ON A CLASS OF OPERATORS OCCURRING IN THE THEORY OF CHAINS OF INFINITE ORDER

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Introduction. Let T, E be two sets and $\mathfrak{T} \subset \mathfrak{P}(T)$,[1] $\mathfrak{E} \subset \mathfrak{P}(E)$ two tribes. For every $n \in N^*$ denote by E^n the product $E^{\{1,\ldots,n\}}$ and by $\mathfrak{E}^n$ the tribe $\mathfrak{E}^{\{1,\ldots,n\}}$. For every $x \in E$ let u_x be a mapping of T into T. For $x = (x_1, \ldots, x_n) \in E^n$ define $u_x = u_{x_n} \circ \ldots \circ u_{x_1}$ and suppose that $\{(t, x_1, \ldots, x_n) | u_{(x_1,\ldots,x_n)}(t) \in A\} \in \mathfrak{T} \oplus \mathfrak{E}^n$ for all $n \in N^*$ and $A \in \mathfrak{T}$.

Let $\mathfrak{M}$ be the Banach space of functions defined on T, real-valued, bounded and $\mathfrak{T}$-measurable with norm

$$\|f\| = \sup_{t \in T} |f(t)|.$$

For any sequence $S = (a_n)_{n \in N^*}$ of positive numbers, denote by $\mathfrak{M}_S$ the part of $\mathfrak{M}$ consisting of the functions f satisfying the inequality $|f(u_x(t_1)) - f(u_x(t_2))| \leqslant a_n$ for every $n \in N^*$, $x \in E^n$ and $t_1, t_2 \in T$.

Let p be a real-valued function defined on $T \times \mathfrak{E}$ having the properties:

(1) $0 \leqslant p(t, A) \leqslant p(t, E) = 1$ for $(t, A) \in T \times \mathfrak{E}$;

(2) $A \to p(t, A)$ is a completely additive measure for every $t \in T$;

(3) $t \to p(t, A)$ belongs, for every $A \in \mathfrak{E}$, to the same set $\mathfrak{M}_S$, where $S = (a_n)_{n \in N^*}$ is such that

$$\sum_{n \in N^*} a_n < \infty.$$

Define on $\mathfrak{M}$ the operator U by the equality

$$Uf(t) = \int_E p(t, dx) f(u_x(t)).$$

U is a linear operator of norm one which maps $\mathfrak{M}$ into $\mathfrak{M}$. Operators such as U occur in the study of certain stochastic models, especially in the theory of chains of infinite order **(1–4; 6–10; 12; 14; 15)**. In this paper, under supplementary hypotheses, two ergodic properties of the sequence $(U^n)_{n \in N}$ will be proved. Under restrictive conditions it will be shown that the functions $t \to p(t, A)$ are conditional probabilities of a stationary mixing stochastic process **(8**, Theorem 6**)**. Two other results, a non-homogeneous ergodic theorem and a central limit theorem, will also be given.

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[1]Some of the notations used in this paper are explained in paragraph 9 at the end of the paper.

1. For every $n \in N$ let $p_{1,n}$ be the function defined on $T \times \mathfrak{E}^n$ by

$$p_{1,n}(t, A) = \int_E p(t, dx_1) \int_E \cdots \int_E p(u_{(x_1,\ldots,x_{n-1})}(t), dx_n)\, \phi_A(x_1, \ldots, x_n), \qquad n > 1;$$

$$p_{1,n} = p, \qquad n = 1.$$

For any bounded sequence $C = (c_n)_{n \in N^*}$ write $\tilde{C} = (\tilde{c}_n)_{n\in N^*}$ where

$$\tilde{c}_n = 4 \sum_{j \geqslant n} a_j + \sup_{j \geqslant n} c_j, \qquad n \in N^*.$$

The following three results will be needed below:

(i) for every $n \in N^*$, $p_{1,n}$ has the properties (1) to (3) if we replace $\mathfrak{E}$ by $\mathfrak{E}^n$ and $S = (a_k)_{k\in N^*}$ by

$$\left(\sum_{k \leqslant j < k+n} a_j \right)_{k \in N^*};$$

(ii) for every $n \in N^*$, $m \in N^*$ and $f \in \mathfrak{M}$, $t \in T$,

$$U^{n+m}f(t) = \int_{E^n} p_{1,n}(t, dx) U^m f(u_x(t));$$

(iii) if $f \in \mathfrak{M}_C$ where $C = (c_k)_{k \in N^*}$ then, for every $n \in N^*$, $U^n f \in \mathfrak{M}_{\|f\| + \tilde{C}}$.

