A Spectral Identity on Jacobi Polynomials and its Analytic Implications

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Abstract. The Jacobi coefficients $c_j^\ell(\alpha, \beta)$ ($1 \le j \le \ell$, $\alpha, \beta > -1$) are linked to the Maclaurin spectral expansion of the Schwartz kernel of functions of the Laplacian on a compact rank one symmetric space. It is proved that these coefficients can be computed by transforming the even derivatives of the Jacobi polynomials $P_k^{(\alpha,\beta)}$ ($k \ge 0, \alpha, \beta > -1$) into a spectral sum associated with the Jacobi operator. The first few coefficients are explicitly computed, and a direct trace interpretation of the Maclaurin coefficients is presented.

1 Introduction

The Jacobi polynomials have a close connection with the Laplace operator on compact rank one symmetric spaces. They represent the spherical functions on these spaces and serve as the key ingredient in describing the spectral projections and the spectral measure associated with the Laplacian. In this note we bring this connection to the fore by showing that a set of spectral and geometric quantities associated with the so-called Jacobi operator fully describe the Maclaurin coefficients relating to the Schwartz kernel of operators in the functional calculus of the Laplacian.

The Jacobi polynomials $P_k^{(\alpha,\beta)}$ (integer $k \ge 0$, real $\alpha, \beta > -1$) constitute an orthogonal family of polynomials with the generating function relation [1]

$$\frac{1}{(1 - z + R)^\alpha (1 + z + R)^\beta} = R 2^{-(\alpha+\beta)} \sum_{k=0}^{\infty} P_k^{(\alpha,\beta)}(t) z^k, \qquad |z| < 1,$$

where $R = \sqrt{1 - 2tz + z^2}$. It is seen that $P_k^{(\alpha,\beta)}$ is a degree k polynomial admitting the truncated series representation

$$P_k^{(\alpha,\beta)}(t) = \frac{\Gamma(\alpha + k + 1)}{\Gamma(\alpha + \beta + k + 1)} \sum_{l=0}^{k} \binom{k}{l} \frac{\Gamma(\alpha + \beta + k + l + 1)}{2^l \Gamma(\alpha + l + 1) k!} (t - 1)^l,$$

and satisfying the classical Rodrigues' representation formula

$$P_k^{(\alpha,\beta)}(t) = \frac{(-1)^k}{2^k k!} (1 - t)^{-\alpha} (1 + t)^{-\beta} \frac{d^k}{dt^k} \left[(1 - t)^{k+\alpha} (1 + t)^{k+\beta} \right].$$

[1]For more information and further reading on this scale of orthogonal polynomials, the interested reader is referred to [4,7,8] and [15,16].

The Jacobi polynomial $y = P_k^{(\alpha,\beta)}$ satisfies a second-order homogenous linear differential equation: the Jacobi equation

$$(1.1) \qquad (1 - t^2)\frac{d^2 y}{dt^2} - \left(\alpha - \beta + (\alpha + \beta + 2)t\right)\frac{dy}{dt} + k(k + \alpha + \beta + 1)y = 0,$$

which in turn constitutes a regular Sturm–Liouville system, where the associated Jacobi operator is a positive, selfadjoint, second order, linear, differential operator in the weighted space $L^2[-1, 1; (1-t)^\alpha (1+t)^\beta dt]$. The spectrum here is discrete and is given by the sequence of eigenvalues and eigenfunctions

$$(1.2) \qquad \lambda_k^{(\alpha,\beta)} = k(k + \alpha + \beta + 1), \qquad y = P_k^{(\alpha,\beta)}(t), \qquad k \geq 0.$$

In particular, and as a result, the Jacobi polynomials satisfy the orthogonality relations

$$c^{(\alpha,\beta)}\delta_{k,m} = \langle P_k^{(\alpha,\beta)}, P_m^{(\alpha,\beta)} \rangle_{L^2[-1,1;(1-t)^\alpha(1+t)^\beta dt]} \qquad k, m \geq 0$$

$$= \int_{-1}^{1} P_k^{(\alpha,\beta)}(t)P_m^{(\alpha,\beta)}(t)(1-t)^\alpha(1+t)^\beta\, dt.$$

Here, $\delta_{k,m}$ is the usual Kronecker delta and the constants $c^{(\alpha,\beta)}$ on the left (that are of relevance only when $k = m$) are given by

$$c^{(\alpha,\beta)} = 2^{\alpha+\beta+1}\frac{(\alpha+1)_k(\beta+1)_k(\alpha+\beta+k+1)}{k!(\alpha+\beta+2)_k(\beta+\alpha+2k+1)}\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}.$$

