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PARTIAL λ-GEOMETRIES AND GENERALIZED HADAMARD MATRICES OVER GROUPS

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Introduction. Section 1 of this paper contains all the work which deals exclusively with generalizations of Hadamard matrices. The non-existence theorem proven here (Theorem 1.10) generalizes a theorem of Hall and Paige [**15**] on the non-existence of complete mappings in certain groups.

In Sections 2 and 3, we consider the duals of (Hanani) transversal designs; these dual structures, which we call (s, r, μ)-nets, are a natural generalization of the much studied (Bruck) nets which in turn are equivalent to sets of mutually orthogonal Latin squares. An (s, r, μ)-net $\mathcal{J}$ is a set of $s^2\mu$ points together with r parallel classes of blocks. Each class consists of s blocks of equal cardinality. Two non-parallel blocks meet in precisely μ points. It has been proven that r is always less than or equal to $(s^2\mu - 1)/(s - 1)$. When $\mu = 1$, $\mathcal{J}$ is just a net of order s; and r is only known to achieve the upper bound when $s^2\mu$ is a prime power. In fact, for $\mu = 1$, only one non-prime-power value of s is known for which r may be made larger than half the bound stated above: two groups totaling six mathematicians have constructed a set of 5 Latin squares of order 12 giving rise to a net of order 12 which has 7 parallel classes. (See [**11**, pp. 479–481] for details.) Using a result of Butson [**6**] on generalized Hadamard matrices, however, we are able to construct many (s, r, μ)-nets where r is nearly as large as the preceding bound even though $s^2\mu$ is not a prime power. (See, e.g., Remark 3.8 (iii) below.) Unfortunately, s itself is a prime power in all of our examples.

Generalized Hadamard matrices are also useful not only for the construction of the "uniform Klingenberg structures" studied by Jungnickel and the present author in [**14**]. These structures are treated briefly in Section 4.

The author is grateful to the referee for a careful reading of the first version of this paper and for suggestions concerning the reorganization of that version. In addition, the referee deserves credit for informing the author that generalized Hadamard matrices had already been examined in the literature. This led to the interesting discovery that Butson's construction could be used to create nearly "complete" (s, r, μ)-nets with point sets of non-prime power cardinality.

A better title for this article would be "Generalized Hadamard matrices and their associated geometries." The actual title was originally chosen, because the study of partial λ-geometries (a generalization of the partial geometries of Bose [**1**]) had led to the author's discovery of the matrices. Though we have

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reorganized the paper (at the referee's suggestion) to feature the matrices, we have retained the original title to forestall possible confusion: this article has already been referenced [**9**], and a preprint of version one has been circulated. Partial λ-geometries are examined in Section 5.

Section 1. Hadamard matrices over groups.

Definition 1.1. Let G be a group of finite order s, $H = [h_{ij}]$ be a square matrix of order r whose entries are elements of G. Then H is said to be a *Hadamard matrix*, briefly a GH-*matrix*, (of *type* r/s) (*over* G) provided:

(i) whenever $i \neq j$, the sequence $\{h_{ix}h_{jx}^{-1}\}$ with $1 \leqq x \leqq r$ contains every element of G equally often;

(ii) H^T, the transpose of H, has property (i).

Definition 1.2. Let C_s be the multiplicative group of all complex s^{th} roots of unity. A square matrix $H = [h_{ij}]$ of order r with elements from C_s is said to be a *Butson Hadamard matrix*, briefly, a BH-*matrix*, (of *type* r/s) (*over* C_s) if $HH^* = rI$. (Here H^* denotes the conjugate transpose of H.)

Remarks 1.3. (i) In the definition of a BH-matrix, the condition $HH^* = rI$ is equivalent to the requirement that $H^*H = rI$.

(ii) Every GH-matrix over C_s is a BH-matrix over C_s.

(iii) If s is a prime, every BH-matrix over C_s (except for the matrix [1] of order 1) is a GH-matrix over C_s.