2. Let us say that p satisfies condition (K) if there is on $\mathfrak{E}$ a completely additive measure μ, with value one on the whole space, and a constant $\lambda > 0$ such that $p(t, A) \geqslant \lambda \mu(A)$ for every $(t, A) \in T \times \mathfrak{E}$.

THEOREM 1. *If p satisfies condition (K) and $f \in \mathfrak{M}_C$ where $C = (c_n)_{n \in N^*}$ has the property*

$$\lim_{n \to \infty} c_n = 0$$

then there is a constant function $U^\infty f$ and a constant $0 < h = h_C < 1$ satisfying for every $n \in N^*$ the inequality

$$(4) \qquad \|U^n f - U^\infty f\| \leqslant \|f\|^+ \inf_{1 \leqslant s \leqslant n} (\tilde{c}_s/(1-h) + 2h^{(n/s)-1}).$$

Choose an $r \in N^*$ such that

$$\sum_{j \geqslant r} a_j \leqslant \tfrac{1}{8}$$

and for every $n \in N^*$ let $\mu_n = \mu^1 \oplus \ldots \oplus \mu^n$ where $\mu^1 = \ldots = \mu^n = \mu$. For any $n \in N^*$ and $(t, A) \in T \times \mathfrak{E}^n$ let $\mu_n(t, A) = \mu_n(A)$ if $n \leqslant r$ and

$$\mu_n(t, A) = \int_{E^r} \mu_r(dx) \int_{E^{n-r}} p_{1,n-r}(u_{(x_1,\ldots,x_r)}(t), d(x_{r+1}, \ldots, x_n))\, \phi_A(x_1, \ldots, x_n)$$

if $n > r$. From the choice of r and property (i) it follows that $|\mu_n(t_1, A) - \mu_n(t_2, A)| \leqslant \frac{1}{4}$ for any $n \in N^*$, $t_1, t_2 \in T$ and $A \in \mathfrak{E}^n$. Using this inequality and condition (K) we obtain

$$(5) \qquad p_{1,n}(t_1, A) \geqslant \lambda^r \mu_n(t_1, A) \geqslant \lambda^r \mu_n(t_2, A) - \tfrac{1}{4}\lambda^r.$$

For every $n \in N^*$, $t_1, t_2 \in T$ and $A \in \mathfrak{F}^n$, write $q_n(t_1, t_2; A) = p_{1,n}(t_1, A) - p_{1,n}(t_2, A)$. Let P, Q be two disjoint $\mathfrak{F}^n$-measurable sets whose union is E^n, such that $q_n(t_1, t_2; A) \geqslant 0$ if $A \subset P$, and $q_n(t_1, t_2; A) \leqslant 0$ if $A \subset Q$; we have then

$$B = q_n(t_1, t_2; P) = q_n(t_2, t_1; Q)$$

because $q_n(t_1, t_2; E^n) = 0$ (P, Q, and B depend on $n \in N^*$ and $t_1, t_2 \in T$). Using the inequality (5) we obtain

$$(6) \qquad B \leqslant \inf\,(1 - p_{1,n}(t_2, P), 1 - p_{1,n}(t_1, Q)) \leqslant h = 1 - \tfrac{1}{4}\lambda^r.$$

