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# HOMOTOPY INVARIANT RESULTS ON COMPLETE GAUGE SPACES

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A fixed point theorem and two homotopy invariant results are presented for generalised contractive maps defined on complete gauge spaces.

## 1. INTRODUCTION

In this paper we present a new fixed point result for generalised contractive multivalued maps defined on a complete gauge space. The ideas rely only on the notions of pseudometric and completeness, and our fixed point theorem extends results in [**2, 3, 6**]. Also we present two general continuation theorems for generalised contractive multivalued maps. Our first result extends a result in [**2, 6**] and is established via Zorn's lemma. The second result is based on an argument in the single valued case, see [**1, 5, 7, 8**]; recently in [**9**] it was observed that single valued ideas could be used to discuss multivalued maps in certain circumstances.

For the remainder of this section we present some notations which will be used in Section 2. Throughout this paper $X = \left(X,\ \{d_\alpha\}_{\alpha\in\Lambda}\right)$ (here $\Lambda$ is a directed set) will denote a gauge space endowed with a complete gauge structure $\{d_\alpha : \alpha \in \Lambda\}$ (see Dugundji [**4**, pp. 198, 308]). For $A \subseteq X$ and $x \in X$ fixed, by $\mathrm{dist}_\alpha(x, A)$, we mean $d_\alpha(x, A)$. For $r = \{r_\alpha\}_{\alpha\in\Lambda} \in (0,\infty)^\Lambda$ and $x \in X$, we define the pseudo–ball centred at $x$ of radius $r$ by

$$B(x,r) = \left\{y \in X : d_\alpha(x,y) \leqslant r_\alpha \ \text{ for all } \ \alpha \in \Lambda\right\}.$$

We denote by $D_\alpha$ the generalised Hausdorff pseudometric induced by $d_\alpha$; that is, for $Z, Y \subseteq X$,

$$D_\alpha(Z,Y) = \inf\Big\{\varepsilon > 0 : \forall\, x \in Z,\ \forall y \in Y,\ \exists x^\star \in Z,\ \exists y^\star \in Y \\ \text{such that } d_\alpha(x,y^\star) < \varepsilon,\ d_\alpha(x^\star,y) < \varepsilon\Big\}$$

with the convention that $\inf(\emptyset) = \infty$.

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## 2. Fixed point and homotopy invariant results

We begin with a new fixed point result for generalised contractive multimaps with closed values defined on a complete gauge space.

**Theorem 2.1.** *Let $X$ be a complete gauge space, $r \in (0,\infty)^\Lambda$, $x_0 \in X$ and $F : B(x_0, r) \to C(X)$; here $C(X)$ denotes the family of nonempty closed subsets of $X$. Suppose there exist constants $q = \{q_\alpha\}_{\alpha\in\Lambda} \in [0,1)^\Lambda$ such that for every $\alpha \in \Lambda$ and every $x$, $y \in B(x_0, r)$, we have*

$$D_\alpha(Fx, Fy) \leqslant q_\alpha \max\Big\{d_\alpha(x,y), \operatorname{dist}_\alpha(x, Fx), \operatorname{dist}_\alpha(y, Fy), \frac{1}{2}\big[\operatorname{dist}_\alpha(x, Fy) + \operatorname{dist}_\alpha(y, Fx)\big]\Big\}.$$

*In addition, assume the following two conditions hold:*

$$\text{(2.1)} \qquad \text{for each } \alpha \in \Lambda, \text{ we have } \operatorname{dist}_\alpha\big(x_0, F(x_0)\big) < (1-q_\alpha)\, r_\alpha$$

*and*

$$\text{(2.2)} \qquad \begin{cases} \text{for every } x \in B(x_0, r) \text{ and every } \varepsilon = \{\varepsilon_\alpha\}_{\alpha\in\Lambda} \in (0,\infty)^\Lambda \\ \text{there exists } y \in F(x) \text{ with } d_\alpha(x,y) \leqslant \operatorname{dist}_\alpha\big(x, F(x)\big) + \varepsilon_\alpha \\ \text{for every } \alpha \in \Lambda. \end{cases}$$