Here and below, $(x)_k = \Gamma(x + k)/\Gamma(x)$ denotes the rising factorial. Now for $m \geq 1$, the Jacobi polynomials satisfy the differential recursion formula

$$(1.3) \qquad \frac{d^m}{dt^m}P_k^{(\alpha,\beta)}(t) = \frac{\Gamma(k + m + \alpha + \beta + 1)}{2^m\Gamma(k + \alpha + \beta + 1)}P_{k-m}^{(\alpha+m,\beta+m)}(t),$$

along with the reflection symmetry and pointwise identities

$$P_k^{(\alpha,\beta)}(-t) = (-1)^k P_k^{(\beta,\alpha)}(t), \quad P_k^{(\alpha,\beta)}(1) = \frac{(\alpha+1)_k}{k!}.$$

The Jacobi, Gegenbauer, and Legendre polynomials are related to one another for a suitable choice of (α, β) parameters, as given by (with $k \geq 0$, $v > -1/2$)

$$P_k^{(0,0)}(t) = C_k^{1/2}(t) = P_k(t), \qquad C_k^v(t) = \frac{(2v)_k}{(v+1/2)_k}P_k^{(v-1/2,v-1/2)}(t).$$

Having all this in place we note that in the sequel it is often more advantageous to use the normalised form of the Jacobi polynomials, written $\mathscr{P}_k^{(\alpha,\beta)}$ and defined by

$$\mathscr{P}_k^{(\alpha,\beta)}(t) = \frac{P_k^{(\alpha,\beta)}(t)}{P_k^{(\alpha,\beta)}(1)} = \frac{k!}{(\alpha+1)_k}P_k^{(\alpha,\beta)}(t).$$

Note that as a result of this normalisation, $\mathscr{P}_k^{(\alpha,\beta)}(1) = 1$.

2 Jacobi Coefficients and a Spectral Identity

We start by proving a differential-spectral identity relating the derivatives of the Jacobi polynomial to a suitable weighted sum of the integer powers of the eigenvalues of the Jacobi operator. Applications and implications of this identity to rank one symmetric spaces will be discussed later on.

Theorem 2.1 (Jacobi coefficients) *The normalised Jacobi polynomial $\mathscr{P}_k^{(\alpha,\beta)}$ with $k \geq 0$, $\alpha, \beta > -1$ satisfies the differential-spectral identity*

$$(2.1) \qquad \frac{d^{2\ell}}{d\theta^{2\ell}} \mathscr{P}_k^{(\alpha,\beta)}(\cos\theta)\Big|_{\theta=0} = \sum_{j=1}^{\ell} c_j^{\ell}(\alpha,\beta)[\lambda_k^{(\alpha,\beta)}]^j = \mathscr{R}_{\ell}(\lambda_k^{(\alpha,\beta)}).$$

Here, $(c_j^{\ell}(\alpha,\beta) : 1 \leq j \leq \ell)$ are suitable constants, $\lambda_k^{(\alpha,\beta)} = k(k + \alpha + \beta + 1)$ with $k \geq 0$ are the eigenvalues of the Jacobi operator from (1.1)–(1.2) and $\mathscr{R}_{\ell} = \mathscr{R}_{\ell}(X)$ is the degree ℓ polynomial defined via (2.1) (see also (2.6)).

Proof Basic considerations shows that for suitable scalars A_m^{ℓ} ($1 \leq m \leq \ell$) and upon invoking the differential-recursion formula (1.3), we can write

(2.2)

$$
\begin{aligned}
\frac{d^{2\ell}}{d\theta^{2\ell}} P_k^{(\alpha,\beta)}(\cos\theta)\Big|_{\theta=0} &= \sum_{m=1}^{\ell} \mathsf{A}_m^{\ell} \frac{d^m}{dt^m} P_k^{(\alpha,\beta)}(t)\Big|_{t=1} \\
&= \sum_{m=1}^{\ell} \frac{\mathsf{A}_m^{\ell} \Gamma(k + m + \alpha + \beta + 1)}{2^m \Gamma(k + \alpha + \beta + 1)} P_{k-m}^{(\alpha+m,\beta+m)}(1) \\
&= \sum_{m=1}^{\ell} \frac{\mathsf{A}_m^{\ell} \Gamma(k + \alpha + 1)}{\Gamma(\alpha + m + 1)} \frac{2^{-m}\Gamma(k + \alpha + \beta + m + 1)}{\Gamma(k + \alpha + \beta + 1)(k - m)!} \\
&= \sum_{m=1}^{\ell} \frac{\mathsf{A}_m^{\ell} \Gamma(\alpha + 1) 2^{-m} \Gamma(k + \alpha + \beta + m + 1) k!}{\Gamma(k + \alpha + \beta + 1)\Gamma(\alpha + m + 1)(k - m)!} P_k^{(\alpha,\beta)}(1).
\end{aligned}
$$