(iv) If $s = pt$ where p is a prime and $t > 1$, then there exists a BH-matrix of order p over C_s, but certainly no GH-matrix of order p over C_s.

Proof. The truth of assertions (i) and (ii) is clear. To prove (iii), let s be a prime, $H = [h_{ij}]$ be a BH-matrix of order r over C_s. Then $\sum_{x=1}^{r} h_{ix}h_{jx}^{-1} = 0$ when $i \neq j$. Now every term $h_{ix}h_{jx}^{-1}$ is $\omega^{k(x)}$ where ω denotes a primitive s^{th} root of unity and $0 \leqq k(x) < s$. Combining terms, one obtains $\sum_0^{s-1} a_k\omega^k = 0$, hence $\sum_0^{s-2}(a_k - a_{s-1})\omega^k = 0$. Since the $\{\omega^k: 0 \leqq k \leqq s-2\}$ is an independent set of vectors for $Q(\omega)$ over Q, every $a_k = a_{s-1}$. Then every ω^k must occur equally often in the sequence of products $h_{ix}h_{jx}^{-1}$.

The existence of a BH-matrix of prime order p over C_p has long been known. (See, e.g., [**6**, footnote on p. 894 or Thm. 3.3] or [**18**, Thm. 9.5] where the example $H = [\omega^{i+j}]$ is given, ω being a primitive p^{th} root of unity. Clearly, H is also a BH-matrix over any C_s where s is any multiple of p.

THEOREM 1.4. (Butson) (See [**6**, Thm. 3.5] or [**21**, Cor. 9.7]) *If p is a prime and m and k are non-negative integers with $m \leqq k$, then there exists a* BH-*matrix H of order $2^m p^k$ over C_p. In view of* 1.3(iii), *H is also a* GH-*matrix over C_p unless $k = 0$.*

PROPOSITION 1.5. *There is a symmetric* GH-*matrix of type* 1 *over every finite elementary abelian group.*

Proof. If G is the elementary abelian group of order q, we may regard G as the additive group of the field $F = \mathrm{GF}(q)$. A multiplication table for F is a GH-matrix of type 1 over G.

PROPOSITION 1.6. *If H and H' are GH-matrices of orders r and r', resp., defined over an abelian group G, then the Kronecker product $H \times H'$ is a GH-matrix of order rr' over G.*

LEMMA 1.7. *Applying row and column permutations to a GH-matrix H over a group G always yields a GH-matrix over G. Multiplying the elements of any row or column of H by a fixed element from the center of G also yields a GH-matrix over G. In particular, if G is abelian, a sequence of such multiplications will always produce a GH-matrix over G in which every entry in the first row and column is the identity element of G.*

PROPOSITION 1.8. *Let $\phi: G \rightarrow G'$ be a group epimorphism, $H = [h_{ij}]$ be a GH-matrix over G. Then $H' = [h_{ij}^{\phi}]$ is a GH-matrix over G'.*

Proposition 1.8 may be used both to construct GH-matrices and to obtain non-existence results. As an example of the former of these two uses, we now obtain

COROLLARY 1.9. *Let G be an elementary abelian group of order p^i, p a prime, j be an arbitrary non-negative integer. Then there exists a GH-matrix H of type p^j over G.*

Proof. By Proposition 1.5, there is a GH-matrix K of type 1 over the elementary abelian group of order p^{i+j}. By Proposition 1.8, K may be modified to yield the desired matrix H.

We next apply Proposition 1.8 to obtain a non-existence theorem. The first paragraph of the proof is due to Hall and Paige [**15**, proof of Theorem 5]. The rest of the proof is adapted from an argument due to the present author [**12**, proof of Corollary 1.4].