Let us write the difference $U^n f(t_1) - U^n f(t_2)$ in the form

$$(7) \int_{E^s} q_s(t_1, t_2; dx) U^{n-s} f(u_x(t_1)) + \int_{E^s} p_{1,s}(t_2, dx)(U^{n-s} f(u_x(t_1)) - U^{n-s} f(u_x(t_2)))$$

where $1 \leqslant s \leqslant n$. The second term in the sum (7) is less than or equal to $||f||^+ \tilde{c}_s$. If $B \neq 0$, the first term in the sum (7) can be written as

$$B\left(\int_P (q_s(t_1, t_2; dx)/B) U^{n-s} f(u_x(t_1)) - \int_Q (q_s(t_2, t_1; dx)/B) U^{n-s} f(u_x(t_1))\right)$$

and it follows from (6) that it is less than or equal to $h(\bar{f}^{n-s} - \underline{f}^{n-s})$. This inequality is obviously true if $B = 0$. Here, for every $k \in N$,

$$\bar{f}^k = \sup_{t \in T} U^k f(t), \ \underline{f}^k = \inf_{t \in T} U^k f(t).$$

We obtain $\bar{f}^n - \underline{f}^n \leqslant ||f||^+ \tilde{c}_s + h(\bar{f}^{n-s} - \underline{f}^{n-s})$. Hence for every integer $p \geqslant 1$ such that $ps \leqslant n$,

$$(8) \qquad \bar{f}^n - \underline{f}^n \leqslant ||f||^+(1 + h + \ldots + h^{p-1})\tilde{c}_s + h^p(\bar{f}^{n-ps} - \underline{f}^{n-ps}).$$

If we remark that the sequence $(\bar{f}^n)_{n \in N}$ is decreasing and that the sequence $(\underline{f}^n)_{n \in N}$ is increasing, then the existence of $U^\infty f$ and the inequality (4) follows from (8).

Remarks. 1° Denote by $\mathfrak{M}_1$ the union of the sets $\mathfrak{M}_C$ where $C = (c_n)_{n \in N^*}$ has the property

$$\lim_{n \to \infty} c_n = 0.$$

$\mathfrak{M}_1$ is a linear space and U^∞ is a linear form on $\mathfrak{M}_1$.

2° For every $n \in N^*$, $k \in N^*$ let us define the function $p^k{}_{1,n}$ on $T \times \mathfrak{F}^n$ by the equalities: $p^k{}_{1,n} = p_{1,n}$ if $k = 1$, and

$$p^k_{1,n}(t, A) = \int_E p(t, dx) p^{k-1}_{1,n}(u_x(t), A), \qquad (t, A) \in T \times \mathfrak{F}^n,$$

if $k > 1$. If we write $E^0 \times A = A$, then for every n, $k \in N^*$, $t \in T$ and $A \in \mathfrak{F}^n$, $p^k{}_{1,n}(t, A) = p_{1,n+k-1}(t, E^{k-1} \times A)$.

We deduce from Theorem 1 that for every $n \in N^*$ there is on $\mathfrak{E}^n$ a measure $p^{\infty}{}_{1,n}$, completely additive and with value one on the whole space, such that $(k \geqslant 1)$

$$|p_n^{k+1}(t, A) - p_{1,n}^{\infty}(A)| \leqslant \inf_{1 \leqslant s \leqslant k} \left(6 \sum_{j > s} \frac{a_j}{1-h} + 2h^{(k/s)-1} \right). \tag{9}$$

3° If for any $n \in N^*$, $a_n = a^n$, $c_n = c^n$ where $0 < a, c < 1$, then the second member in the inequality (4) is dominated by $\| f \|^+ A_1 \exp(-q\sqrt{n})$ where $A_1 = A_1(a, c, \lambda)$ and $q = q(a, c, \lambda) > 0$.

