integers.* ∎

Using Theorem 3.1 we are able to show (1) of Theorem 4.1 is removed if we further assume that Γ does not contain a free subgroup of finite index. As remarked in the Section 1, having a free subgroup of finite index is equivalent to being a graph of groups where all vertex groups are finite. Denote by $\mathcal{H}$ the class of such groups with a unique vertex.

THEOREM 4.2. *Let* Γ *be a finitely generated non-elementary subgroup of* $\mathrm{SL}(2,\mathbf{C})$ *whose traces consist of algebraic numbers and which contains an element whose trace is not an algebraic integer. Suppose in addition that* $\Gamma\notin\mathcal{H}$*.*

Then Γ *splits as a non-trivial free product with amalgamation.*

The proof requires the following result in [9] (Theorem 3 on p. 79).

THEOREM 4.3. *Let k be a finite extension of* $\mathbf{Q}_p$ *and G a subgroup of* $\mathrm{SL}(2,k)$*. If G is dense in* $\mathrm{SL}(2,k)$ *then G splits as a non-trivial free product with amalgamation.* ∎

PROOF OF THEOREM 4.2. Firstly if Γ is virtually free, then since it is not in $\mathcal{H}$ there will be a non-trivial free product with amalgamation decomposition. Thus we now assume that Γ is not virtually free. The non-elementary assumption implies that Γ cannot be conjugate to a group of upper triangular matrices, and the existence of a trace which is not an algebraic integer implies that Γ is not conjugate into $\mathrm{SL}(2,\mathbf{A})$.

Since Γ is assumed to have algebraic traces, we can conjugate so that entries of elements of Γ are algebraic. Let k be the field generated over $\mathbf{Q}$ by the coefficients of matrices in Γ. Since Γ is finitely generated, k is a finitely generated extension algebraic extension of $\mathbf{Q}$. In particular k is a finite extension of $\mathbf{Q}$. Using the existence of an element g whose trace is not algebraic integer, we choose a valuation ν on k such that $\nu\big(\mathrm{tr}(g)\big)<0$. Denote by k_ν the completion of k using the ν-adic metric. By the classification theorem of local fields this is a finite extension of $\mathbf{Q}_p$ for some prime p. Completion induces a faithful representation i of Γ into $\mathrm{SL}(2,k_\nu)$. Let Γ_ν denote the closure of $i(\Gamma)$ in $\mathrm{SL}(2,k_\nu)$.

Now we claim that $i(\Gamma)$ is not discrete. For if it were, then by passing to a torsion-free subgroup of finite index, we may apply Ihara's theorem (see [6] and [9] Chapter II, p. 82–83) and deduce that $i(\Gamma)$ is virtually free, contradicting our assumption.

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As a closed subgroup of a p-adic Lie group Γ_ν is a p-adic Lie group (see [10], p. 155). The existence of the element g implies that Γ_ν is noncompact. Exactly as in [7] Lemma 5.2, the nonelementary assumption, together with the fact that Γ_ν is non-discrete implies that Γ_ν is nonsolvable.

By Theorem 3.1 we can conjugate Γ_ν (over a finite extension of k_ν) so that $\Gamma_\nu = \mathrm{SL}(2,\ell)$ for some subfield ℓ. Summarizing we obtain a faithful representation of Γ into $\mathrm{SL}(2,\ell)$ which is dense in $\mathrm{SL}(2,\ell)$. By Theorem 4.3 it follows that Γ splits as a non-trivial free product with amalgamation. ∎

By the results of [3], we may deduce that Γ (as in Theorem 4.2) admits a non-trivial free product with amalgamation decomposition where the vertex and edge stabilizers are finitely generated.

A special case of Theorem 4.2 is:

THEOREM 4.4. *Let Γ be a finitely generated non-elementary subgroup of* $\mathrm{SL}(2,\mathbf{C})$ *whose traces consist of algebraic numbers and which contains an element whose trace is not an algebraic integer. Suppose in addition that Γ is torsion-free.*

Then Γ splits as a non-trivial free product with amalgamation. ∎

As a final remark we observe that the condition about "virtual freeness" in 4.2 is used to guarantee that the image of Γ under the inclusion map into $\mathrm{SL}(2,k)$ is not discrete. The following theorem replaces this assumption.

THEOREM 4.5. *Let Γ be a finitely generated non-elementary subgroup of* $\mathrm{SL}(2,\mathbf{C})$ *whose traces consist of algebraic numbers and which contains an element whose trace is not an algebraic integer. Assume further that there is an element x of infinite order whose trace is an algebraic integer. Then Γ splits as a non-trivial free product with amalgamation.*

PROOF. The proof follows the arguments above, the only point to check is that (in the notation of the proof of Theorem 4.2) $i(\Gamma)$ is not discrete in $\mathrm{SL}(2,k_\nu)$, for then the rest of the argument follows directly as in the proof of Theorem 4.2).