Now, focusing on the last line above, using the basic properties of the Gamma function and a subsequent expansion, it follows that for suitable choice of constants $\mathsf{B}_j^m = \mathsf{B}_j^m(\alpha,\beta)$, we can write

$$
\begin{aligned}
\frac{\Gamma(k + \alpha + \beta + m + 1) k!}{\Gamma(k + \alpha + \beta + 1)(k - m)!} &= \prod_{p=0}^{m-1}(k + \alpha + \beta + p + 1) \times \prod_{p=0}^{m-1}(k - p) \\
&= \prod_{p=0}^{m-1}\big(k(k + \alpha + \beta + 1) - p(p + \alpha + \beta + 1)\big) \\
&= \sum_{j=1}^{m} \mathsf{B}_j^m\big[k(k + \alpha + \beta + 1)\big]^m = \sum_{j=1}^{m} \mathsf{B}_j^m\big[\lambda_k^{(\alpha,\beta)}\big]^j,
\end{aligned}
$$

where the penultimate identity follows from a straightforward induction on m. Substituting back into (2.2) gives the required conclusion. ∎

Table 1: The Parameters a, b, N, α, β and ρ associated with $\mathscr{X}$

$\mathscr{X}$	N	a	b	α	β	ρ
$\mathbb{S}^n$	n	$n-1$	0	$(n-2)/2$	$(n-2)/2$	$(n-1)/2$
$\mathbf{P}^n(\mathbb{R})$	n	$n-1$	0	$(n-2)/2$	$(n-2)/2$	$(n-1)/2$
$\mathbf{P}^n(\mathbb{C})$	$2n$	1	$2(n-1)$	$n-1$	0	$n/2$
$\mathbf{P}^n(\mathbb{H})$	$4n$	3	$4(n-1)$	$2n-1$	1	$n+1/2$
$\mathbf{P}^2(\mathrm{Cay})$	16	7	8	7	3	$11/2$

Now let $-\Delta_{\mathscr{X}}$ denote the (positive) Laplace operator on a compact rank one symmetric space $\mathscr{X}$. Then by basic spectral theory upon taking Φ in the functional calculus of $-\Delta_{\mathscr{X}}$ the Schwartz kernel of $\Phi(-\Delta_{\mathscr{X}})$ can be expressed by the spectral sum

$$(2.3) \qquad K_\Phi(x,y) = \sum_{k=0}^{\infty} \frac{M_k \Phi(\lambda_k^{(\alpha,\beta)})}{\mathrm{Vol}(\mathscr{X})} \mathscr{P}_k^{(\alpha,\beta)}(\cos\theta).$$

Here, θ is the geodesic distance between x, y, $\mathrm{Vol}(\mathscr{X})$ is the volume of $\mathscr{X}$, and

$$\lambda_k(\mathscr{X}) = \lambda_k^{(\alpha,\beta)} = k(k+\alpha+\beta+1)$$

are the numerically distinct eigenvalues of $-\Delta_{\mathscr{X}}$ with multiplicities $M_k = M_k(\mathscr{X})$ where $k \geq 0$. (See below for more.)

Before proceeding further let us note that the compact rank one symmetric spaces that are of particular interest are the sphere $\mathbb{S}^n = \mathbf{SO}(n+1)/\mathbf{SO}(n)$, the real projective space $\mathbf{P}^n(\mathbb{R}) = \mathbb{S}^n/\{\pm\} = \mathbf{SO}(n+1)/\mathbf{O}(n)$, the complex projective space $\mathbf{P}^n(\mathbb{C}) = \mathbf{SU}(n+1)/\mathbf{S}(\mathbf{U}(n) \times \mathbf{U}(1))$ (of real dimension $2n$), the quaternionic projective space $\mathbf{P}^n(\mathbb{H}) = \mathbf{Sp}(n+1)/\mathbf{Sp}(n) \times \mathbf{Sp}(1)$ (of real dimension $4n$), and the Cayley projective plane $\mathbf{P}^2(\mathrm{Cay}) = \mathbf{F}_4/\mathbf{Spin}(9)$ (of real dimension 16).[2]