*Then $F$ has a fixed point (that is, there exists $x \in B(x_0, r)$ with $x \in F(x)$).*

Proof: From (2.1) and (2.2) we may choose a $x_1 \in F(x_0)$ with

$$d_\alpha(x_1, x_0) < (1-q_\alpha) r_\alpha \quad \text{for every } \alpha \in \Lambda.$$

Notice $x_1 \in B(x_0, r)$. Next for $\alpha \in \Lambda$, choose $\varepsilon_\alpha > 0$ so that

$$\text{(2.3)} \qquad q_\alpha\, d_\alpha(x_1, x_0) + \frac{\varepsilon_\alpha}{1-q_\alpha} < q_\alpha\,(1-q_\alpha) r_\alpha.$$

Then choose $x_2 \in F(x_1)$ so that for every $\alpha \in \Lambda$, we have

$$\begin{aligned} d_\alpha(x_1, x_2) &\leqslant \operatorname{dist}_\alpha\big(x_1, F(x_1)\big) + \varepsilon_\alpha \\ &\leqslant D_\alpha\big(F(x_0), F(x_1)\big) + \varepsilon_\alpha \\ &\leqslant q_\alpha \max\Big\{d_\alpha(x_0, x_1), \operatorname{dist}_\alpha(x_0, Fx_0), \operatorname{dist}_\alpha(x_1, Fx_1), \\ &\qquad \frac{1}{2}\big[\operatorname{dist}_\alpha(x_0, Fx_1) + \operatorname{dist}_\alpha(x_1, Fx_0)\big]\Big\} + \varepsilon_\alpha. \end{aligned}$$

If the maximum in the brackets on the right hand side is

$$\frac{1}{2}\big[\operatorname{dist}_\alpha(x_0, Fx_1) + \operatorname{dist}_\alpha(x_1, Fx_0)\big],$$

then

$$d_\alpha(x_1, x_2) \leqslant \frac{q_\alpha}{2}\left[d_\alpha(x_0, x_2) + 0\right],$$

so

$$d_\alpha(x_1, x_2) \leqslant \frac{q_\alpha}{2 - q_\alpha} d_\alpha(x_0, x_1) + \frac{\varepsilon_\alpha}{2 - q_\alpha} \leqslant q_\alpha d_\alpha(x_0, x_1) + \frac{\varepsilon_\alpha}{1 - q_\alpha}.$$

The other cases are treated similarly and so we have

$$d_\alpha(x_1, x_2) \leqslant q_\alpha\, d_\alpha(x_0, x_1) + \frac{\varepsilon_\alpha}{1 - q_\alpha} \quad \text{for all} \quad \alpha \in \lambda.$$

This together with (2.3) implies that we have chosen $x_2 \in F x_1$ so that

$$d_\alpha(x_1, x_2) < q_\alpha(1 - q_\alpha)\, r_\alpha \quad \text{for all} \quad \alpha \in \lambda.$$

Notice $x_2 \in B(x_0, r)$ since (we give an argument here which can be used in the general step) for $\alpha \in \Lambda$, we have

$$d_\alpha(x_0, x_2) \leqslant (1 - q_\alpha)\, r_\alpha + q_\alpha(1 - q_\alpha)\, r_\alpha \leqslant (1 - q_\alpha) r_\alpha [1 + q_\alpha + q_\alpha^2 + \cdots] = r_\alpha.$$

Next for $\alpha \in \Lambda$, choose $\delta_\alpha > 0$ such that

$$q_\alpha\, d_\alpha(x_1, x_2) + \frac{\delta_\alpha}{1 - q_\alpha} < q_\alpha^2(1 - q_\alpha)\, r_\alpha. \tag{2.4}$$

Then choose $x_3 \in F(x_2)$ so that for every $\alpha \in \Lambda$, we have

$$d_\alpha(x_2, x_3) \leqslant \operatorname{dist}_\alpha\big(x_2, F(x_2)\big) + \delta_\alpha.$$

