ON A DIOPHANTINE PROBLEM

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Introduction. If $a_1, a_2, \ldots, a_k$ are relatively prime positive integers then the equation

$$(1) \qquad a_1x_1 + \ldots + a_kx_k = n$$

always has solutions in non-negative x_i for n sufficiently large. Sylvester called the number of non-negative solutions of (1) the *denumerant* of the equation. We shall denote the denumerant by $n(a_1, \ldots, a_k)$.

We define $N_j(a_1, \ldots, a_k)$ to be the smallest positive integer such that $n(a_1, \ldots, a_k) \geqslant j$ for all $n \geqslant N_j(a_1, \ldots, a_k)$.

Using the known result

$$n(1, 2, 3) = [(n^2 + 6n + 15)/12]$$

we can readily show that

$$N_j(1, 2, 3) = [\sqrt{(12j - 6)} - 2].$$

In general the computation of $N_j(a_1, \ldots, a_k)$ is a difficult job. Our interest here is in $N_1(a_1, \ldots, a_k)$. The present author has given **(4)** the value of $N_1(a_1, \ldots, a_k)$ when the a_i are in arithmetic progression. Brauer and Seelbinder **(2; 3)** have given various upper bounds for $N_1(a_1, \ldots, a_k)$ in the general case. In this paper we concern ourselves with an upper bound for $N_1(m, m + a, m + b)$ where $(a, b) = 1$. Our upper bound is in many cases best possible. We also give a theorem concerning an upper bound for

$$N_1(m, m + a_1, \ldots, m + a_k)$$

where $a_1, a_2, \ldots, a_k$ are relatively prime positive integers.

1. Three lemmas. Throughout this section

$$0 < a < b, \ (a, b) = 1, \ P = (a - 1)(b - 1).$$

Since $N_1(a, b) = P$ **(1**, p. 124**)**, the equation $ax + by = n$ is solvable in non-negative integers for all $n \geqslant P$. The non-negative solution with smallest x is denoted by x_n, y_n. If x, y is any non-negative solution of $ax + by = n$ then

$$x + y \geqslant x_n + y_n.$$

Also

$$x_{n+b} + y_{n+b} = x_n + y_n + 1.$$

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Since $(a, b) = 1$, the numbers αa for $0 \leqslant \alpha \leqslant b - 1$ constitute a complete system of residues modulo b. Hence $n \equiv \alpha a \pmod b$ has a unique solution with $0 \leqslant \alpha \leqslant b - 1$. Denote this solution by α_n.

LEMMA 1. $x_n = \alpha_n, \quad y_n = (n - a\alpha_n)/b.$

Proof. Clearly

$$a\alpha_n + b((n - a\alpha_n)/b) = n.$$

Hence all solutions of $ax + by = n$ are given by

$$\alpha_n + bt, \ (n - a\alpha_n)/b - at.$$

For non-negative solutions we must have

$$t \geqslant -\alpha_n/b > -1.$$

Hence all non-negative solutions have $x \geqslant \alpha_n$.

LEMMA 2. *If $n > P$, $\alpha_n = b - 1$ and $1 \leqslant c \leqslant n - P$ then*

$$x_n + y_n \geqslant x_{n-c} + y_{n-c} + 1.$$

Proof. We have

$$\begin{aligned} x_n + y_n &= \alpha_n + (n - a\alpha_n)/b = (n + \alpha_n(b - a))/b \\ &= (n + (b - 1)(b - a))/b > (n - c + \alpha_{n-c}(b - a))/b \\ &= x_{n-c} + y_{n-c}. \end{aligned}$$

LEMMA 3.

$$\max_{n \in K} (x_n + y_n) \leqslant b - 2 + [m/b]$$

where K consists of those n satisfying $P \leqslant n \leqslant P + m - 1$, $m \geqslant 1$.