THEOREM 1.10. *Let G be a finite group of even order with a cyclic Sylow 2-subgroup T. Let $s = |G|$, t be an odd integer. Then there is no $(3 \times st)$-matrix H with entries from G which satisfies axiom (i) of Definition 1.1; a. fort., no GH-matrix of type t over G.*

Proof. The order of the automorphism group of T, hence also the order of every automorphism of T, is a power of 2. Then, in G, T is in the center of its normalizer. By a theorem of Burnside (See, e.g., [**22**, p. 169]), T is a homomorphic image of G.

Let $\phi: G \rightarrow T$ be an epimorphism. Assume the existence of a $(3 \times st)$-matrix $H = [h_{ij}]$ with entries from G which satisfies axiom (i). Then $H' = [h_{ij}^{\phi}]$ is a

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$(3 \times ev)$-matrix over T which satisfies axiom (i): here e denotes $|T|$, and v is an odd integer.

Let $T = \langle b \rangle$. For $(i, j) \in \mathbf{Z} \times \mathbf{Z}$, define $f(i, j)$ to be the number of columns of H' which begin $(z, b^i z, b^j z)$, z arbitrary. Then for arbitrary i, j, k, and $x = 0, 1, \ldots, e - 1$,

$$v = \sum_x f(x, j) = \sum_x f(i, x) = \sum_x f(x, x + k).$$

It follows that

$$\sum_{j=0}^{e-1} j \sum_{x=0}^{e-1} f(x, j) - \sum_{i=0}^{e-1} i \sum_{x=0}^{e-1} f(i, x) + \sum_{k=1}^{e} (e - k) \sum_{x=0}^{e-1} f(x, x + k) = (e - 1)ev/2 = \sum_{a=0}^{e-2} \sum_{b=a+1}^{e-1} e \cdot f(a, b).$$

Then e must divide $(e - 1)ev/2$; but, since e is a power of 2 and v is odd, this is impossible.

COROLLARY 1.11. (Hall, Paige [**15**, Theorem 5]) *If G is a finite group of even order with a cyclic Sylow 2-subgroup, then G has no complete mapping.*

Proof. Assume that $\theta: G \to G$ is a complete mapping; i.e., a bijection such that $\phi: x \to xx^\theta$ is also a bijection. Then, if $G = \{g_1, \ldots, g_s\}$,

$$\begin{bmatrix} 1 & \ldots & 1 \\ g_1 & \ldots & g_s \\ (g_1{}^\theta)^{-1} & \ldots & (g_s{}^\theta)^{-1} \end{bmatrix}$$

is a $(3 \times s)$-matrix over G which satisfies axiom (i) of Definition 1.1.

Section 2. (s, r, μ)-**nets.** In this section, we shall consider incidence structures whose duals have been studied under the names *transversal designs* and *semi-regular group divisible designs*. We shall see that GH-matrices may be used to construct these "(s, r, μ)-nets."

Here and throughout this paper, all incidence structures considered will be tacitly assumed to be finite. À la Dembowski [**10**], we shall write $[p_1, \ldots, p_n]$ to denote the number of blocks that contain the point set $\{p_1, \ldots, p_n\}$, $[G_1, \ldots, G_n]$ for the dual notion. A *parallelism* on an incidence structure is an equivalence relation on the set of blocks such that each equivalence class (called a *parallel class*) partitions the point set.

Definition 2.1. Let $\mathscr{I}$ be an incidence structure. Define $B || G$ for blocks B, G of $\mathscr{I}$ to mean that either $B = G$ or $[B, G] = 0$. Then $\mathscr{I}$ is called a *net* or an (s, r, μ)-*net* provided:

(i) $||$ is a parallelism;

(ii) $G \nparallel H$ implies $[G, H] = \mu$;

(iii) there is at least one point, some parallel class has $s \geqq 2$ blocks, and there are $r \geqq 3$ parallel classes.

$\mathscr{J}$ is called an *affine resolvable partial plane* (briefly, an ARPP or an (s, r, μ)-ARPP) if, in addition, there exists an integer λ such that

(iv) $[p, q] = 0$ or λ whenever $p \neq q$.