Now proceed as above to obtain

$$d_\alpha(x_2, x_3) \leqslant q_\alpha\, d_\alpha(x_1, x_2) + \frac{\delta_\alpha}{1 - q_\alpha}$$

and this together with (2.4) yields

$$d_\alpha(x_3, x_2) < q_\alpha^2(1 - q_\alpha)\, r_\alpha \quad \text{for every} \quad \alpha \in \Lambda.$$

Notice $x_3 \in B(x_0, r)$. Proceed inductively to obtain $x_n \in F(x_{n-1})$, $n = 4, 5, \ldots$, with $d_\alpha(x_{n+1}, x_n) < q_\alpha^n(1 - q_\alpha) r_\alpha$ for all $\alpha \in \Lambda$ and $x_n \in B(x_0, r)$. Now since $0 \leqslant q_\alpha < 1$, it is immediate that $(x_n)$ is a Cauchy sequence and hence converges to $x \in B(x_0, r)$ since $X$ is complete. It remains to show $x \in F(x)$. Notice for $\alpha \in \Lambda$, that

$$\begin{aligned}
\operatorname{dist}_\alpha\big(x, F(x)\big) &\leqslant d_\alpha(x, x_n) + \operatorname{dist}_\alpha\big(x_n, F(x)\big) \\
&\leqslant d_\alpha(x, x_n) + D_\alpha\big(F(x_{n-1}), F(x)\big) \\
&\leqslant d_\alpha(x, x_n) + q_\alpha \max\Big\{ d_\alpha(x, x_{n-1}), \operatorname{dist}_\alpha\big(x, F(x)\big), \\
&\qquad \operatorname{dist}_\alpha\big(x_{n-1}, F(x_{n-1})\big), \frac{1}{2}\Big[\operatorname{dist}_\alpha(x, F(x_{n-1}) + \operatorname{dist}_\alpha\big(x_{n-1}, F(x)\big)\Big] \Big\} \\
&\leqslant d_\alpha(x, x_n) + q_\alpha \max\Big\{ d_\alpha(x, x_{n-1}), \operatorname{dist}_\alpha\big(x, F(x)\big), \\
&\qquad d_\alpha(x_{n-1}, x_n), \frac{1}{2}\Big[d_\alpha(x, x_n) + d_\alpha(x_{n-1}, x) + \operatorname{dist}_\alpha\big(x, F(x)\big)\Big] \Big\}.
\end{aligned}$$

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Fix $\alpha \in \Lambda$. Let $n \to \infty$, to obtain

$$\operatorname{dist}_\alpha\big(x, F(x)\big) \leqslant q_\alpha \operatorname{dist}_\alpha\big(x, F(x)\big).$$

That is $\operatorname{dist}_\alpha\big(x, F(x)\big) = 0$ for each $\alpha \in \Lambda$ so $x \in \overline{F(x)} = F(x)$, and we are finished. ∎

REMARK 2.1. Theorem 2.1 immediately guarantees a result for generalised contractions $F : X \to C(X)$. We leave the details to the reader.

Next we obtain a homotopy result (see [**5, 6**]) via Zorn's lemma.

**THEOREM 2.2.** *Let $X$ be a complete gauge space with $U$ an open subset of $X$. Suppose $H : \overline{U} \times [0,1] \to C(X)$ is a closed map (that is, has closed graph) with the following conditions satisfied:*

(a) $x \notin H(x,t)$ *for* $x \in \partial U$ *and* $t \in [0,1]$;

(b) *there exist constants* $q = \{q_\alpha\}_{\alpha\in\Lambda} \in [0,1)^\Lambda$ *such that for all* $t \in [0,1]$, *every* $\alpha \in \Lambda$ *and* $x, y \in \overline{U}$, *we have*

$$D_\alpha\big(H(x,t), H(y,t)\big) \leqslant q_\alpha \max\Big\{d_\alpha(x,y), \operatorname{dist}_\alpha\big(x, H(x,t)\big), \operatorname{dist}_\alpha\big(y, H(y,t)\big), \\ \frac{1}{2}\Big[\operatorname{dist}_\alpha\big(x, H(y,t)\big) + \operatorname{dist}_\alpha\big(y, H(x,t)\big)\Big]\Big\};$$