3. Let us suppose in this paragraph that E is a finite set, and that for every $n \in N^*$ and $t \in T$ there is $x \in E^n$ and $t_n \in T$, such that $u_x(t_n) = t$. Under these hypotheses we can prove:

THEOREM 2. *For every $f \in \mathfrak{M}_1$ there is a function $U^{\infty}f \in \mathfrak{M}_1$ which satisfies the equality*

$$\lim_{n \to \infty} \left\| \frac{1}{n} \sum_{j=1}^{n} U^j f - U^{\infty} f \right\| = 0. \tag{10}$$

Let $t_0 \in T$ and denote by T_0 the set $\{u_x(t_0) \mid x \in E^n, n \in N^*\}$. As T_0 is denumerable, there is a strictly increasing subsequence of N^*, $(n_j)_{j \in N^*}$ such that the sequence

$$\left(\frac{1}{n_j} \sum_{i=1}^{n_j} U^i f(t) \right)_{1 \leqslant j < \infty}$$

is convergent for every $t \in T_0$. Using property (iii) we deduce that there exists a function $U^{\infty}f \in \mathfrak{M}_1$ satisfying the equality

$$\lim_{j \to \infty} \left\| \frac{1}{n_j} \sum_{i=1}^{n_j} U^i f - U^{\infty} f \right\| = 0.$$

As $\|U^j\| = 1$ for every $j \in N$, the mean ergodic theorem of Yosida and Kakutani **(16)** implies (10).

U^{∞} can be extended uniquely to the closure of $\mathfrak{M}_1$ in $\mathfrak{M}$, $\bar{\mathfrak{M}}_1$, and (10) remains valid for $f \in \bar{\mathfrak{M}}_1$.

4. For any $n \in Z$ let p_n be a real-valued function defined on $T \times \mathfrak{E}$, having the properties:

(11) $0 \leqslant p_n(t, A) \leqslant p_n(t, E) = 1$ for $(t, A) \in T \times \mathfrak{E}$;

(12) $A \to p_n(t, A)$ is a completely additive measure for every $t \in T$;

(13) $t \to p_n(t, A)$ belongs, for every $n \in Z$ and $A \in \mathfrak{E}$, to the same set $\mathfrak{M}_S$ where $S = (a_k)_{k \in N^*}$ is such that

$$\sum_{n \in N^*} a_n < \infty.$$

For every $n \in Z$ define on $\mathfrak{M}$ the operator $U^{n-1,n}$ by the equality

$$U^{n-1,n} f(t) = \int_E p_n(t, dx) f(u_x(t)).$$

For $(n, m) \in \mathfrak{Z} = \{(n, m) | n \in Z, m \in Z, n \leqslant m\}$ write $U^{n,m} = U^{n,n+1} \circ \ldots \circ U^{m-1,m}$ if $n < m$ and $U^{n,m} = I$ if $n = m$. Let us say that the family $(p_n)_{n \in Z}$ satisfies condition (K) if there is on $\mathfrak{E}$ a family $(\mu_n)_{n \in Z}$ of completely additive measures, having value one on the whole space, and a constant $\lambda > 0$ such that $p_n(t, A) \geqslant \lambda \mu_n(A)$ for every $(t, A) \in T \times \mathfrak{E}$ and $n \in Z$. By an argument similar to the one used in the proof of Theorem 1, but somewhat more involved, we can obtain:

THEOREM 3. *If the family $(p_n)_{n \in Z}$ satisfies condition (K) and $f \in M_C$, then there is a constant $0 < h = h_C < 1$ satisfying for every $(n, m) \in \mathfrak{Z}$, $n < m$, and t_1, t_2 the inequality*

$$(14) \quad |U^{n,m}f(t_1) - U^{n,m}f(t_2)| \leqslant \|f\|^+ \inf_{1 \leqslant s \leqslant m-n} (\tilde{c}_s/(1-h) + h^{((m-n)/s)-1}).$$

Here C does not necessarily satisfy any supplementary condition.

If the sequence $C = (c_n)_{n \in N^*}$ is such that

$$\lim_{n \to \infty} c_n = 0$$

then it follows from (14) that

$$\lim_{m-n \to \infty} (U^{n,m}f(t_1) - U^{n,m}f(t_2)) = 0$$

uniformly with respect to $t_1, t_2 \in T$ and f in a given bounded part of $\mathfrak{M}_C$.