For the sake of future reference we now present some of the main spectral geometric quantities associated with the above spaces. The formulation of these, in the simply-connected case are given, in turn, by the eigenvalues (with $k \geq 0$ and $\rho = (a+b/2)/2$; see Table 1)

$$(2.4) \qquad \lambda_k(\mathscr{X}) = (\rho+k)^2 - \rho^2 = k(k+a+b/2);$$

the radial part of the Laplacian $\partial_\theta^2 + (a\cot\theta + (1/2)b\cot(\theta/2))\partial_\theta$; the multiplicity of the eigenvalue $\lambda_k(\mathscr{X})$ (with $k \geq 0$ and $N = a+b+1$) given by

$$M_k(\mathscr{X}) = \frac{2(k+\rho)\Gamma(k+2\rho)\Gamma((a+1)/2)\Gamma(k+N/2)}{k!\Gamma(2\rho+1)\Gamma(N/2)\Gamma(k+(a+1)/2)},$$

and the volume

$$(2.5) \qquad \mathrm{Vol}(\mathscr{X}) = \pi^{\frac{N}{2}} \frac{2^N \Gamma((a+1)/2)}{\Gamma((N+a+1)/2)}.$$

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[2]With the exception of the sphere $\mathbb{S}^1$ and the real projective space $\mathbf{P}^n(\mathbb{R})$ (with $n \geq 1$) all these spaces are simply-connected. Indeed $\pi_1(\mathbb{S}^1) \cong \pi_1(\mathbf{P}^1(\mathbb{R})) \cong \mathbb{Z}$, whilst $\pi_1(\mathbf{P}^n(\mathbb{R})) \cong \mathbb{Z}_2$ for $n \geq 2$.

In the non simply-connected case $\mathbf{P}^n(\mathbb{R})$ the counterparts of these quantities are obtained upon using standard arguments from those of its double cover $\mathbb{S}^n$. In Table 1 we gather together the values of the parameters a, b, N, α, β and ρ for the symmetric space $\mathcal{X}$. Note that here N is the real dimension of $\mathcal{X}$, $\alpha = (N-2)/2$ and $\beta = (a-1)/2$.

Table 2: Data for the symmetric spaces $\mathbb{S}^n$, $\mathbf{P}^n(\mathbb{R})$ and $\mathbf{P}^n(\mathbb{C})$

$\mathcal{X}$	$\mathbb{S}^n$	$\mathbf{P}^n(\mathbb{R})$	$\mathbf{P}^n(\mathbb{C})$
$\lambda_k(\mathcal{X})$	$k(k+n-1)$	$2k(2k+n-1)$	$k(k+n)$
$M_k(\mathcal{X})$	$(2k+n-1)\frac{(k+n-2)!}{k!(n-1)!}$	$(4k+n-1)\frac{(2k+n-2)!}{(2k)!(n-1)!}$	$\frac{2k+n}{n}\left[\frac{\Gamma(k+n)}{\Gamma(n)k!}\right]^2$
$\mathrm{Vol}(\mathcal{X})$	$\frac{2\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)}$	$\frac{\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)}$	$\frac{4^n\pi^n}{n!}$

Table 3: Data for the symmetric spaces $\mathbf{P}^n(\mathbb{H})$ and $\mathbf{P}^2(\mathrm{Cay})$

$\mathcal{X}$	$\mathbf{P}^n(\mathbb{H})$	$\mathbf{P}^2(\mathrm{Cay})$
$\lambda_k(\mathcal{X})$	$k(k+2n+1)$	$k(k+11)$
$M_k(\mathcal{X})$	$\frac{(2k+2n+1)(k+2n)}{(2n)(2n+1)(k+1)}\left[\frac{\Gamma(k+2n)}{k!\Gamma(2n)}\right]^2$	$6(2k+11)\frac{\Gamma(k+8)\Gamma(k+11)}{7!11!k!\Gamma(k+4)}$
$\mathrm{Vol}(\mathcal{X})$	$\frac{(4\pi)^{2n}}{\Gamma(2n+2)}$	$\frac{3!}{11!}(4\pi)^8$

Tables 2 and 3 illustrate some of the main spectral geometric quantities [*cf.* (2.4)–(2.5)] associated with the symmetric spaces described above. For further reference and discussion, see also [1, 3, 5, 6, 16, 17].