Thus assume that $i(\Gamma)$ is discrete. By conjugating $i(\Gamma)$ in $\mathrm{GL}(2,k_\nu)$ if necessary we can assume that the "integral" element x given by hypothesis lies in $\mathrm{SL}(2,R_\nu)$, where R_ν is the ring of ν-adic integers in k. But then all powers of x will lie in $\mathrm{SL}(2,R_\nu)$. However, $\mathrm{SL}(2,R_\nu)$ is compact and if $i(\Gamma)$ is discrete, then we must have $i(\Gamma)\cap\mathrm{SL}(2,R_\nu)$ is finite, and this contradicts x being of infinite order. ∎

5. **Applications.** We now give some specific applications of Theorem 4.2 to 3-manifold groups. The results here are already known but this offers a different proof.

We begin with some lemmas. The first is well-known so we omit the proof.

LEMMA 5.1. *Let V be an algebraic set defined over $\mathbf{Q}$ which has dimension* 0. *Then V consists of a finite collection of points, all of whose coordinates are algebraic numbers.*

∎

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Recall if G is a finitely generated group we denote by $\mathrm{Hom}\big(G, \mathrm{SL}(2, \mathbf{C})\big)$ the set of homomorphisms of G into $\mathrm{SL}(2, \mathbf{C})$. It is a standard fact that $\mathrm{Hom}\big(G, \mathrm{SL}(2, \mathbf{C})\big)$ has the structure of an affine algebraic set defined over $\mathbf{Q}$.

THEOREM 5.2. *Let* $G \notin H$ *be a finitely generated group for which* $\mathrm{Hom}\Big(G, \mathrm{SL}\big(2, (\mathbf{C})\big)\Big)$ *has a component V which consists of faithful, irreducible representations. Suppose further that V has dimension at least* 4.

Then G splits as a non-trivial free product with amalgamation.

PROOF. That the dimension of V is at least 4 means that V contains more than one conjugacy class of representation. Suppose that we add extra $\mathbf{Z}$ polynomials to obtain an algebraic subset of V containing a component of dimension zero. It follows from Lemma 5.1 that the coordinates of this point ρ are algebraic. We wish to arrange that there is an element α of G such that $\mathrm{tr}(\rho\alpha)$ is not an algebraic integer. For then we can apply Theorem 4.2 to obtain a splitting as a free product with amalgamation for $\rho(G)$ and hence G.

Given α in G the function $f_\alpha \colon V \longrightarrow \mathbf{C}$ given by $f_\alpha(\rho) = \mathrm{tr}(\rho\alpha)$ is polynomial. If for every α this function is constant then every pair of representations in V have the same character as a given irreducible representation, and therefore are all conjugate. But this contradicts the dimension of V being at least 4.

Choose α for which this function is not constant and choose an algebraic non-integer z in the image of f_α. This is possible since f_α dominates $\mathbf{C}$, so that the image can omit only finitely many values. Setting V_1 to be the preimage $f_\alpha^{-1}(z)$, we see that this is an integrally defined subalgebraic set of V consisting of faithful irreducible representations. If V_1 contains a pair of nonconjugate representations, then we repeat this argument.

The argument terminates in an integrally defined subalgebraic set V' consisting of one conjugacy class of representation, which therefore contains a representation of the form:

$$\rho'(\alpha) = \begin{pmatrix} 0 & 1 \\ -1 & a \end{pmatrix} \quad \rho'(\beta) = \begin{pmatrix} b & c \\ -c^{-1} & 0 \end{pmatrix}.$$

We may add extra $\mathbf{Z}$ polynomials to produce a subset of V' containing only this representation. This completes the proof. ∎

In the same vein we have:

THEOREM 5.3. *Let G be a finitely generated group for which* $\mathrm{Hom}\Big(G, \mathrm{SL}\big(2, (\mathbf{C})\big)\Big)$ *has a component V of dimension at least* 5.

Then G splits as a non-trivial free product with amalgamation.

PROOF. As above V consists of more than one conjugacy class of irreducible representation. We subdivide into two cases:

Suppose first that there is some element of the commutator subgroup α for which the function f_α is nonconstant. Then as above we find a subalgebraic set V_1 of V where $\mathrm{tr}\big(\rho(\alpha)\big) = z$ for every $\rho \in V_1$ where z is some algebraic noninteger. Since reducible

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representations take the value 2 on any element of the commutator subgroup (*cf.* [4]), we see that V_1 contains only irreducible representations, moreover, it has dimension at least 4. It follows that there is at least one further element β whose trace varies on V_1, and we may arrange that the trace of this element is some algebraic integer, not of the form $\omega + \bar{\omega}$ where ω is any root of unity. This guarantees that the representations in this subset map β into $\mathrm{SL}(2, \mathbf{C})$ as an element of infinite order.