5. Suppose now that:

(15) E is metric complete and separable and $\mathfrak{E}$ is the tribe of Borel parts of E;

(16) $T = E^{-N}$ and $\mathfrak{T} = \mathfrak{E}^{-N}$ where $-N = \{\ldots, -1, 0\}$;

(17) $u_x, x \in E$, is defined on T by: $u_x((\ldots, x_{-1}, x_0)) = (\ldots, x_0, x)$.

For every $(n, m) \in \mathfrak{Z}$ define the function $p^{\infty}{}_{n,m}$ on $\mathfrak{E}^{\{n,\ldots,m\}}$ by the equality (we identify $\mathfrak{E}^{m-n+1}$ with $\mathfrak{E}^{\{n,\ldots,m\}}$): $p^{\infty}{}_{n,m}(A) = p^{\infty}{}_{1,m-n+1}(A)$, and for every $(n, m) \in \mathfrak{Z}$ and $r \in N^*$ define the function $p^r{}_{n,m}$ on $T \times \mathfrak{E}^{\{n,\ldots,m\}}$ by: $p_{n,m}{}^r(t, A) = p_{1,m-n+1}{}^r(t, A)$.

THEOREM 4. *If p satisfies condition (K), then there is one and only one stochastic process $(E^Z, \mathfrak{E}^Z, p^Z)$ such that the equality*

$$(18) \quad p^Z\{\overset{-1}{pr}_{n+1}(A) | pr_{\{\ldots,n\}}(\omega) = t\} = p(t, A)$$

is satisfied almost everywhere for any $n \in Z$ and $A \in \mathfrak{E}$. The stochastic process $(E^Z, \mathfrak{E}^Z, p^Z)$ is stationary and strongly mixing.

Let us remark that if $(n, m), (n', m') \in \mathfrak{Z}$ and

$$\overset{-1}{pr}_{\{n',\ldots,m'\}}(B) = \overset{-1}{pr}_{\{n,\ldots,m\}}(A) \in \mathfrak{E}^Z$$

then

$$p^{\infty}_{n,m}(A) = p^{\infty}_{n',m'}(B).$$

Hence there is one and only one stochastic process $(E^Z, \mathfrak{E}^Z, p^Z)$ such that $p^Z(pr^{-1}_{\{n,\ldots,m\}}(A)) = p^{\infty}_{n,m}(A)$ for $(n, m) \in \mathfrak{Z}$ and $pr^{-1}_{\{n,\ldots m\}}(A) \in \mathfrak{E}^Z$. Hypothesis (15) is used here only. We leave to the reader to verify that the process is stationary.

For every $(s, m) \in \mathfrak{Z}$ let us define the function $\tilde{p}_{s,m}$ on $E^{\{s,\ldots,m\}} \times \mathfrak{E}$ by the equality $\tilde{p}_{s,m}(x, A) = p(u_x(t_0), A)$ where $t_0 \in T$ is a fixed element. For $n \leqslant s \leqslant m$, $A \in \mathfrak{E}$ and $M \in \mathfrak{E}^{\{n,\ldots,m\}}$ we have then

$$\int p^Z(d\omega)p(pr_{\{\ldots,m\}}(\omega), A) = \theta_1 a_{m-s+1} + \int p^Z(d\omega)\tilde{p}_{s,m}(pr_{\{s,\ldots,m\}}(\omega), A)$$

$$= \theta_1 a_{m-s+1} + \int_M p_{n,m}(dx)\tilde{p}_{s,m}(x_{s,m}, A) = \theta_1 a_{m-s+1} + \lim_{r\to\infty}\int_M p^{r+1}_{n,m}(t, dx)\tilde{p}_{s,m}(x_{s,m}, A)$$

where the first two integrals are taken over $pr^{-1}_{\{n,\ldots,m\}}(M)$, $|\theta_1| \leqslant 1$, and $x_{s,m} = (x_s, \ldots, x_m)$ if $x = (x_{n'}, \ldots, x_m)$ and $n' \leqslant s \leqslant m$. For any $r \in Z$

$$\int_M p^{r+1}_{n,m}(t, dx)\tilde{p}_{s,m}(x_{s,m}, A) = \int_{M(r)} p_{n-r,m}(t, dx)\tilde{p}_{s,m}(x_{s,m}, A)$$