One calls s the *order*, r the *degree*, μ the *type* of $\mathscr{J}$; in the case of ARPP's, λ is called the *index* of $\mathscr{J}$.

Remark 2.2. The term (s, r, μ)-net was introduced by Drake and Jungnickel in [**14**], since these structures are obviously generalizations of the well known nets of Bruck [**5**]: nets in the sense of Bruck are simply the $(s, r, 1)$-nets. The duals of (s, r, μ)-nets have been called *transversal designs* by Hanani [**16**]. These dual structures are also special cases of the semi-regular group divisible designs of Bose and Connor [**3**]. See also [**7**], [**10**].

PROPOSITION 2.3. (Drake, Jungnickel [**14**, Prop. 5.2, Cor. 5.4]) *Let $\mathscr{J}$ be an (s, r, μ)-net. Then $\mathscr{J}$ has $v = s^2\mu$ points, $b = sr$ blocks, s blocks in every parallel class, and $k = s\mu$ points per block. If $\mathscr{J}$ is an* ARPP, $(\lambda - 1)(s\mu - 1) = (r - 1)(\mu - 1)$.

As a consequence of Proposition 2.3, the truth of each of the following conditions for an ARPP implies the truth of all: $r = k$, $b = v$, $\lambda = \mu$ (except that $\lambda = \mu = 1$ yields no further equalities). Henceforth, call an ARPP *quasi-symmetric* if $\lambda = \mu$; *symmetric* if $r = k$. Call an (s, r, μ)-net $\mathscr{J}$ *quasi-symmetric* if it satisfies: $[p, q] = 0$ or μ whenever $p \neq q$. If the dual of $\mathscr{J}$ is also an (s, r, μ)-net with the same invariants s, r, μ as $\mathscr{J}$, we shall say that $\mathscr{J}$ is a *symmetric* (s, r, μ)*-net*.

LEMMA 2.4. *An incidence structure $\mathscr{J}$ is a quasi-symmetric (s, r, μ)-*ARPP *if and only if it is a quasi-symmetric (s, r, μ)-net. $\mathscr{J}$ is a symmetric (s, r, μ)-*ARPP *if and only if it is a symmetric (s, r, μ)-net.*

Proof. The first assertion is clearly true. Then let $\mathscr{J}$ be a symmetric (s, r, μ)-ARPP, $\mathscr{J}^d$ be the dual of $\mathscr{J}$. Clearly, $\mathscr{J}^d$ satisfies axioms (ii) and (iv) of Definition 2.1 (with $\lambda = \mu$). Let $[p, q] = 0 = [q, m]$ for points p, q, m of $\mathscr{J}$. Assume $p \neq m$ and $[p, m] > 0$. We take G to be a block containing both p and m. Counting flags (x, Y) with $x \in G$ and $q \in Y$, one sees that q is joined to exactly $k - 1$ points of G, hence to at least one of p, m. From the contradiction, we conclude that $\|$ is an equivalence relation on the blocks of $\mathscr{J}^d$. Now let G be any block of $\mathscr{J}$, $\{p_\alpha\}$ be a complete parallel class of points of $\mathscr{J}$. Each p_α lies on a different one of the s blocks which are parallel to G. Further, a given p_α is joined to $r(k - 1)/\lambda = s(s\mu - 1) = v - s$ points, so $|\{p_\alpha\}| = s$. Then every block which is parallel to G (including G itself) contains exactly one point of $\{p_\alpha\}$. Thus $\mathscr{J}^d$ satisfies axioms (i) and (iii). The truth of the converse is obvious.

The construction which yields the following result was discovered by Bose and Bush [**2**, Theorem 3] in the setting of orthogonal arrays.