Now as by (2.3), K_Φ is an even function of the geodesic distance its Maclaurin expansion about $\theta = 0$ takes the form

$$K_\Phi = \sum_{\ell=0}^{\infty} \frac{\theta^{2\ell}}{(2\ell)!} \frac{\partial^{2\ell}}{\partial\theta^{2\ell}} K_\Phi \bigg|_{\theta=0} = \sum_{\ell=0}^{\infty} b_{2\ell}^n \frac{\theta^{2\ell}}{(2\ell)!}$$

with $b_{2\ell}^n = b_{2\ell}^n[\Phi]$ ($\ell \geq 0$) the associated Maclaurin coefficients. A direct calculation upon invoking (2.1) now shows that

$$
\begin{aligned}
b_{2\ell}^n = \frac{\partial^{2\ell}}{\partial\theta^{2\ell}} K_\Phi \bigg|_{\theta=0} &= \sum_{k=0}^{\infty} \frac{M_k \Phi(\lambda_k^{(\alpha,\beta)})}{\mathrm{Vol}(\mathcal{X})} \frac{d^{2\ell}}{d\theta^{2\ell}} \mathscr{P}_k^{(\alpha,\beta)}(\cos\theta) \bigg|_{\theta=0} \\
&= \sum_{k=0}^{\infty} \frac{M_k \Phi(\lambda_k^{(\alpha,\beta)})}{\mathrm{Vol}(\mathcal{X})} \sum_{j=1}^{\ell} c_j^\ell(\alpha,\beta)[\lambda_k^{(\alpha,\beta)}]^j \\
&= \frac{1}{\mathrm{Vol}(\mathcal{X})} \mathrm{tr}[\mathscr{R}_\ell \Phi](-\Delta_{\mathcal{X}}),
\end{aligned}
$$

where $\mathscr{R}_\ell = \mathscr{R}_\ell(X)$ is the polynomial of degree ℓ built out of the Jacobi coefficients $(c_j^\ell(\alpha,\beta) : 1 \le j \le \ell)$, specifically,

$$(2.6) \qquad \mathscr{R}_\ell(X) = \sum_{j=1}^{\ell} c_j^\ell(\alpha,\beta) X^j.$$

Note in particular that in the case of the heat semigroup with $\Phi_t(X) = e^{-tX}$, the Maclaurin coefficients of the heat kernel $K_t(x,y) := K_{\Phi_t}(x,y)$ can be expressed as $(t > 0)$:

$$b_{2\ell}^n(t) = \sum_{k=0}^{\infty} \frac{M_k e^{-t\lambda_k^{(\alpha,\beta)}}}{\text{Vol}(\mathscr{X})} \sum_{j=1}^{\ell} c_j^\ell [\lambda_k^{(\alpha,\beta)}]^j = \frac{1}{\text{Vol}(\mathscr{X})} \text{tr}\{\mathscr{R}_\ell(-\Delta_\mathscr{X}) e^{t\Delta_\mathscr{X}}\},$$

and so in this case one can alternatively express

$$\text{Vol}(\mathscr{X}) b_{2\ell}^n(t) = \sum_{k=0}^{\infty} M_k \Big\{ \sum_{j=1}^{\ell} (-1)^j c_j^\ell \frac{d^j}{dt^j} \Big\} e^{-t\lambda_k^{(\alpha,\beta)}} = \Big\{ \mathscr{R}_\ell\Big(-\frac{d}{dt}\Big) \Big\} \text{tr}\, e^{t\Delta_\mathscr{X}}.$$

The above analysis nicely underlines the role of the polynomials $\mathscr{R}_\ell$ and the Jacobi coefficients c_j^ℓ in expressing the Maclaurin coefficients $b_{2\ell}^n$ associated with the Schwartz kernel K_Φ of $\Phi(-\Delta_\mathscr{X})$. (For related but different results and discussions, see also [1–3] as well as [9–14].)

3 Explicit Calculations of the First Few Coefficients $(c_j^\ell : 1 \le j \le \ell)$ and the Polynomials $\mathscr{R}_\ell$

The proof of Theorem 2.1 does not reveal the Jacobi coefficients in explicit form. The aim here is to compute the first few in the sequence explicitly. For the sake of brevity, hereafter we set $y = P_k^{(\alpha,\beta)}$. Notice that the scalars A_m^ℓ below refer to those in the proof of Theorem 2.1.

