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THE LIE RING OF SYMMETRIC DERIVATIONS OF A RING WITH INVOLUTION

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Abstract

In this paper we investigate how the ideal structure of the Lie ring of symmetric derivations of a ring with involution is determined by the ideal structure of the ring.

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1. Introduction

C. R. Jordan and D. A. Jordan (1978a) showed that the Lie ring of derivations of a prime (respectively semiprime) 2-torsion-free associative ring is a prime (respectively semiprime) Lie ring. The aim of this paper is to prove analogous results for the Lie ring, $SD(R)$, of symmetric derivations of a ring R with involution, a symmetric derivation being one which commutes with the involution. The cases where R is commutative and where R is non-commutative will be treated separately. In the commutative case it will be shown that if R is *-prime (respectively semiprime) and 2-torsion-free then $SD(R)$ is prime (respectively semiprime). Certain well-known examples, which will be described in Section 3, indicate that for such results to hold in the non-commutative case further conditions on R are required.

In the prime case the best one can hope for is that $SD(R)$ is prime whenever R is 2-torsion-free, *-prime and does not satisfy the standard identity S_8 of 4×4 matrices. That this is true will be established using ideas of Lanski (1976, 1977, 1978). In the semiprime case two approaches are possible. One is to impose conditions on the *-prime factor rings of R, for example that R is S_8-free in the sense of Lanski (1978), then show that $SD(R)$ is semiprime. Alternatively one can aim to show that there is a particular ideal of $SD(R)$ which must always be semiprime whenever R is semiprime and 2-torsion-free. The latter approach is the one taken here, the ideal concerned being the set of all symmetric derivations of R which annihilate all antisymmetric elements of R. The results obtained can then be applied to show that if R is semiprime, 2-torsion-free and S_4-free then $SD(R)$ is semiprime. The definitions of prime and semiprime Lie ring and ideal are analogous to those for associative rings and may be found in C. R. Jordan and D. A. Jordan (1978a).

2. Notation and preliminary remarks

Throughout R will be an associative ring which is 2-torsion-free and * will be an involution of R. The set of symmetric elements of R and the set of antisymmetric elements of R will be denoted by S and K respectively. The centre of a ring T will be denoted by $Z(T)$ and Z will denote $Z(R)$.

DEFINITION. A derivation δ of R is said to be *symmetric* if $\delta(r^*) = \delta(r)^*$ for all $r \in R$. A derivation δ of R is said to be *antisymmetric* if $\delta(r^*) = -\delta(r)^*$ for all $r \in R$.

By $D(R)$ we denote the Lie ring of all derivations of R, by $SD(R)$ we denote the set of all symmetric derivations of R and by $KD(R)$ we denote the set of antisymmetric derivations of R.

REMARKS.

(i) $SD(R)$ is a Lie subring of $D(R)$ and $KD(R)$ is an $SD(R)$-submodule of $D(R)$.

(ii) $SD(R)$ is a $Z \cap S$-module in a natural way: if $z \in Z \cap S$ and $\delta \in SD(R)$ then $z\delta$ maps a typical element r of R to $z\delta(r)$.

(iii) If $\frac{1}{2} \in R$ then, as $SD(R)$-modules, $D(R) = SD(R) \oplus KD(R)$. See the note following Proposition 3 of C. R. Jordan and D. A. Jordan (1978b).

(iv) For $r \in R$ let i_r denote the inner derivation of R induced by r. Thus

$$i_r(s) = rs - sr \quad \text{for all } s \in R.$$

If $r \in S$ then $i_r \in KD(R)$ and if $r \in K$ then $i_r \in SD(R)$. If $i_r \in SD(R)$ then $(r + r^*) \in Z$ so that, if $\frac{1}{2} \in R$, then $i_r = i_k$ where $k = \frac{1}{2}(r - r^*) \in K$.

NOTATION. By $I_K(R)$ we denote the set $\{i_k \colon k \in K\}$. Thus $I_K(R)$ is a Lie subring of $SD(R)$.