PROPOSITION 2.5. (Bose, Bush) *The existence of a* GH*-matrix H of order $r \geqq 3$ over an abelian group G of order $s \neq 1$ implies the existence of a symmetric $(s, r, r/s)$-net.*

Proof. Let the matrix $A = [a_{ij}]$ be a multiplication table for the abelian group G. For each $x \in G$, obtain a matrix H_x from H by replacing every x in H by 1 and all other entries by 0. One now defines M by setting

$$M = [H_{(a_{ij})}].$$

Then M is the incidence matrix of a symmetric $(s, r, r/s)$-net.

PROPOSITION 2.6. *For $\lambda > 1$, a symmetric $(2, 2\lambda, \lambda)$-net $\mathscr{I}$ exists if and only if there is an ordinary Hadamard matrix H of order 2λ.*

Proof. That the existence of H implies the existence of $\mathscr{I}$ is a special case of Proposition 2.5. Assume then the existence of $\mathscr{I}$. By Proposition 2.3, $v = b = 4\lambda$; $r = k = 2\lambda$; and each parallel class of blocks of $\mathscr{I}$ consists of two blocks. The points are also partitioned into "parallel classes" of two points each. Now let M be an incidence matrix for $\mathscr{I}$, so arranged that the first 2λ rows represent one point from each parallel pair of points and the first 2λ columns represent one block from each parallel pair of blocks. Form a Hadamard matrix H of order 2λ from the upper left quarter of M by replacing each 0 by -1.

Section 3. Size of the replication numbers of (s, r, μ)-nets. In this section, we shall see that the existence of (s, r, μ)-nets is equivalent to the existence of orthogonal arrays. This connection will allow us to apply a theorem of Bose and Bush to obtain a bound on the size of r in terms of s and μ. We shall then apply the results of Section 1 on the existence of GH-matrices to obtain (s, r, μ)-nets for which r is equal or nearly equal to the Bose-Bush bound.

Definition 3.1. An $r \times N$ matrix A with entries from a set of s symbols is called an *orthogonal array* of *strength* 2, *size* N, r *constraints* and s *levels* if each $2 \times N$ submatrix contains every possible 2×1 column vector with frequency μ. (Clearly, $N = s^2\mu$.) More briefly, A is called an $(N, r, s, 2)$-*array*.

PROPOSITION 3.2. *The existence of an $(s^2\mu, r, s, 2)$-array A with $r \geqq 3$, $s \geqq 2$ is equivalent to the existence of an (s, r, μ)-net $\mathscr{I}$.*

Proof. One identifies the columns of A with the points of $\mathscr{I}$, the rows of A with the parallel classes of blocks of $\mathscr{I}$.

The following popular result has been proven independently by Plackett and Burman [**18**] and by Hanani [**16**, Lemma 5]. See also Bose-Bush [**2**, pp. 508–512] and Drake-Jungnickel [**14**, Prop. 5.3] for additional comments and proofs.

PROPOSITION 3.3. *In every* (s, r, μ)*-net,*

$$r < (s^2\mu - 1)/(s - 1) = f(s, \mu).$$

In the setting of orthogonal arrays, Bose and Bush [**2**] have obtained an improvement of the preceding proposition. Applying Proposition 3.2, we may restate their result as follows:

THEOREM 3.4. (Bose, Bush) *Assume the existence of an* (s, r, μ)*-net such that* $s - 1$ *does not divide* $\mu - 1$. *Define* a, b *by requiring that*

$$\mu - 1 = a(s - 1) + b, \quad 0 < b < s - 1.$$

Writing $[x]$ *to denote the greatest integer not exceeding* x, *one has*

$$r \leqq g(s, \mu) = [(s^2\mu - 1)/(s - 1)] - [\theta] - 1$$

where

$$2\theta = (1 + 4s(s - 1 - b))^{1/2} - (2s - 2b - 1).$$

An incidence structure is said to be *resolvable* if it possesses a parallelism. The following result was proved by Hanani [**16**, proof of Lemma 6] and by Bose and Bush [**2**, second part of Theorem 3 and following comments].