(c) *for every* $t \in [0,1]$ *and for every* $\varepsilon = \{\varepsilon_\alpha\}_{\alpha\in\Lambda} \in (0,\infty)^\Lambda$, *there exists* $y \in H(x,t)$ *with* $d_\alpha(x,y) \leqslant \operatorname{dist}_\alpha\big(x, H(x,t)\big) + \varepsilon_\alpha$ *for every* $\alpha \in \Lambda$; *and*

(d) *there exists* $M \in (0,\infty)^\Lambda$ *and there exists a continuous increasing function* $\phi : [0,1] \to \mathbf{R}$ *such that* $D_\alpha\big(H(x,t), H(x,s)\big) \leqslant M_\alpha\big|\phi(t) - \phi(s)\big|$ *for all* $t, s \in [0,1]$, $x \in \overline{U}$ *and for every* $\alpha \in \Lambda$.

*Then $H(\,.\,,0)$ has a fixed point if and only if $H(\,.\,,1)$ has a fixed point.*

PROOF: Suppose $H(\,.\,,0)$ has a fixed point. Consider

$$Q = \big\{(t,x) \in [0,1] \times U : x \in H(x,t)\big\}.$$

Now $Q$ is nonempty since $H(\,.\,,0)$ has a fixed point. On $Q$, define the partial order

$$(t,x) \leqslant (s,y) \quad \text{if and only if} \quad t \leqslant s \quad \text{and} \quad d_\alpha(x,y) \leqslant 2\frac{M_\alpha}{1-q_\alpha}\big[\phi(s) - \phi(t)\big]$$

for every $\alpha \in \Lambda$. Let $P$ be a totally ordered subset of $Q$ and let

$$t^\star = \sup\big\{t : (t,x) \in P\big\}.$$

Take a sequence $\big\{(t_n, x_n)\big\}$ in $P$ such that $(t_n, x_n) \leqslant (t_{n+1}, x_{n+1})$ and $t_n \to t^\star$. We have

$$d_\alpha(x_m, x_n) \leqslant 2\frac{M_\alpha}{1-q_\alpha}\big[\phi(t_m) - \phi(t_n)\big] \quad \text{for all} \quad m > n \quad \text{and every} \quad \alpha \in \Lambda,$$

and so $(x_m)$ is a Cauchy sequence, which converges to some $x^\star \in \overline{U}$. Now since $H$ is a closed map we have $(t^\star, x^\star) \in Q$ (note $x^\star \in H(x^\star, t^\star)$, by closedness and (a) implies $x^\star \in U$). It is also immediate from the definition of $t^\star$ and the fact that $P$ is totally ordered that

$$(t, x) \leqslant (t^\star, x^\star) \text{ for every } (t, x) \in P.$$

Thus $(t^\star, x^\star)$ is an upper bound of $P$. By Zorn's Lemma $Q$ admits a maximal element $(t_0, x_0) \in Q$.

We *claim* $t_0 = 1$ (if our claim is true then we are finished). Suppose our claim is false. Then, choose $r = \{r_\alpha\}_{\alpha\in\Lambda} \in (0,\infty)^\Lambda$ and $t \in (t_0, 1]$ with

$$B(x_0, r) \subseteq U \text{ and } r_\alpha = 2\frac{M_\alpha}{1-q_\alpha}\left[\phi(t) - \phi(t_0)\right] \text{ for every } \alpha \in \Lambda.$$

Notice for every $\alpha \in \Lambda$, that

$$\begin{aligned}\operatorname{dist}_\alpha\bigl(x_0, H(x_0, t)\bigr) &\leqslant \operatorname{dist}_\alpha\bigl(x_0, H(x_0, t_0)\bigr) + D_\alpha\bigl(H(x_0, t_0)\bigr), H(x_0, t)\\ &\leqslant M_\alpha\bigl[\phi(t) - \phi(t_0)\bigr] = \frac{1}{2}(1-q_\alpha)r_\alpha < (1-q_\alpha)r_\alpha.\end{aligned}$$