(v) There is a natural isomorphism of Lie rings:

$$I_K(R) \simeq K/K \cap Z.$$

(vi) An ideal I or R is said to be a **-ideal* of R if $I^* = I$. The ring R is said to be **-prime* if for every pair of non-zero *-ideals A, B of R the product AB is non-zero. If I is a *-ideal then there is an induced involution, also denoted *, on the ring R/I. The *-ideal I of R is said to be **-prime* if the ring R/I is *-prime. It is easy to show that a *-prime ring or ideal is semiprime.

3. The non-commutative case

Throughout this section R, in addition to being 2-torsion-free, will be assumed to be non-commutative. The involution * will be said to be of the first kind if $Z \subseteq S$. Otherwise * is of the second kind. Following Lanski (1978) we shall say that a prime ring satisfies S_{2n} if it is an order in a simple algebra of dimension at most n^2 over its centre. A semiprime ring will be said to satisfy S_{2n} if it is a subdirect product of prime rings, each of which satisfy S_{2n}.

Before passing to the general theory we describe some well-known examples which indicate that some further conditions on R may be required if positive results are to be obtained.

(i) Let $R = k_2$ be the ring of 2×2 matrices over a prime field k of characteristic not 2. By Lemma 1 of Kawada (1952) every derivation of R is inner so, by remarks (iv) and (v) of Section 2, $SD(R) \simeq K$, which, in this case, is an abelian Lie ring. Thus $SD(R)$ cannot be prime or semiprime although R is prime.

(ii) Let $R = k_4$ be the ring of 4×4 matrices over a prime field k of characteristic not 2. As in Example (i) $SD(R) \simeq K$. In this case K has a pair of non-zero ideals I and J such that $K = I \oplus J$. Thus K cannot be prime although R is prime. Note, however, that K is semiprime.

We now consider the case where R is *-prime, the aim being to show that $SD(R)$ is a prime Lie ring. The above examples indicate that we should avoid the case where R satisfies S_8.

THEOREM 1. *Let R be *-prime, not satisfying S_8. Then $K \cap Z$ is a prime ideal of K. Equivalently $I_K(R)$ is a prime Lie ring.*

PROOF. This is a special case of Theorem 7 of Lanski (1978).

LEMMA 1. *Let R be *-prime and $0 \neq \delta \in SD(R)$. Then $\delta(K) \nsubseteq Z$.*

PROOF. We adapt an argument used in the proof of Theorem 7 of Lanski (1977). Suppose that $\delta(K)\subseteq Z$. Then $\delta([K,K])=0$. For $k\in K$ and $t\in[K,K]$, $ktk\in K$ so that $\delta(ktk)\in Z$, that is, since $\delta(t)=0$ and $\delta(k)\in Z$, $(kt+tk)\,\delta(k)\in Z$. Suppose that $\delta(k)\neq 0$. Since $\delta\in SD(R)$, $\delta(K)\subseteq K$ so that $\delta(k)\in K\cap Z$. But R is *-prime so it follows that $\delta(k)$ is a non-zero-divisor and, hence, that $kt+tk\in Z$. Applying δ, $2\delta(k)\,t\in Z$ whence $t\in Z$. Thus $\delta(K)\neq 0$ implies that $[K,K]\subseteq Z$. Suppose now that $\delta(K)=0$. Then $\delta(K^2)=0$ and, since K^2 is a Lie ideal of R (see Lanski (1976), p. 735), it follows that $\delta([K^2,R])=0$ whence $K^2\,\delta(R)=0$. But $\delta(R)+\delta(R)\,R$ is a non-zero *-ideal of R and hence $K^2=0$. In particular, $[K,K]=0$. So we can certainly assume that $[K,K]\subseteq Z$. But then, by Theorem 1, $K\subseteq Z$. But $K\subseteq Z$ implies that $[2R,2R]\subseteq[S+K,S+K]\subseteq[S,S]\subseteq K\subseteq Z$. By Lemma 1 of Herstein (1970) it follows, since R is *-prime and hence semiprime, that R is commutative, a contradiction.

THEOREM 2. *If R is *-prime not satisfying S_8 then $SD(R)$ is a prime Lie ring.*

PROOF. By Theorem 1 it suffices to show that for any non-zero ideal A of $SD(R)$, $A\cap I_K(R)\neq 0$. Let $0\neq\delta\in A$. By Lemma 1 there exists $k\in K$ such that $\delta(k)\notin Z$. Then $0\neq i_{\delta(k)}=[\delta,i_k]\in A$. The result follows.