- ($\ell = 1$) Indeed,

$$\frac{d^2}{d\theta^2} y(\cos\theta)\Big|_{\theta=0} = \mathsf{A}_1^1 y'(1) = \mathsf{A}_1^1 \frac{(k+\alpha+\beta+1)\Gamma(k+\alpha+1)}{2\Gamma(\alpha+2)(k-1)!}$$
$$= \mathsf{A}_1^1 \frac{k(k+\alpha+\beta+1)}{2(\alpha+1)} y(1),$$

hence

$$c_1^1(\alpha,\beta) = -\frac{1}{2(\alpha+1)}.$$

- ($\ell = 2$) Here we have

$$\frac{d^4}{d\theta^4} y(\cos\theta)\Big|_{\theta=0} = \mathsf{A}_1^2 y'(1) + \mathsf{A}_2^2 y''(1)$$
$$= \mathsf{A}_1^2 \frac{(k+\alpha+\beta+1)\Gamma(k+\alpha+1)}{2\Gamma(\alpha+2)(k-1)!}$$
$$+ \mathsf{A}_2^2 \frac{(k+\alpha+\beta+2)\Gamma(k+\alpha+1)}{4\Gamma(\alpha+3)(k-2)!}(k+\alpha+\beta+1).$$

Simplifying further, we see that

$$
\begin{aligned}
y''(1) &= \frac{(k-1)(k+\alpha+\beta+2)}{4(\alpha+1)(\alpha+2)} k(k+\alpha+\beta+1) y(1) \\
&= \left\{ \frac{[k(k+\alpha+\beta+1)]^2}{4(\alpha+1)(\alpha+2)} - \frac{(\alpha+\beta+2)k(k+\alpha+\beta+1)}{4(\alpha+1)(\alpha+2)} \right\} y(1).
\end{aligned}
$$

Hence, we obtain

$$
\begin{aligned}
\frac{d^4}{d\theta^4} y(\cos\theta)\Big|_{\theta=0} &= \left\{ \frac{A_1^2(2\alpha+4) - A_2^2(\alpha+\beta+2)}{4(\alpha+1)(\alpha+2)} k(k+\alpha+\beta+1) \right. \\
&\quad \left. + \frac{A_2^2[k(k+\alpha+\beta+1)]^2}{4(\alpha+1)(\alpha+2)} \right\} y(1).
\end{aligned}
$$

Clearly, basic differentiation gives $A_1^2 = 1$, $A_2^2 = 3$, and so we obtain

$$
c_1^2(\alpha,\beta) = -\frac{\alpha+3\beta+2}{4(\alpha+1)(\alpha+2)}, \qquad c_2^2(\alpha,\beta) = \frac{3}{4(\alpha+1)(\alpha+2)}.
$$

- ($\ell = 3$) Here we have

$$
\begin{aligned}
(3.1) \qquad \frac{d^6}{d\theta^6} y(\cos\theta)\Big|_{\theta=0} &= A_1^3 y'(1) + A_2^3 y''(1) + A_3^3 y'''(1) \\
&= A_1^3 y'(1) + A_2^3 y''(1) + A_3^3 (k+\alpha+\beta+1) \\
&\quad \times \frac{(k+\alpha+\beta+2)(k+\alpha+\beta+3)\Gamma(k+\alpha+1)}{8\Gamma(\alpha+4)(k-3)!}.
\end{aligned}
$$

Further simplification gives

$$
\begin{aligned}
\frac{y'''(1)}{y(1)} &= \frac{(k-1)(k-2)(k+\alpha+\beta+2)(k+\alpha+\beta+3)}{8(\alpha+1)(\alpha+2)(\alpha+3)} k(k+\alpha+\beta+1) \\
&= \left\{ \frac{[k(k+\alpha+\beta+1)]^3}{8(\alpha+1)(\alpha+2)(\alpha+3)} - \frac{(3\alpha+3\beta+8)[k(k+\alpha+\beta+1)]^2}{8(\alpha+1)(\alpha+2)(\alpha+3)} \right. \\
&\quad \left. + \frac{2(\alpha+\beta+3)(\alpha+\beta+2)k(k+\alpha+\beta+1)}{8(\alpha+1)(\alpha+2)(\alpha+3)} \right\}.
\end{aligned}
$$

So it follows that

$$
\begin{aligned}
(3.1) &= \left\{ c_1^3(\alpha,\beta)k(k+\alpha+\beta+1) \right. \\
&\quad \left. + c_2^3(\alpha,\beta)\left[k(k+\alpha+\beta+1)\right]^2 + c_3^3(\alpha,\beta)\left[k(k+\alpha+\beta+1)\right]^3 \right\} y(1),
\end{aligned}
$$

where we have

$$c_1^3(\alpha, \beta) = \frac{4A_1^3(\alpha + 2)(\alpha + 3) - 2A_2^3(\alpha + 3)(\alpha + \beta + 2)}{8(\alpha + 1)(\alpha + 2)(\alpha + 3)} +$$