Now Theorem 2.1 guarantees that $H(\,.\,, t)$ has a fixed point $x \in B(x_0, r)$. Thus $(x, t) \in Q$, and notice since

$$d_\alpha(x_0, x) \leqslant r_\alpha = 2\frac{M_\alpha}{1-q_\alpha}\bigl[\phi(t) - \phi(t_0)\bigr] \text{ for every } \alpha \in \Lambda \text{ and } t_0 < t,$$

we have $(t_0, x_0) < (t, x)$. This contradicts the maximality of $(t_0, x_0)$. ∎

Next ideas from the single valued case [**1, 9**] are used to obtain another continuation theorem for generalised contractive maps.

**Theorem 2.3.** *Let $X$ be a complete gauge space with $U$ an open subset of $X$. Suppose $H : \overline{U} \times [0,1] \to C(X)$ satisfies the following conditions:*

(e) *$x \notin H(x, t)$ for $x \in \partial U$ and $t \in [0,1]$;*

(f) *there exist constants $q = \{q_\alpha\}_{\alpha\in\Lambda} \in [0,1)^\Lambda$ such that for all $t \in [0,1]$, every $\alpha \in \Lambda$ and $x, y \in \overline{U}$, we have*

$$\begin{aligned}D_\alpha\bigl(H(x,t), H(y,t)\bigr) \leqslant q_\alpha \max\Bigl\{&d_\alpha(x,y), \operatorname{dist}_\alpha\bigl(x, H(x,t)\bigr), \operatorname{dist}_\alpha\bigl(y, H(y,t)\bigr),\\ &\frac{1}{2}\Bigl[\operatorname{dist}_\alpha\bigl(x, H(y,t)\bigr) + \operatorname{dist}_\alpha\bigl(y, H(x,t)\bigr)\Bigr]\Bigr\};\end{aligned}$$

(g) *for every $t \in [0,1]$ and for every $\varepsilon = \{\varepsilon_\alpha\}_{\alpha\in\Lambda} \in (0,\infty)^\Lambda$, there exists $y \in H(x,t)$ with $d_\alpha(x,y) \leqslant \operatorname{dist}_\alpha\bigl(x, H(x,t)\bigr) + \varepsilon_\alpha$ for every $\alpha \in \Lambda$;*

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(h) *for every* $\varepsilon = \{\varepsilon_\alpha\}_{\alpha\in\Lambda} \in (0,\infty)^\Lambda$, *there exists* $\delta = \delta(\varepsilon) > 0$ *(which does it not depend on* $\alpha$*) such that for* $t, s \in [0,1]$ *with* $|t-s| < \delta$, *then* $D_\alpha\big(H(x,t), H(x,s)\big) < \varepsilon_\alpha$ *for all* $x \in \overline{U}$ *and for all* $\alpha \in \Lambda$*; and*

(i) *there exists* $\alpha \in \Lambda$, *with* $\inf\Big\{\operatorname{dist}_\alpha\big(x, H_t(x)\big) : x \in \partial U, t \in [0,1]\Big\} > 0$*;*

*here* $H_t(\,.\,) = H(\,.\,,t)$. *Then* $H(\,.\,,0)$, *has a fixed point if and only if* $H(\,.\,,1)$, *has a fixed point.*

PROOF: Suppose $H(\,.\,,0)$ has a fixed point. Let

$$A = \Big\{\lambda \in [0,1] : \ x \in H(x,\lambda) \ \text{ for some } \ x \in U\Big\}.$$

Now since $H(\,.\,,0)$, has a fixed point (and (e) holds) we have that $0 \in A$, so $A$ is nonempty. We shall show $A$ is both closed and open in $[0,1]$, and so by the connectedness of $[0,1]$, we are finished since $A = [0,1]$.