'We now pass to the case where R is semiprime. We intend to avoid imposing further conditions on R, such as 'R does not satisfy S_4', as long as possible.

NOTATION. Let $C(K^2)=\{r\in R\colon [K^2,r]=0\}$. As in the proof of Lemma 1, K^2 is a Lie ideal of R. Hence $C(K^2)$ is a Lie ideal of R.

LEMMA 2. *If R is semiprime then $K^2\cap C(K^2)\subseteq Z$.*

PROOF. This is immediate from Lemma 1 of Herstein (1970).

For convenience we quote Lemma 1 of Lanski (1976).

LEMMA 3. *Let R be a semiprime ring. If $x\in K$ and $xKx=0$ then $x=0$.*

LEMMA 4. *Let R have a prime ideal P such that $P\cap P^*=0$. Let A be an ideal of K such that $[A,A]\subseteq Z$. Then $[A,K]=0$.*

PROOF. Consider first the case where * is of the first kind, $Z\subseteq S$. In this case the result holds under the weaker hypothesis that R is semiprime. The proof is based loosely on that of Lemma 9 of Lanski (1976). Since $Z\subseteq S$ it follows that $K\cap Z=0$ and hence that $[A,A]=0$. For all $x\in A$ and $k\in K$, $[x,k]\in A$ so that $[x,[x,k]]=0$

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that is

(1) $$x^2k+kx^2-2xkx=0 \quad \text{for all } x\in A,\ k\in K.$$

Let $a\in A$, $k\in K$ and $\delta=i_a$ be the inner derivation induced by a. Then $\delta(K)\subseteq A$ and $\delta^2(K)=0$. But $k\delta(k)k\in K$ so that $\delta^2(k\delta(k)k)=0$, that is $\delta(k)^3=0$. But $\delta(k)\in A$ so that, by (1), $2\delta(k)^2K\delta(k)^2=0$. Let $y=\delta(k)$ so that $y\in A$ and $y^2Ky^2=0$. Let $z\in A$. Then, because $[A,A]=0$, $yz=zy$ so that $y^2z=zy^2$ and $y^2zKy^2z=0$. But $y^2z=yzy\in K$ so that $y^2z=0$ by Lemma 3. Thus $y^2A=0$ whence $y^2[A,K]=0$ so that $y^2KA=0$. It follows that

(2) $$(yKy)(yKy)\subseteq yKy^2KA=0.$$

Let $j\in K$. Then $kj\delta(k)+\delta(k)jk-k\delta(j)k\in K$ and, since $\delta^2(K)=0$,

$$\delta(kj\delta(k)+\delta(k)jk-k\delta(j)k)=2\delta(k)j\delta(k)=2yjy.$$

Thus $2yKy\subseteq\delta(K)\subseteq A$. Let $w\in 2yKy$. Then $w\in A$ and, by (2), $w^2=0$ so that, by (1) and Lemma 3, $w=0$. Another application of Lemma 3 gives that $y=0$, that is $[a,k]=0$. This holds for all $a\in A$, $k\in K$ and the result follows, in the case where * is of the first kind.

Suppose now that * is of the second kind. We adapt the proof of Theorem 2 of Lanski (1977). Since R is *-prime the non-zero elements of $Z\cap S$ are regular. By a standard argument there is no loss of generality in assuming that R has been localized at $Z\cap S\setminus\{0\}$ so that the non-zero elements of $Z\cap S$ are units. For $0\neq k\in Z\cap K$, $0\neq k^2\in Z\cap S$ so that the non-zero elements of $Z\cap K$ are units also. Choose $z\in Z$ such that $(z-z^*)\neq 0$. Then $(z-z^*)$ is a unit and, for $r\in R$, $(z-z^*)r\in K+z^*K$ because

$$(z-z^*)r=(zr-z^*r^*)+z^*(r^*-r).$$

Consequently $R=K+z^*K$. Let $I=\{i\in R\colon [A,i]\subseteq Z\}$. Then, since $R=K+z^*K$, I is a Lie ideal of R. Let $J=\{j\in R\colon [I,j]\subseteq Z\}$ so that J is a Lie ideal of R and $A\subseteq J$. By Lemma 1 of Herstein (1970), $I\cap J\subseteq Z$. But $A\subseteq J$ and $A\subseteq I$ since $[A,A]\subseteq Z$. Thus $A\subseteq Z$. This completes the proof of Lemma 4.