PROPOSITION 3.5. (Bose, Bush, Hanani) *Let* $\mathscr{J}$ *be an* (s, r, μ)*-net for which* $\mathscr{J}^d$ *is resolvable. Then an additional parallel class of blocks can be adjoined to the block set of* $\mathscr{J}$ *which will "extend"* $\mathscr{J}$ *to an* $(s, r + 1, \mu)$*-net.*

For r-nets of order s, it is well known that r can achieve the bound $f(s, 1) = s + 1$ whenever s is a prime power. The existence of $(s + 1)$-nets of order s where s is not a prime power is still an open question. To date, however, there is only one non-prime-power value of s for which an r-net of order s with $r \geqq s/2$ is known to exist; namely, the 7-net of order 12 mentioned in the Introduction. In contrast to this situation for $\mu = 1$, we now obtain infinitely many symmetric transversal designs with $r > f(s, \mu)/2$, $\mu > 1$ where $s^2\mu$ is not a prime power.

PROPOSITION 3.6. *Let* p *be a prime number;* m, k *be non-negative integers with* $k \geqq \max(1, m)$. *Then there exists a symmetric* (s, r, μ)*-net with* $s = p$, $r = 2^m p^k$, $\mu = 2^m p^{k-1}$ *(unless* $r = 2$*).*

Proof. Apply Theorem 1.4 and Proposition 2.5.

PROPOSITION 3.7. *Let* p *be a prime number;* i, j *be integers with* $i \geqq 1$, $j \geqq 0$. *Then there exists a symmetric* (s, r, μ)*-net with* $s = p^i$, $r = p^{i+j}$, $\mu = p^j$.

Proof. Apply Corollary 1.9 and Proposition 2.5.

Remarks 3.8. (i) The symmetric (s, r, μ)-nets of Proposition 3.6 have $r > f(s, \mu)(p - 1)/p$; those of Proposition 3.7 have $r > f(s, \mu)(p^i - 1)/p^i$.

(ii) The application of Proposition 3.5 enables one to increase the size of r by 1 in Propositions 3.6 and 3.7 when the adjective "symmetric" is omitted.

(iii) Applying Propositions 3.6 and 3.5 with $m = k = 1, p > 2$, one obtains the existence of (s, r, μ)-nets with $s = p, \quad r = 2p + 1, \quad \mu = 2$. Here r coincides with the Bose-Bush bound $g(s, \mu)$ of Theorem 3.4.

(iv) Proposition 3.7 improves a result of Hanani [**16**, proof of Theorem 2]; Hanani obtained the same conclusion under the additional assumption that i is a multiple of j.

Section 4. Uniform Klingenberg c-structures. In this brief section, we apply the preceding results to the construction of "uniform c-K-structures." The uninterested reader may go directly to Section 5 with no loss of comprehension. The study of c-K-structures was initiated by Jungnickel and Drake in [**13**] and [**14**]. We refer the reader to these papers for the appropriate definitions and to [**14**, Prop. 6.19], in particular, for a proof of the following result.

PROPOSITION 4.1. *Let Π' be a connected incidence structure with at least 3 points per block and dually, c and t be integers with $1 \neq c \neq t$. Then there exists a c-K-structure Π "over" Π' which has parameter t and is uniform of index λ if and only if: Π' is a tactical configuration with $k = r = [(\lambda - 1)(t - 1)/(c - 1)] + 1$, and there exists a $(t/c, r, c)$-ARPP of index λ.*

By definition, a symmetric (s, r, μ)-net is an (s, r, μ)-ARPP of index μ. To obtain existence results for c-K-structures, one may thus combine Proposition 4.1 either with 2.6 and the known existence results for Hadamard matrices (see, e.g., [**21**]) or with 2.5 in conjunction with 1.4 and 1.9. We make the first of these two connections in the following corollary which is an immediate consequence of 4.1 and 2.6.

