# PRIMARY IDEALS AND PRIME POWER IDEALS

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**1. Introduction.** This paper is concerned with the ideal theory of a commutative ring $R$. We say $R$ has Property ($\alpha$) if each primary ideal in $R$ is a power of its (prime) radical; $R$ is said to have Property ($\delta$) provided every ideal in $R$ is an intersection of a finite number of prime power ideals. In (**2**, Theorem 8, p. 33) it is shown that if $D$ is a Noetherian integral domain with identity and if there are no ideals properly between any maximal ideal and its square, then $D$ is a Dedekind domain. It follows from this that if $D$ has Property ($\alpha$) and is Noetherian (in which case $D$ has Property ($\delta$)), then $D$ is Dedekind. This suggests the following question: In the definition of a Dedekind domain (i.e., every ideal is a product of prime powers) can "product" be replaced by "intersection"? This paper answers this question in the affirmative. In fact, Theorem 11 shows that ($\delta$) holds in a commutative ring with identity if and only if $R$ is a Z.P.I. ring (i.e., every ideal is a product of prime ideals). We note that this implies that Property ($\alpha$) follows from Property ($\delta$) in case $R$ has an identity (Theorem 8 shows that Property ($\delta$) implies Property ($\alpha$) in any commutative ring).

In addition some results concerning the implications of Property ($\alpha$) are obtained. For example, if the ascending chain condition for prime ideals holds in the domain $D$ with identity, then ($\alpha$) holds in $D$ if and only if $D_P$ is a discrete valuation ring for each proper prime $P$ of $D$. This result is related to (**1**, Theorem 1.0) which shows for an integral domain $J$ with identity that $J_P$ is a discrete rank-one valuation ring if and only if each ideal of $J$ with prime radical is a prime power. Another result in this vein is (**9**, Theorem 3.8): if the ascending chain condition for prime ideals holds in the integral domain $J$ with identity, then $J$ is a Prüfer domain (i.e. $J_P$ is a valuation ring for each prime ideal $P$ of $J$) if and only if every primary ideal of $J$ is a valuation ideal (**13**, p. 340).

The notation and terminology are those of (**12**; **13**) with one exception: $\subseteq$ denotes containment and $\subset$ denotes proper containment. As stated previously, all rings considered are assumed to be commutative.

**2. Rings with Property ($\alpha$).** In this section we derive some consequences of Property ($\alpha$). We see at once that if $R$ has Property ($\alpha$), then so does any homomorphic image $R/A$ of $R$ and any quotient ring $R_M$ of $R$. We say that a prime ideal $P$ of a ring $R$ is *unbranched* if $P$ itself is the only $P$-primary ideal of $R$; otherwise we say $P$ is *branched*. $R$ is a *u-ring* if $R$ is unbranched. $R$ is

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said to have *dimension* $n$ if there is a strictly ascending chain of $n+1$ prime ideals ($\neq R$) of $R$ but no such chain of $n+2$ prime ideals.

THEOREM 1. *Suppose* ($\alpha$) *holds in the ring* $R$ *and* $M$ *is a proper ideal of* $R$ *such that* $M$ *is a minimal prime of* $P+(x)$ *for some prime* $P$ *of* $R$ *and some* $x \in M-P$. *Then the powers of* $M$ *properly descend,* $P \subseteq \bigcap_1^\infty M^n$ *and* $\bigcap_1^\infty M^n$ *is the intersection of all* $M$*-primary ideals of* $R$.

*Proof.* For $i$ a positive integer, $M$ is a minimal prime of $P+(x^i)$. If $Q_i$ is the isolated primary component of $P+(x^i)$ belonging to $M$, then $Q_1 \supset Q_2 \supset \ldots$. (We have $x^i \in Q_i - Q_{i+1}$.) For each $i$, $Q = M^{n_i}$ for some $n_i$. Hence $n_1 < n_2 < \ldots$ so that the powers of $M$ properly descend. Further,

$$P \subseteq \bigcap_1^\infty M^{n_i} = \bigcap_1^\infty M^i.$$

If $Q$ is the intersection of all $M$-primary ideals, then because $M^{n_i}$ is $M$-primary,

$$\bigcap_1^\infty M^{n_i} \supseteq Q.$$

Since each $M$-primary ideal is a prime power, then

$$Q \supseteq \bigcap_1^\infty M^j.$$

Hence

$$Q = \bigcap_1^\infty M^j.$$

THEOREM 2. *Suppose* ($\alpha$) *holds in the ring* $R$ *and* $P$ *and* $M$ *are prime ideals of* $R$ *such that* $P \subset M \subset R$. *Then*

$$P \subseteq \bigcap_1^\infty M^n.$$

*Proof.* Let $m \in M-P$ and let $P_0$ be a minimal prime of $P+(m)$ contained in $M$. By Theorem 1,

$$P \subseteq \bigcap_1^\infty P_0{}^n \subseteq \bigcap_1^\infty M^n.$$

COROLLARY 1. *If* ($\alpha$) *holds in the ring* $R$ *and* $P \supset \bar{P}$ *are prime ideals in* $R$ *with* $R \neq P$, *then* $P$ *is idempotent if and only if* $P$ *is the union of a chain of primes properly contained in* $P$.