If R is semiprime then R is a subdirect product of a family $\{R_i\}$ of rings with involution induced by * and each satisfying the hypothesis of Lemma 4. The images in R_i of elements of K are antisymmetric so that an immediate consequence of Lemma 4 is the following.

LEMMA 5. *If R is semiprime and A is an ideal of K such that $[A,A]\subseteq Z$ then* $[A,K]=0$.

NOTATION. Let $\ell(K)=\{k\in K\colon [K,k]=0\}$. Lemma 5 now says that $[A,A]\subseteq Z$ implies that $A\subseteq\ell(K)$. Note that $\ell(K)$ is an ideal of K.

THEOREM 3. *If R is semiprime then $\ell(K)$ is a semiprime ideal of K.*

PROOF. Let A be an ideal of K such that $[A, A] \subseteq \ell(K)$. Then $[[A, A], K^2] = 0$ so that $[A, A] \subseteq K^2 \cap C(K^2) \subseteq Z$ by Lemma 2. By Lemma 5 it follows that $A \subseteq \ell(K)$. Thus $\ell(K)$ is a semiprime ideal of K.

NOTATION. Let $\ell(SD(R)) = \{\delta \in SD(R): \delta(K) = 0\}$. Then $\ell(SD(R))$ is an ideal of R.

THEOREM 4. If R is semiprime then $\ell(SD(R))$ is a semiprime ideal of $SD(R)$.

PROOF. Let A be an ideal of $SD(R)$ such that $[A, A] \subseteq \ell(SD(R))$. Let $\delta \in A$, $z \in Z \cap S$. Then $z\delta \in SD(R)$ so that $[\delta, z\delta] \in A$, that is $\delta(z)\delta \in A$. It follows that $\delta^2(z)\delta = [\delta, \delta(z)\delta] \in [A, A]$ so that $\delta^2(z)\delta(K) = 0$. Replacing z by z^2 gives $2\delta(z)\delta(z)\delta(K) = 0$ so that $\delta(z)\delta(K)R\delta(z)\delta(K) = 0$ since $\delta(z) \in Z$. But R is semiprime so that

(1) $$\delta(z)\delta(K) = 0 \quad \text{for all } z \in Z \cap S.$$

Let $J = \{j \in K: i_j \in A\}$. Then J is an ideal of K and $[[J, J], K] = 0$ so that, by Theorem 3, $[J, K] = 0$. But for $\delta \in A$ and $k \in K$, $i_{\delta(k)} = [\delta, i_k] \in A$ so that $\delta(K) \subseteq J$ for all $\delta \in A$. Thus

(2) $$[\delta(K), K] = 0 \quad \text{for all } \delta \in A.$$

It follows that $[\delta(K), K^2] = 0$ and, hence, that $[\delta(K)\delta(K), K^2] = 0$. Thus

$$\delta(K)\delta(K) \subseteq K^2 \cap C(K^2) \subseteq Z$$

by Lemma 2. Let $k \in K$. Then $\delta(k)\delta(k) \in Z \cap S$ so that, by (1), $\delta(\delta(k)\delta(k))\delta(K) = 0$ and, in particular, $\delta(\delta(k)\delta(k))\delta(k) = 0$. It follows from (2) that