COROLLARY 4.2. *Let Π' be a connected incidence structure with at least 3 points per block and dually. Then, for $c \neq 1$, there exists a c-K-structure Π over Π' with parameter $t = 2c$ which is uniform of index c if and only if (i) Π' is a tactical configuration with $k = r = 2c$ and (ii) there exists a Hadamard matrix of order 2c.*

Section 5. Partial λ-geometries. Partial λ-geometries are a generalization of the partial geometries of Bose [**1**] and a subclass of the partial geometric designs of Bose, Shrikhande and Singhi [**4**]. We shall see that all partial λ-geometries with $\lambda \neq 1$ are "symmetric." A symmetric partial λ-geometry whose index of parallelism is 0 is simply a symmetric block design. Cameron and Drake [**9**] have investigated partial λ-geometries of large index. In this final section, we shall prove that symmetric partial λ-geometries of index 1 are just the symmetric (s, r, λ)-nets. In addition, we shall obtain a characterization of the class of all symmetric partial λ-geometries in terms of their incidence graphs.

Definition 5.1. For $\lambda > 0$, a *partial λ-geometry* (with *nexus* $e > 0$) is an incidence structure with v points and b blocks which satisfies:

(i) $[p, q] = 0$ or λ for each point pair (p, q) with $p \neq q$;

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(ii) $[G, H] = 0$ or λ for each block pair (G, H) with $G \neq H$;

(iii) for each non-incident point-block pair (p, G), there exist precisely e blocks X with $p \in X$ and $[X, G] \neq 0$;

(iv) $\lambda < [p] < b$ for every p, and $\lambda < [G] < v$ for every G; if $\lambda = 1$, we also assume the existence of integers k, r such that $[G] = k$ and $[p] = r$ for all G, p.

Partial 1-geometries were first studied by Bose who called them simply "partial geometries." He proved the following result in the special case that $\lambda = 1$ [**1**, p. 398]:

LEMMA 5.2. *Let $\mathscr{I}$ be a partial λ-geometry. Then there are integers k, r such that* $[G] = k$ and $[p] = r$ *for all G, p. Further,*

$$v = [k(r-1)(k-\lambda)/e\lambda] + k,$$

$$b = [r(r-1)(k-\lambda)/e\lambda] + r.$$

The truth of the preceding lemma for $\lambda \neq 1$ follows immediately from the following result:

LEMMA 5.3. *Let $\mathscr{I}$ be a partial λ-geometry, $\lambda \neq 1$. Then*

(i) $b = v$;

(ii) $[p] = [G] = k$ *for some fixed positive integer k for all p and all G;*

(iii) $v = [k(k-1)(k-\lambda)/e\lambda] + k$.

Proof. First, observe that the non-triviality assumptions assure that $\mathscr{I}$ is connected; i.e., that every point and block of $\mathscr{I}$ is joined to every other point and block by a sequence of flags. Now, for a given flag (p, G), let k denote $[G]$, r denote $[p]$. Count flags (x, Y) such that $x \in G - \{p\}$ and $p \in Y \neq G$ to obtain $(k-1)(\lambda-1) = (r-1)(\lambda-1)$, hence $k = r$. The desired general result follows from the connectivity of $\mathscr{I}$. To see that $b = v$, one counts all flags of $\mathscr{I}$. To compute v, one fixes a block G and counts the double flags (x, y, Z) with x in G and y not in G.