$$+ \frac{2A_3^3(\alpha + \beta + 3)(\alpha + \beta + 2)}{8(\alpha + 1)(\alpha + 2)(\alpha + 3)},$$

$$c_2^3(\alpha, \beta) = \frac{2A_2^3(\alpha + 3) - A_3^3(3\alpha + 3\beta + 8)}{8(\alpha + 1)(\alpha + 2)(\alpha + 3)},$$

$$c_3^3(\alpha, \beta) = \frac{A_3^3}{8(\alpha + 1)(\alpha + 2)(\alpha + 3)}.$$

Clearly, by a basic differentiation, we see that $A_1^3 = -1$, $A_2^3 = A_3^3 = -15$, and thus

$$c_1^3(\alpha, \beta) = -\frac{4\alpha^2 + 30\alpha\beta + 30\beta^2 + 20\alpha + 60\beta + 24}{8(\alpha + 1)(\alpha + 2)(\alpha + 3)},$$

$$c_2^3(\alpha, \beta) = \frac{15(\alpha + 3\beta + 2)}{8(\alpha + 1)(\alpha + 2)(\alpha + 3)},$$

$$c_3^3(\alpha, \beta) = -\frac{15}{8(\alpha + 1)(\alpha + 2)(\alpha + 3)}.$$

- ($\ell = 4$) Indeed, from (2.2) we have

$$(3.2) \qquad \frac{d^8}{d\theta^8} y(\cos\theta)\bigg|_{\theta=0} = A_1^4 y'(1) + A_2^4 y''(1) + A_3^4 y'''(1) + A_4^4 y^{(4)}(1),$$

where $y'(1)$, $y''(1)$ and $y'''(1)$ are as above, while

$$(3.3) \qquad \frac{y^{(4)}(1)}{y(1)} = k(k + \alpha + \beta + 1)(k - 1)(k - 2)(k - 3)$$

$$\times \frac{(k + \alpha + \beta + 2)(k + \alpha + \beta + 3)(k + \alpha + \beta + 4)}{16(\alpha + 1)(\alpha + 2)(\alpha + 3)(\alpha + 4)}.$$

Therefore, we have

$$(3.3) = \left\{ \frac{[k(k + \alpha + \beta + 1)]^4 - 2(3\alpha + 3\beta + 10)[k(k + \alpha + \beta + 1)]^3}{16(\alpha + 1)(\alpha + 2)(\alpha + 3)(\alpha + 4)} \right.$$

$$+ \frac{[11(\alpha^2 + \beta^2) + 70(\alpha + \beta) + 22\alpha\beta + 108][k(k + \alpha + \beta + 1)]^2}{16(\alpha + 1)(\alpha + 2)(\alpha + 3)(\alpha + 4)}$$

$$\left. - \frac{6(\alpha + \beta + 2)(\alpha + \beta + 3)(\alpha + \beta + 4)k(k + \alpha + \beta + 1)}{16(\alpha + 1)(\alpha + 2)(\alpha + 3)(\alpha + 4)} \right\}.$$

It now follows that

$$(3.2) = \left\{ c_1^4(\alpha, \beta)k(k + \alpha + \beta + 1) + c_2^4(\alpha, \beta)[k(k + \alpha + \beta + 1)]^2 + \right.$$

$$\left. + c_3^4(\alpha, \beta)[k(k + \alpha + \beta + 1)]^3 + c_4^4(\alpha, \beta)[k(k + \alpha + \beta + 1)]^4 \right\} y(1),$$

where

$$c_1^4(\alpha,\beta) = \frac{8A_1^4(\alpha+2)(\alpha+3)(\alpha+4) - 4A_2^4(\alpha+3)(\alpha+4)(\alpha+\beta+2)}{16(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)}$$
$$+ 4\frac{A_3^4(\alpha+4)(\alpha+\beta+2)(\alpha+\beta+3)}{16(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)}$$
$$- \frac{6A_4^4(\alpha+\beta+2)(\alpha+\beta+3)(\alpha+\beta+4)}{16(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)},$$
$$c_2^4(\alpha,\beta) = \frac{4A_2^4(\alpha+3)(\alpha+4) - 2A_3^4(\alpha+4)(3\alpha+3\beta+8)}{16(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)}$$
$$+ \frac{A_4^4[11(\alpha^2+\beta^2) + 70(\alpha+\beta) + 22\alpha\beta + 108]}{16(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)},$$
$$c_3^4(\alpha,\beta) = \frac{2A_3^4(\alpha+4) - 2A_4^4(3\alpha+3\beta+10)}{16(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)},$$
$$c_4^4(\alpha,\beta) = \frac{A_4^4}{16(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)}.$$