(3) $$\delta(k)\delta^2(k)\delta(k) = 0.$$

Also by (2),

(4) $$\delta(k)\delta^2(k)K \subseteq K.$$

Together (2) and (3) give that

$$(\delta(k)\delta^2(k)K)K(\delta(k)\delta^2(k)K) = 0$$

so that by Lemma 3 and (4) $\delta(k)\delta^2(k)K = 0$. In particular $u = \delta(k)\delta^2(k)\delta^2(k) = 0$ and $v = \delta(k)\delta^2(k)\delta^3(k) = 0$. It follows, using (2), that

$$0 = \delta(u) = \delta^2(k)^3 + 2v = \delta^2(k)^3.$$

Thus $\delta^2(k)^3 = 0$ so that, by (2),

$$(\delta^2(k)\, K\delta^2(k))\, K(\delta^2(k)\, K\delta^2(k)) = 0.$$

Two applications of Lemma 3 now give that $\delta^2(k) = 0$ for all $k \in K$. But then $\delta^2(k\,\delta(k)\,k) = 0$, that is, $2\delta(k)^3 = 0$ whence $\delta(k)^3 = 0$. Repeating the argument used above for $\delta^2(k)$ it follows that $\delta(k) = 0$. Thus $A \subseteq \ell(SD(R)$.

Example (i) at the beginning of Section 3 indicates that if $SD(R)$ is to be semiprime then further conditions on R are required. The condition which we shall impose is the following. Suppose that R is semiprime and let $\mathscr{P}$ denote the set of *-prime ideals of R of the form $P \cap P^*$ where P is prime and $2R \subseteq P$. Let $X = \cap\{Q \in \mathscr{P}\colon R/Q$ does not satisfy $S_4\}$ and $Y = \cap\{Q \in \mathscr{P}\colon R/Q$ does satisfy $S_4\}$. We shall say that R is *independent of* S_4 if, all $r \in R$, r is central modulo X implies that $r \in Z$.

Lanski (1978) defines the term 'S_8-free'. Replacing 8 by $2n$ throughout Lanski's definition one obtains the notion of an S_{2n}-free semiprime ring. It is straightforward to check that if R is semiprime and S_4-free then R is independent of S_4 in the above sense.

LEMMA 6. *If R is semiprime and independent of S_4 and if $r \in R$ is such that $[K^2, r] = 0$ then $r \in Z$.*

PROOF. Let P be any prime ideal of R such that $2R \subseteq P$. Denote images in $\bar{R} = R/P$ using $\bar{\ }$. Then $[\overline{K^2}, \bar{r}] = \bar{0}$ so by Lemma 8 of Lanski and Montgomery (1972) either $\overline{K^2} \subseteq Z(\bar{R})$ or $\bar{r} \in Z(\bar{R})$. Suppose that $\overline{K^2} \subseteq Z(\bar{R})$. Then $[K^2, R] \subseteq P$ and it follows immediately that $[K^2, R] \subseteq P \cap P^*$. Let $Q = P \cap P^*$ and $\hat{R} = R/Q$. Denote images in $\hat{R}$ using $\hat{\ }$. Then $[\hat{K}^2, \hat{R}] = \hat{0}$. But * induces an involution, also denoted *, on $\hat{R}$. Let $\tilde{K}$ denote the set of antisymmetric elements of $\hat{R}$ under the induced involution. Let $\hat{y} = [y+Q] \in \tilde{K}$. Then $[y+Q] = [-y^*+Q]$ so that $[2y+Q] = [y-y^*+Q]$. Thus $2\tilde{K} \subseteq \hat{K}$. But $[\hat{K}^2, \hat{R}] = \hat{0}$ and it follows, since $2R \subseteq P$, that $[\tilde{K}^2, \hat{R}] = 0$. In particular $[\tilde{K}^2, \tilde{K}^2] = 0$ and, by Lemma 2 of Lanski (1976) every 2-torsion-free prime factor ring of R satisfies S_4 so that R satisfies S_4. Thus $\bar{r} \in Z(\bar{R})$ or R/Q satisfies S_4. Applying this to P^* rather than P we obtain that $\hat{r} \in Z(\hat{R})$ or R/Q satisfies S_4. It follows from the definition of independence of S_4 that $r \in Z$.

THEOREM 5. *If R is semiprime and independent of S_4 then*

(i) *$K \cap Z$ is a semiprime ideal of K;*

(ii) *$SD(R)$ is a semiprime Lie ring.*

PROOF. (i) By Theorem 3, $\ell(K)$ is a semiprime ideal of K. Let $k \in \ell(K)$ so that $[k, K] = 0$. Then $[k, K^2] = 0$ and, by Lemma 6, $k \in Z$. Thus $l(K) = K \cap Z$ and the result follows.