Call a partial λ-geometry *symmetric* if $r = k$. Clearly, the incidence graph $\mathscr{G}$ of a symmetric partial λ-geometry $\mathscr{I}$ is bipartite and regular with valence k. If $e = k$, $\mathscr{G}$ has diameter 3; if $e < k$, $\mathscr{G}$ has diameter 4. Cameron has proved [**8**, p. 90] that a regular bipartite graph is "metrically regular" (for a definition, see [**8**, p. 41]) if and only if the following condition is satisfied: for every vertex pair (x, y) at distance i, $1 < i < \operatorname{diam} \mathscr{G}$, the number h_i of vertices at distance 1 from x and distance $i - 1$ from y depends only upon i. Clearly, $h_2 = \lambda$ and $h_3 = e$ for the incidence graph $\mathscr{G}$ of any partial λ-geometry $\mathscr{I}$. Given $\mathscr{G}$, $\mathscr{I}$ and its dual can be easily recaptured. We have proved:

PROPOSITION 5.4. *Let $\mathscr{I}$ be an incidence structure with incidence graph $\mathscr{G}$. Then $\mathscr{I}$ is a symmetric partial λ-geometry on v points with block size k and nexus e if and only if $\mathscr{G}$ is a metrically regular graph on $2v$ vertices with diameter 3 or 4 and valence $k < v$ such that $k > h_2 = \lambda > 0$, $h_3 = e > 0$.*

By axiom (when $\lambda = 1$) or Lemma 5.3 (when $\lambda \neq 1$), there is an integer r such that $[p] = r$ for every point p of a partial λ-geometry $\mathscr{J}$. Then, for each given non-flag (p, G), there are precisely $I = r - e$ blocks X with $p \in X$ and $[X, G] = 0$. We call I the *index* (*of parallelism*) of $\mathscr{J}$. In contrast to the investigations of Cameron and Drake cited above, we are here concerned with partial λ-geometries which have *small* index; namely, $I = 1$.

PROPOSITION 5.5. *An incidence structure $\mathscr{J}$ is a partial λ-geometry with index* 1, *replication number $r \geqq 3$ and block size k if and only if $\mathscr{J}$ is a quasi-symmetric $(k/\lambda, r, \lambda)$-net. If $\mathscr{J}$ is a partial λ-geometry of index* 1, *then $\mathscr{J}$ is symmetric as a partial λ-geometry if and only if it is symmetric as an (s, r, μ)-net.*

Proof. Assume first that $\mathscr{J}$ is a partial λ-geometry of index 1 with $r \geqq 3$. By the comments of the preceding paragraph, the relation $\|$ of Definition 2.1 is a parallelism for $\mathscr{J}$. Using Lemma 5.2 with $e = r - 1$, one obtains $(v - k)(r - 1)\lambda = k(r - 1)(k - \lambda)$, hence $v\lambda = k^2$. We now define s by demanding that $k = s\lambda$, hence that $v = s^2\lambda$. Clearly then, every parallel class consists of $s = k/\lambda$ blocks, and $\mathscr{J}$ is a quasi-symmetric $(k/\lambda, r, \lambda)$-net. Conversely, assume that $\mathscr{J}$ is a quasi-symmetric $(k/\lambda, r, \lambda)$-net, and set $s = k/\lambda$. The truth of axioms (i)–(iii) of Definition 5.1 (with $e = r - 1$) is clear. The existence of integers b, v, k, r such that $[p] = r$ for all p, $[G] = k$ for all G, $r < b$, $\lambda < k < v$ follows from Proposition 2.3. Since $k \geqq k/\lambda = s \geqq 2$, there is at least one pair of points joined by λ blocks. Then $r \geqq \lambda$. If $r = \lambda$, all blocks through a given point would be incident with the same set of k points. Then one would have $\lambda = k$, a contradiction.

The truth of the second assertion follows from Lemma 2.4.

Added in proof. Since the submission of the present paper, three related articles have been written. In [**17**] D. Jungnickel simplifies the proof of the Butson theorem (Theorem 1.4 above), in the process generalizing it to treat GH-matrices over elementary abelian groups. He also proves that the two axioms for GH-matrices are equivalent to each other. Using ideas due to D. Rajkundlia [**19**], J. Seberry proves [**20**] the existence of GH-matrices of order $q(q - 1)$ over the elementary abelian group of order q whenever $q - 1$ is a prime power.

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