$$= \theta_2 a_{m-s+1} + \int_{M(r)} p_{n-r,m}(t, dx)p(u_x(t), A) = \theta_2 a_{m-s+1} + p^{r+1}_{n,m+1}(t, M \times A)$$

where $M(r) = E^{\{n-r,\ldots,n-1\}} \times M$, and $|\theta_2| \leqslant 1$. It follows that

$$\int p^Z(d\omega)p(pr_{\{\ldots,m\}}(\omega), A) = \theta_3 a_{m-s+1} + \lim_{r\to\infty} p^r_{n,m+1}(t, M \times A)$$

$$= \theta_3 a_{m-s+1} + p^{\infty}_{n,m+1}(M \times A) = \theta_3 a_{m-s+1} + p^Z(pr^{-1}_{\{n,\ldots,m\}}(M) \cap pr^{-1}_{m+1}(A)),$$

the integral being taken over $pr^{-1}_{\{n,\ldots,m\}}(M)$, and $|\theta_3| \leqslant 2$. But

$$pr^{-1}_{\{n,\ldots,m\}}(M) = pr^{-1}_{\{n',\ldots,m\}}(E^{\{n',\ldots,n-1\}} \times M) \qquad \text{if } n' < n;$$

hence s can be allowed to tend to $-\infty$ in the above formula. It follows that (18) is satisfied almost everywhere.

Suppose now that $(E^Z, \mathfrak{E}^Z, \bar{p})$ is a stochastic process such that the equality $\bar{p}\{pr^{-1}_{n+1}(A)|pr^{-1}_{\{\ldots,n\}}(\omega) = t\} = p(t, A)$ is satisfied almost everywhere for any $n \in Z$ and $A \in \mathfrak{E}$. Then for $(n, m) \in \mathfrak{Z}$, $r \geqslant 1$ and $A \in \mathfrak{E}^{\{n,\ldots,m\}}$ we have

$$\int_{E^Z} \bar{p}(d\omega)p_{n-r,m}(pr_{\{\ldots,n-r-1\}}(\omega), E^{\{n-r,\ldots,n-1\}} \times A) = \bar{p}(pr^{-1}_{\{n,\ldots,m\}}(A)).$$

If we let r tend to ∞ and use (9) we obtain $\bar{p}(pr^{-1}_{\{n,\ldots,m\}}(A)) = p^Z(pr^{-1}_{\{n,\ldots m\}}(A))$; therefore $\bar{p} = p^Z$.

It remains to prove that $(E^Z, \mathfrak{E}^Z, p^Z)$ is strongly mixing. For this it is sufficient to show that

$$\lim_{n\to\infty} p^Z(\tau^n(A) \cap B) = p^Z(A)p^Z(B)$$

for every $A = pr^{-1}_{\{v,\ldots,z\}}(A_1) \in \mathfrak{E}^Z$ and $B = pr^{-1}_{\{s,\ldots,t\}}(B_1) \in \mathfrak{E}^Z$. But if we remark that $\tau^n(A) = pr^{-1}_{\{v-n,\ldots,z-n\}}(A_1)$ we obtain that

$$\lim_{n\to\infty} p^Z(\tau^n(A) \cap B) = \lim_{n\to\infty}\int_{\tau^n(A)} p^Z(d\omega)p_{z-n+1,t}(pr_{\{\ldots,z-n\}}(\omega), E^{\{z-n+1,\ldots,s-1\}} \times B_1)$$

is equal to $p^Z(A)p^Z(B)$. Hence the process is strongly mixing and so the theorem is proved.