First we show $A$ is open in $[0,1]$. Let $\lambda_0 \in A$ and $x_0 \in U$ with $x_0 \in H(x_0,\lambda_0)$. Since $U$ is open we know there exists $\delta_1,\dots,\delta_m$, in $(0,\infty)$ with

$$U(x_0,\delta_1) \cap \ldots \cap U(x_0,\delta_m) \subseteq U;$$

here $U(x_0,\delta_i) = \big\{x : d_{\alpha_i}(x,x_0) \leqslant \delta_i\big\}$ for $i = 1,2,\dots,m$ (here $\alpha_i \in \Lambda$ for $i \in \{1,\dots,m\}$). As a result there exists $\delta = \{\delta_\alpha\}_{\alpha\in\Lambda} \in (0,\infty)^\Lambda$ with $B(x_0,\delta) \subseteq U$. Fix $\alpha \in \Lambda$. Now $(h)$ guarantees that there exists $\eta = \eta(\delta) > 0$, with

$$\operatorname{dist}_\alpha\big(x_0, H(x_0,\lambda)\big) \leqslant D_\alpha\big(H(x_0,\lambda_0), H(x_0,\lambda)\big) < (1-q_\alpha)\delta_\alpha$$

for $\lambda \in [0,1]$ and $|\lambda - \lambda_0| \leqslant \eta$. Now Theorem 2.1 guarantees that there exists $x_\lambda \in B(x_0,\delta) \subseteq U$ with $x_\lambda \in H_\lambda(x_\lambda)$ for $\lambda \in [0,1]$ and $|\lambda-\lambda_0| \leqslant \eta$. Thus $A$ is open in $[0,1]$.

Next we show $A$ is closed in $[0,1]$. Let $(\lambda_k)$ be a sequence in $A$ with $\lambda_k \to \lambda \in [0,1]$ as $k \to \infty$. By definition for each $k$, there exists a $x_k \in U$ with $x_k \in H(x_k,\lambda_k)$.

We *claim*

$$\inf_{k\geqslant 1} \operatorname{dist}_\alpha(x_k, \partial U) > 0 \quad (\text{here } \ \alpha \ \text{ is as in (i)}). \tag{2.5}$$

Suppose our claim is true. Then there exists $\varepsilon_\alpha > 0$ with $d_\alpha(x_k,z) > \varepsilon_\alpha$ for all $k \geqslant 1$ and for all $z \in \partial U$. As a result there exists $\varepsilon = \{\varepsilon_\alpha\}_{\alpha\in\Lambda} \in (0,\infty)^\Lambda$ with $B(x_k,\varepsilon) \subseteq U$ for $k \geqslant 1$. Fix $\alpha \in \Lambda$. This together with (h) implies that there exists an integer $n_0$, (which does not depend on $\alpha$) with

$$\operatorname{dist}_\alpha\big(x_{n_0}, H_\lambda(x_{n_0})\big) \leqslant D_\alpha\big(H(x_{n_0},\lambda_{n_0}), H(x_{n_0},\lambda)\big) < (1-q_\alpha)\varepsilon_\alpha.$$

Now Theorem 2.1 guarantees that $H_\lambda$ has a fixed point $x_{\lambda,n_0} \in B(x_{n_0},\varepsilon) \subseteq U$. As a result $\lambda \in A$ so $A$ is closed in $[0,1]$.

It remains to show (2.5). Suppose (2.5) is false, that is, suppose

$$\inf_{k\geqslant 1} \operatorname{dist}_\alpha(x_k, \partial U) = 0 \quad (\text{here } \alpha \text{ is as in (i)}).$$

Fix $i \in \{1,2,\dots\}$. Then there exists $n_i \in \{1,2,\dots\}$ and $y_{n_i} \in \partial U$ with $d_\alpha(x_{n_i}, y_{n_i}) < 1/i$. As a result there exists a subsequence $S_\alpha$, of $\{1,2,\dots\}$ and a sequence $(y_i)$ in $\partial U$ (for $i \in S_\alpha$) with

$$d_\alpha(x_i, y_i) < \frac{1}{i} \quad \text{for } i \in S_\alpha. \tag{2.6}$$

This together with (i) implies

$$0 < \inf\left\{\operatorname{dist}_\alpha\big(x, H_t(x)\big) : x \in \partial U,\, t \in [0,1]\right\} \leqslant \liminf_{i\to\infty \text{ in } S_\alpha} \operatorname{dist}_\alpha\big(y_i, H_{\lambda_i}(y_i)\big). \tag{2.7}$$