Further differentiation gives $A_1^4 = 1$, $A_2^4 = 63$, $A_3^4 = 210$, $A_4^4 = 105$, and we get

$$c_1^4(\alpha,\beta) = -\frac{34\alpha^3 + 462\alpha^2\beta + 1050\alpha\beta^2 + 630\beta^3 + 306\alpha^2}{16(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)} +$$
$$- \frac{2184\alpha\beta + 2310\beta^2 + 884\alpha + 2604\beta + 816}{16(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)},$$
$$c_2^4(\alpha,\beta) = \frac{147\alpha^2 + 1050\alpha\beta + 1155\beta^2 + 714\alpha + 2310\beta + 924}{16(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)},$$
$$c_3^4(\alpha,\beta) = -\frac{210\alpha + 630\beta + 420}{16(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)},$$
$$c_4^4(\alpha,\beta) = \frac{105}{16(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)}.$$

References

[1] R. O. Awonusika and A. Taheri, *On Jacobi polynomials* ($\mathscr{P}_k^{(\alpha,\beta)} : \alpha, \beta > -1$) *and Maclaurin spectral functions on rank one symmetric spaces*, J. Analysis 25(2017), no. 1, 139–166.
http://dx.doi.org/10.1007/s41478-017-0038-5

[2] D. Bakry, I. Gentil, and M. Ledoux, *Analysis and geometry of Markov diffusion operators*. Grundlehren der Mathematischen Wissenschaften, 348, Springer, Cham, 2014.
http://dx.doi.org/10.1007/978-3-319-00227-9

[3] M. Berger, P. Gauduchon, E. Mazet, *Le spectre dúne variétè Riemannienne*, Springer, 1971.

[4] I. S. Gradshtejn, I. M. Ryzhik, *Table of Integrals, Series and Products*, Academic Press, 2007.
http://dx.doi.org/10.1090/S0025-5718-1982-0669666-2

[5] S. Helgason, *Eigenspaces of the Laplacian; integral representations and irreducibility.* J. Funct. Anal. **17**(1974), 328–353.   http://dx.doi.org/10.1016/0022-1236(74)90045-7

[6] _____, *Topics in harmonic analysis on homogeneous spaces.* Progress in Mathematics, 13, Birkhäuser, Boston, MA, 1981.

[7] T. H. Koornwinder, *The addition formula for Jacobi polynomials: I Summary of results.* Indag. Math. **34**(1972), 188–191.   http://dx.doi.org/10.1016/1385-7258(72)90011-X

[8] _____, *A new proof of a Paley-Wiener type theorem for the Jacobi transform.* Ark. Mat. **13**(1975), 145–159.   http://dx.doi.org/10.1007/BF02386203

[9] H. McKean and I. M. Singer, *Curvature and the eigenvalues of the Laplacian.* J. Differential Geometry **1**(1967), no. 1, 43–69.   http://dx.doi.org/10.4310/jdg/1214427880

[10] C. Morris and A. Taheri, *On Weyl's asymptotics and remainder term for the orthogonal and unitary groups.* J. Fourier Anal. Appl., to appear.

[11] B. Osgood, R. Phillips, and P. Sarnak, *Extremals and determinants of Laplacians.* J. Funct. Anal. **80**(1988), 148–211.   http://dx.doi.org/10.1016/0022-1236(88)90070-5

[12] P. Sarnak, *Determinants of Laplacians.* Comm. Math. Phys. **110**(1987), no. 1, 113–120. http://dx.doi.org/10.1007/BF01209019

[13] _____, *Determinants of Laplacians: heights and finiteness.* In: Analysis, et cetera, Academic Press, Boston, MA, pp. 601–622.

[14] R. T. Seeley, *Complex powers of an elliptic operator.* In: Singular Integrals (Proc. Sympos. Pure Math., Chicago, III, 1966), American Mathematical Society Providence, RI, 1966, pp. 288–307.

[15] A. Taheri, *Function spaces and partial differential equations.* I & II, Oxford Lecture Series in Mathematics and Its Applications, 40 & 41, Oxford University Press, 2015. http://dx.doi.org/10.1093/acprof:oso/9780198733157.003.0013

[16] N. J. Vilenkin, *Special functions and the theory of group representations.* Translations of Mathematical Monographs, 22, American Mathematical Society, Providence, RI, 1968.

[17] G. Warner, *Harmonic analysis on semisimple Lie groups.* Die Grundlehren der mathematischen Wissenschaften, Band 188. Springer-Verlag, New York-Heidelberg, 1972.

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