6. Let us suppose that the conditions under which Theorem 4 has been proved are satisfied and also that

$$\sum_{n>1} n\Big(\sum_{j>\sqrt{n}} a_j\Big)^{\frac{1}{2}} < \infty .$$

Let f be a function, real-valued, $\mathfrak{F}^r$-measurable, defined on E^r. For every $n \in N^*$ write $f_n = f o pr_{\{n,\ldots,n+r-1\}}$ (we identify $E^{\{n,\ldots,n+r-1\}}$ with E^r and $\mathfrak{F}^{\{n,\ldots,n+r-1\}}$ with $\mathfrak{F}^r$). We have then:

THEOREM 5. *Suppose* $E(f_1) = 0$ *and* $E(|f|^\alpha) < \infty$ *for an* $\alpha > 2$. *Then*:

(j) *the series*

$$D = E(f_1^2) + 2 \sum_{i \in N^*} E(f_1 f_{1+i})$$

converges absolutely and, for $n \to \infty$, $E((f_1 + \ldots + f_n)^2/n) = D + 0\,(1/n)$;

(jj) *if* $D \neq 0$ *we have uniformly in a*

$$(19)\quad \lim_{n\to\infty} p^Z\Big(\frac{(f_1 + \ldots + f_n)}{\sqrt{n}} < a\Big) = (1/(2\pi D)^{\frac{1}{2}}) \int_{-\infty}^{a} \exp(-\,t^2/2D)dt.$$

The expectations are calculated with respect to the measure p^Z. Once the existence of the stationary process $(E^Z, \mathfrak{F}^Z, p^Z)$ is established, the theorem can be obtained by the method used by Doob to prove the central limit theorem for Markoff process (**5**, 221–32). We shall not give details here.

7. The first explicit and systematic study of chains of infinite order was made in (**15**). The transition probabilities of the chains studied in (**15**, 6–11), as well as the transition probabilities of chains of type (A) introduced in (**4**) and of chains of type (B) introduced in (**2**), (**3**), and (**4**) satisfy conditions (1)–(3). It follows that the theorems A and D (**2**), the ergodic theorem proved in (**15**, 6–11), the formulas given in (**4**, 139) (the evaluations are slightly different from those given by formula (9)), and the theorem I_3, (**6**, 423–6) (in the case when $|\phi_i| < 1$ for every i) are particular cases of Theorem 1. The convergence property of the transition probabilities, established in Theorem II, (**4**, 137) is also a consequence of Theorem 1. For chains of type (A) some stronger results, expressed by formula (22), are valid. Under different conditions the C_1 convergence of the sequence $(p^r{}_{1,n})_{r\in N^*}$ has been proved in (**8**, Theorem 6,c). This result is not contained in, nor does it contain the one proved in Theorem 2. If E is a finite set, results similar to Theorem 4 are given in (**8**), under weaker conditions. Various kinds of central limit theorems, having points of contact with Theorem 5 have been given in (**2**; **3**; **7**; **14**).

8. Suppose now that T is a compact metric space, $\mathfrak{T}$ the tribe of Borel parts of T and p a real-valued function defined on $T \times \mathfrak{E}$ having the properties (1), (2) and:

$$|p(t_1, A) - p(t_2, A)| \leqslant Kd(t_1, t_2) \tag{20}$$

for every $A \in \mathfrak{E}$ and $t_1, t_2 \in T$. Suppose further that there is a constant $0 < r < 1$ such that

$$d(u_x(t_1), u_x(t_2)) \leqslant rd(t_1, t_2)$$

for any $x \in E$ and $t_1, t_2 \in T$. It follows then that p satisfies condition (3) if we take $a_n = Mr^n$, where $M = K \times$ diameter of T, for every $n \in N^*$.

Denote by $\mathfrak{CL}$ the Banach space of complex-valued functions defined on T satisfying the Lipschitz condition, the norm being given by $||f||_1 = ||f|| + m(f)$ where $||f|| = \sup_{t \in T}|f(t)|$ and

$$m(f) = \sup_{t_1 = t_2} \frac{|f(t_1) - f(t_2)|}{d(t_1, t_2)}.$$

We remark that the real and the imaginary part of every function $f \in \mathfrak{CL}$ belongs to $\mathfrak{M}_{m(f)S}$ where $S = (a_n)_{n \in N}$; in particular they belong to $\mathfrak{M}_1$. Define the operator U on $\mathfrak{CL}$ by

$$Uf(t) = \int_E p(t, dx) f(u_x(t)). \tag{21}$$

Then **(12; 13)** U maps CL into CL, U is quasi-compact, the sequence $(||U^n||_1)_{n \in N}$ is bounded and 1 is a characteristic value of U.