(ii) Suppose that $0 \neq A$ is an ideal of $SD(R)$ such that $[A, A] = 0$. Then, by Theorem 4, $\delta(K) = 0$ for all $\delta \in A$. Let $0 \neq \delta \in A$. Then $\delta(K^2) = 0$ so that, since K^2 is a Lie ideal of R, $[K^2, \delta(R)] = 0$. By Lemma 6 $\delta(R) \subseteq Z$. Since $\delta(K) = 0$ it follows that $\delta(R) \subseteq S$ so that $\delta(R) \subseteq Z \cap S$. Let $z \in Z \cap S$. Then $z\delta \in SD(R)$ so that $[\delta, [\delta, z\delta]] \in [I, I] = 0$, that is $\delta^2(z)\,\delta = 0$. In particular $\delta^2(z)^2 = 0$ so that $\delta^2(z) = 0$ since R is semiprime and $\delta^2(z)$ is central. Replacing z by z^2 we obtain $2\delta(z)^2 = 0$ and hence, $\delta(z) = 0$. Since $\delta(R) \subseteq Z \cap S$ it follows that $\delta^2(R) = 0$. For $r \in R$, $\delta(r) \in Z$ so one can replace z by r in the above argument to obtain that $\delta(r) = 0$ for all $r \in R$, contradicting the choice of δ. The result follows.

REMARKS. (i) Lanski (1978) has shown that if R is semiprime and S_8-free then $K \cap Z$ is a semiprime ideal of K. Theorem 5(i) shows that 8 may be reduced to 4.

(ii) An advantage of the approach given here is that the examples satisfying S_4 which are excluded in Theorem 5 remain part of the general theory through Theorems 3 and 4.

4. The commutative case

Throughout this section it will be assumed that R is commutative and 2-torsion-free.

LEMMA 7. *If R has an identity element and is *-prime and if $0 \neq \delta \in SD(R)$ then the Lie subring $S\delta = \{s\delta : s \in S\}$ of $SD(R)$ is a prime Lie ring.*

PROOF. See C. R. Jordan and D. A. Jordan (1978b), Theorem 7.

The proof of the next theorem is based on that of Theorem 9 of the same paper.

THEOREM 6. *If R is *-prime then $SD(R)$ is a prime Lie ring.*

PROOF. There is no loss of generality in assuming that R has an identity element. Suppose that there exist non-zero ideals A, B of $SD(R)$ such that $[A, B] = 0$. Choose $0 \neq \delta \in A$, $0 \neq \gamma \in B$ and $s \in S$ such that $\gamma(s) \neq 0$. The choice of s is justified by Proposition 6 (vii) of C. R. Jordan and D. A. Jordan (1978b). Since $\delta \in A \cap S\delta$ it follows from Lemma 7 that $B \cap S\delta = 0$. But $[s\delta, \gamma] \in B$ and, since $[\delta, \gamma] = 0$, $[s\delta, \gamma] = \gamma(s)\,\delta \neq 0$, giving a contradiction. The result follows.

THEOREM 7. *If R is semiprime then $SD(R)$ is a semiprime Lie ring.*

PROOF. Suppose that A is an ideal of $SD(R)$ such that $[A, A] = 0$. Let $\delta \in A$, $s \in S$. Then $\delta^2(s)\,\delta = [\delta, [\delta, s\delta]] = 0$ so that $\delta^2(s)^2 = 0$. As in the proof of Theorem 5(ii), it follows that $\delta(S) = 0$. Let $k \in K$. Then $k^2 \in S$ so that $\delta(k^2) = 0$, that is $2k\delta(k) = 0$, whence $k\delta(k) = 0$. Since δ is symmetric $\delta(k) \in K$ so that $\delta(k)\,\delta^2(k) = 0$. It follows that $0 = \delta(k\delta(k)\,\delta(k)) = \delta(k)^3$ and, hence since R is semiprime, that $\delta(k) = 0$. Thus $\delta(K) = 0$ and, because $\delta(S) = 0$, it follows that $\delta(R) = 0$. The result follows.

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