Proof. Let B be the smallest integer greater than $P + m - 1$ which has $\alpha_B = b - 1$. Then

$$B > P + m - 1 \geqslant n$$

and therefore $B - n \geqslant 1$. Also $B - n \leqslant B - P$. Hence taking the n, c of Lemma 2 to be B, $B - n$ respectively, we see that

$$x_B + y_B \geqslant x_{B-(B-n)} + y_{B-(B-n)} + 1 = x_n + y_n + 1.$$

This is true for all $n \in K$, so

$$\max_{n \in K} (x_n + y_n) \leqslant x_B + y_B - 1 = (B + \alpha_B(b - a))/b - 1.$$

For arbitrary m we have

$$B = P + [m/b]\, b - 1 + b.$$

Hence

$$\begin{aligned} \max_{n \in K} (x_n + y_n) &\leqslant (P + [m/b]\, b - 1 + b + (b - 1)(b - a))/b - 1 \\ &= b - 2 + [m/b]. \end{aligned}$$

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2. The main theorem.

THEOREM 1. $N_1(m, m+a, m+b) \leqslant m(b-2+[m/b]) + (a-1)(b-1)$ *where* $0 < a < b$, $(a, b) = 1$, $m \geqslant 2$.

Proof. Let P, K, x_n, y_n be as in §1. Define

$$Q = \max_{n \in K} (x_n + y_n).$$

Now define $\bar{x}_j$, $\bar{y}_j$, $\bar{z}_j$ for $j \geqslant P$ by the following:

$$\bar{x}_j = x_i, \; \bar{y}_j = y_i \text{ for } j \equiv i \pmod{m}, \qquad P \leqslant i \leqslant P+m-1,$$
$$\bar{z}_j = Q + [(j-P)/m] - \bar{x}_j - \bar{y}_j.$$

Then

$$(2) \quad \begin{aligned} m\bar{z}_j + (m+a)\bar{x}_j + (m+b)\bar{y}_j &= m(\bar{z}_j + \bar{x}_j + \bar{y}_j) + a\bar{x}_j + b\bar{y}_j \\ &= mQ + m[(j-P)/m] + a\bar{x}_j + b\bar{y}_j. \end{aligned}$$

As j runs over

$$P + sm, \; P + sm + 1, \ldots, P + sm + m - 1$$

for $s \geqslant 0$, the right side of this equation runs over

$$mQ + ms + P, \; mQ + ms + P + 1, \ldots, mQ + ms + P + m - 1.$$

Hence as j runs over the integers greater than or equal to P and s runs over the integers greater than or equal to 0, the left side of (2) runs over the integers greater than or equal to $P + mQ$. Replacing Q by its upper bound from Lemma 3 now gives the desired result.

Essentially the same proof yields the following:

THEOREM 2. $N_1(m, m+a_1, \ldots, m+a_k) \leqslant P + mQ$ *where* $a_1, a_2, \ldots, a_k$ *are relatively prime positive integers,*

$$P = N_1(a_1, \ldots, a_k), \quad Q = \max_{n \in K} (x_{1n} + \ldots + x_{kn}),$$

K is the set of n such that $P \leqslant n \leqslant P + m - 1$, *and* $x_{1n}, \ldots, x_{kn}$ *is a non-negative solution of* (1) *with smallest sum.*

3. Special cases. Let $\tilde{N}_1(m, m+a, m+b)$ denote the upper bound in Theorem 1. We compare $\tilde{N}_1$ with N_1 in a few special cases.

(a) By the main result in (**4**; see also **2**, Theorem 7, p. 310),

$$N_1(m, m+1, m+2) = m[\tfrac{1}{2}m] = \tilde{N}_1(m, m+1, m+2).$$

(b) By a result stated in (**4**),

$$N_1(m, m+1, m+b) = \tilde{N}_1(m, m+1, m+b)$$

for

$$m \equiv -1 \pmod{b}, \qquad m \geqslant b^2 - 5b + 3;$$

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and

$$N_1(m, m+1, m+b) = \tilde{N}_1(m, m+1, m+b) - (m - b[m/b])$$

for

$$m \equiv -1 \pmod{b}, \qquad m \geqslant b^2 - 4b + 2.$$

(c) By direct evaluation it is not difficult to show

$$N_1(m, m+2, m+3) = \tilde{N}_1(m, m+2, m+3).$$

(d) By computation for $2 \leqslant m \leqslant 16$ we find

$$N_1(m, m+2, m+5) = \tilde{N}_1(m, m+2, m+5)$$

for

$$m = 5, 8, 10, 13, 14, 15$$

and not for

$$m = 2, 3, 4, 6, 7, 9, 11, 12, 16.$$

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