We now proceed as above cutting down dimensions until we obtain a single algebraic representation which satisfies the hypothesis of Theorem 4.5, to deduce the requisite splitting for the image group, whence the original group.

The second case is only marginally different; suppose that every character of an element of the commutator subgroup is constant. Notice that there must be at least one such element whose trace is not 2, else all the representations on V are reducible. Whence consideration of this character shows that none of the representations on V are reducible.

Now we argue as in the first paragraph, choosing some nonconstant character and pulling back some nonalgebraic integer value. The irreduciblity of representations in V_1 has now already been guaranteed and we choose the second nonconstant character as above. ■

As a corollary of 5.2 we have the following result first proved by Shalen [12] when no boundary component was a torus, and extended in [4] to allow tori. Both these results require the hypothesis that $H_1(\partial M; \mathbf{Q})$ surjects onto $H_1(M; \mathbf{Q})$.

COROLLARY 5.4. *Let M be a compact orientable irreducible* 3*-manifold with incompressible boundary for which every incompressible torus is boundary parallel. Assume ∂M does not consist entirely of tori. Then $\pi_1(M)$ admits a splitting as a non-trivial free product with amalgamation.*

PROOF. By Thurston's hyperbolization theorem for Haken manifolds, the interior of M admits a hyperbolic structure. Since there is at least one boundary component of genus at least 2, Teichmüller theory dictates that the subset of $\mathrm{Hom}\big(\pi_1(M), \mathrm{SL}(2, \mathbf{C})\big)$ consisting of holonomy representations of complete hyperbolic structures on M is at least 4. The corollary now follows from Theorem 5.2. ■

As is standard in 3-manifold topology, the existence of a splitting as a free product with amalgamation of $\pi_1(M)$ determines an incompressible surface in M. However, in general, one cannot deduce that there is a *separating* incompressible surface. For instance if one takes a Seifert fibered space M over a torus with a single cone point with cone angle $2\pi/n$, the results of [7] imply that $\pi_1(M)$ splits as a non-trivial free product with amalgamation. However since any incompressible surface in M is horizontal or vertical it is easy to see in this case that an incompressible surface must be non-separating. The hypothesis mentioned above in [12], [4] guarantee a separating surface.

Omitted from Corollary 5.4 was the case where all boundary components are tori. One can get a similar result if there is component V of $\mathrm{Hom}\big(\pi_1(M), \mathrm{SL}(2, \mathbf{C})\big)$ of large dimension (*cf.* Theorem 5.3). We simply do the following case, first proved in [4].

COROLLARY 5.5. *Let $K \subset S^3$ be a knot whose complement admits a complete hyperbolic structure of finite volume. Then $S^3 \setminus \eta(K)$ contains a separating incompressible surface, where $\eta(K)$ denotes an open tubular neighbourhood of K.*

PROOF. We first fix some notation. Let L be a longitude for K and let $\Gamma = \pi_1(S^3 \setminus K)$. From [13], the component V of $\mathrm{Hom}\big(\Gamma, \mathrm{SL}(2, \mathbf{C})\big)$ containing the faithful discrete representation has dimension 4. Furthermore, since $\mathrm{tr}\big(\rho(L)\big)$ is known to vary on V, the method of proof of Theorem 5.2 implies we can find a representation $\rho \in V$ for which $\rho(\Gamma)$ has traces which are algebraic numbers, and for which $\mathrm{tr}\big(\rho(L)\big)$ is not an algebraic integer.

The representation ρ need not be faithful, so we must deal with the possibility that $\rho(\Gamma) \in H$. Assume that this is the case. Then since Γ admits a unique map to $\mathbf{Z}$ it follows that the graph G in question is a single loop with a unique vertex. So that there is a map $\rho(\Gamma) \longrightarrow \mathbf{Z}$. The group Γ admits only one map to the integers, and this map kills the longitude. However, the composition

$$\Gamma \longrightarrow \rho(\Gamma) \longrightarrow \mathbf{Z}$$

does not kill the longitude, a contradiction.

We deduce that, by Theorem 4.2, $\rho(\Gamma)$, and therefore Γ, splits as a non-trivial free product with amalgamation. The only issue remaining is to ensure that after compression the surface separates. To see this note that as above, the splitting constructed ensures that L does not map into an edge stabilizer under the action on the tree of SL(2). ∎

It seems harder to guarantee that the surface produced above has non-empty boundary, as is done in [4].

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