It follows from Theorem 1 that if p satisfies condition (K), then for every $f \in \mathfrak{CL}$ the sequence $(U^n f)_{n \in N}$ converges uniformly to a constant function $U^\infty f$. But this result implies that the only characteristic value of U of modulus one is 1 and that this characteristic value is simple. Using the properties of U mentioned above we deduce that there are two constants M and $\nu > 0$ satisfying the inequality

$$||U^n - U^\infty||_1 \leqslant \frac{M}{(1 + \nu)^n} \tag{22}$$

for every $n \in N^*$.

The operator U can be defined by formula (21) also for $f \in \mathfrak{C}$. For every $n \in N^*$, $||U^n|| = 1$. As $\mathfrak{CL}$ is dense in $\mathfrak{C}$, it follows that U^∞ can be extended uniquely to $\mathfrak{C}$, and that

$$\lim_{n \to \infty} ||U^n f - U^\infty f|| = 0 \qquad \text{for every } f \in \mathfrak{C}. \tag{23}$$

This proposition contains some results proved in **(9**, §6**)**.

Let us make one more remark. Suppose, in addition, that:

(α) E is a topological space and $\mathfrak{E}$ contains the open sets;

(β) the mapping $x \to u_x(t)$ is continuous for every $t \in T$;

(γ) for every open set $V \subset T$ there is $n(V) \in N^*$ and $x(V) \in E^n(V)$ such that $u_{x(V)}(t) \in V$ for every $t \in T$.

The conditions (α)–(γ) are satisfied in the case of chains of type (A) **(3; 12)**. If p satisfies condition (K): $p(t, A) \geqslant \lambda\mu(A)$ for every $(t, A) \in T \times \mathfrak{E}$ where $\lambda > 0$ and $\mu(A) > 0$ for every open set A, then U is strongly positive with respect to the cone $\{f \mid f \in \mathfrak{C}\mathfrak{R}, f \geqslant 0\}$. We can then obtain[2] (22), more directly, using a slight modification of Theorem 6.3, (a) and (c), 70–3, **(11)**.

9. We shall explain in this paragraph some of the notations used in the paper.

For each set X, $\mathfrak{P}(X)$ is the set of parts of X. $N = \{0, 1, \ldots\}$, $N^* = \{1, 2, \ldots\}$, $Z = \{\ldots, -1, 0, 1, \ldots\}$. A part $\mathfrak{T} \subset \mathfrak{P}(X)$ is a tribe if $\mathfrak{T} \ni X$, $\mathfrak{T} \ni X - A$ if $\mathfrak{T} \ni A$ and $\mathfrak{T} \ni \bigcup_{n \in N} A_n$ if $\mathfrak{T} \ni A_n$ for every $n \in N$.

For every $I \subset Z$ we denote by E^I the product

$$\prod_{j \in I} E_j$$

where $E_j = E$ for $j \in I$. By $\mathfrak{E}^I \subset \mathfrak{P}(E^I)$ we denote the smallest tribe containing the sets of the form

$$\prod_{j \in I} A_j$$

where $A_j \in \mathfrak{E}$ for $j \in I$.

For every real number α we write $\alpha^+ = \sup(\alpha, 1)$. If α is a real number and $\tilde{C} = (\tilde{c}_n)_{n \in N^*}$, then $\alpha\tilde{C} = (\alpha\tilde{c}_n)_{n \in N^*}$.

τ is the mapping of E^Z into E^Z defined by the equality: $\tau((x_n)_{n \in Z}) = (x_{n+1})_{n \in Z}$.

$\mathfrak{C}$ is the Banach space of continuous complex-valued functions defined on T with the norm $\|f\| = \sup_{t \in T} |f(t)|$.

[2]The details were given recently in the Functional Analysis Seminar.

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