We shall now show

$$\liminf_{i\to\infty \text{ in } S_\alpha} \operatorname{dist}_\alpha\big(y_i, H_{\lambda_i}(y_i)\big) = 0. \tag{2.8}$$

If (2.8) is true, then we have a contradiction from (2.7), and as a result (2.5) is true. To see (2.8), notice

$$\begin{aligned}
\liminf_{i\to\infty \text{ in } S_\alpha} \operatorname{dist}_\alpha\big(y_i, H_{\lambda_i}(y_i)\big) &\leqslant \liminf_{i\to\infty \text{ in } S_\alpha} \Big[d_\alpha(y_i, x_i) + \operatorname{dist}_\alpha\big(x_i, H_{\lambda_i}(y_i)\big)\Big] \\
&\leqslant \liminf_{i\to\infty \text{ in } S_\alpha} \Big[\frac{1}{i} + D_\alpha\big(H(x_i,\lambda_i), H(y_i,\lambda_i)\big)\Big] \\
&= \liminf_{i\to\infty \text{ in } S_\alpha} D_\alpha\big(H(x_i,\lambda_i), H(y_i,\lambda_i)\big), \\
&\leqslant q_\alpha \liminf_{i\to\infty \text{ in } S_\alpha} \Big[\max\Big\{d_\alpha(x_i, y_i), \\
&\qquad \operatorname{dist}_\alpha\big(x_i, H(x_i,\lambda_i)\big), \operatorname{dist}_\alpha\big(y_i, H(y_i,\lambda_i)\big), \\
&\qquad \frac{1}{2}\Big[\operatorname{dist}_\alpha\big(x_i, H(y_i,\lambda_i)\big) + \operatorname{dist}_\alpha\big(y_i, H(x_i,\lambda_i)\big)\Big]\Big\}\Big] \\
&\leqslant q_\alpha \liminf_{i\to\infty \text{ in } S_\alpha} \Big[\max\Big\{d_\alpha(x_i, y_i), 0, \\
&\qquad \operatorname{dist}_\alpha\big(y_i, H(y_i,\lambda_i)\big), d_\alpha(x_i,y_i) + \frac{1}{2}\operatorname{dist}_\alpha\big(y_i, H(y_i,\lambda_i)\big)\Big\}\Big],
\end{aligned}$$

and so we have (2.8); to see this suppose

$$\begin{aligned}
\max\Big\{d_\alpha(x_i,y_i), 0, \operatorname{dist}_\alpha\big(y_i, H(y_i,\lambda_i)\big), d_\alpha(x_i,y_i) + \frac{1}{2}\operatorname{dist}_\alpha\big(y_i, H(y_i,\lambda_i)\big)\Big\} \\
= d_\alpha(x_i,y_i) + \frac{1}{2}\operatorname{dist}_\alpha\big(y_i, H(y_i,\lambda_i)\big),
\end{aligned}$$

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then (2.6) yields

$$\begin{aligned}\liminf_{i\to\infty \text{ in } S_\alpha} \operatorname{dist}_\alpha\big(y_i, H_{\lambda_i}(y_i)\big) &\leqslant q_\alpha \liminf_{i\to\infty \text{ in } S_\alpha} \left[\frac{1}{i} + \frac{1}{2}\operatorname{dist}_\alpha\big(y_i, H(y_i,\lambda_i)\big)\right] \\ &= \frac{q_\alpha}{2} \liminf_{i\to\infty \text{ in } S_\alpha} \operatorname{dist}_\alpha\big(y_i, H(y_i,\lambda_i)\big),\end{aligned}$$

and so

$$\left(1 - \frac{q_\alpha}{2}\right) \liminf_{i\to\infty \text{ in } S_\alpha} \operatorname{dist}_\alpha\big(y_i, H_{\lambda_i}(y_i)\big) \leqslant 0.$$

∎

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