*Proof.* If $P \neq P^2$, Theorem 2 shows that $P$ is not the union of such a chain. Conversely, if $P$ is not the union of such a chain, then Zorn's lemma implies that we can find a prime ideal $M \subset P$ such that there are no prime ideals properly between $M$ and $P$. Thus if $p \in P-M$, $P$ is a minimal prime of $M+(p)$. Then by Theorem 1, $P \supset P^2$.

THEOREM 3. *Suppose* ($\alpha$) *holds in the rings* $R$ *and* $M$ *is a proper prime ideal of* $R$ *such that* $M \supset P_1$ *for some prime ideal* $P_1$. *Then*

$$P_0 = \bigcap_1^\infty M^n$$

*is a prime ideal containing each prime ideal properly contained in* $M$. *Further, each* $M^i$ *is primary.*

*Proof.* The theorem is obvious if $M = M^2$. Suppose $M \supset M^2$ and let $\mathfrak{S}$ be the collection of prime ideals properly contained in $M$. By assumption, $\mathfrak{S}$ is not empty and Theorem 2 shows that $\mathfrak{S}$ is inductive under $\subseteq$. Hence $\mathfrak{S}$ contains a maximal element $P$ such that $P$ is prime in $R$, $P \subset M$, and there are no primes properly between $P$ and $M$. Now if $\bar{R} = R_M/P^e$, where "$e$" denotes extension with respect to the quotient ring $R_M$ (**12**, p. 218), $\bar{R}$ has Property ($\alpha$). In $\bar{R}$, each proper ideal is $\bar{M}$-primary where $\bar{M} = M^e/P^e$ and hence is a power of $\bar{M}$. Therefore $\bar{R}$ is a Dedekind domain. Consequently

$$P^e/P^e = \bigcap_1^\infty \bar{M}^k = \bigcap_1^\infty (M^e/P^e)^k = \bigcap_1^\infty ([M^k]^e + P^e/P^e)$$
$$= \bigcap_1^\infty (M^k + P)^e/P^e = \bigcap_1^\infty (M^k)^e/P^e.$$

It follows that

$$P^e = \bigcap_1^\infty (M^k)^e \quad \text{and} \quad P = \bigcap_1^\infty M^{(k)}.$$

This implies that the symbolic powers of $M$ properly descend, and it follows by induction that $M^n = M^{(n)}$ for every positive integer $n$ (for, if $M^k = M^{(k)}$ then $M^k = M^{(k)} \supset M^{(k+1)} \supseteq M^{k+1}$ and $M^{(k+1)} = M^{k+1}$ since $M^{(k+1)}$ is a power of $M$). This means that each power of $M$ is primary, $\bigcap_1^\infty M^k = P$ is a prime ideal, and each prime ideal properly contained in $M$ is contained in $P$.

COROLLARY 2. *If* ($\alpha$) *holds in the domain* $D$, *then prime power ideals of* $D$ *are primary.*

THEOREM 4. *Suppose* ($\alpha$) *holds in the ring* $R$. *If the prime ideal* $M$ *is non-maximal in the set of proper prime ideals of* $R$ *or if* $M$ *is not a minimal prime of* (0), *then*

$$M \subseteq \bigcap_1^\infty R^i.$$

*If* $M$ *is both maximal in the set of proper prime ideals of* $R$ *and a minimal prime of* (0), *and if* $M$ *is not contained in* $\bigcap_1^\infty R^i$, *then* $M$ *is unbranched.*

*Proof.* Suppose $M$ is non-maximal in the set of proper prime ideals of $R$. Let $P$ be a proper prime of $R$ properly containing $M$, let $p \in P - M$, and let

$P_0$ be a minimal prime of $M + (p)$ contained in $P$. By Theorem 1, $P_0 \supset P_0^2$. By Theorem 3,

$$M \subseteq \bigcap_1^\infty P_0^n \subseteq \bigcap_1^\infty R^n.$$

Now suppose $M$ is maximal in the set of proper prime ideals of $R$ but not a minimal prime of (0). If $M = M^2$, then

$$M = \bigcap_1^\infty M^n \subseteq \bigcap_1^\infty R^n.$$

If $M \supset M^2$, consider $x \in R - M$. For each integer $i$, $Q_i = M + (x^i)$ is $R$-primary since any ideal with radical $R$ is $R$-primary. If $Q_i = Q_{i+1}$ for some $i$, then $R/M$ contains an identity; for if $x^i = m + rx^{i+1} + \lambda x^{i+1}$, (with $\lambda$ an integer, $r \in R$, $m \in M$), then $sx^i \equiv s(rx + \lambda x)x^i \pmod M$ for each $s \in R$. Since $M$ is a prime ideal and $x^i \notin M$, then $s \equiv s(rx + \lambda x) \pmod M$ for all $s \in R$. Now Theorem 3 implies that $M^2$ is $M$-primary, so that $R/M^2$ also contains an identity (**5**, Lemma 3, p. 75). Therefore

$$[R/M^2] = [R/M^2]^2 = [R^2 + M^2/M^2] = R^2/M^2$$

and $R = R^2$. Hence

$$M \subset R = \bigcap_1^\infty R^n.$$

If $Q_i \supset Q_{i+1}$ for each $i$, then $Q_i = R^{n_i}$ where $n_1 < n_2 < \ldots$ so that

$$M \subseteq \bigcap_1^\infty Q_i = \bigcap_1^\infty R^{n_i} = \bigcap_1^\infty R^j.$$

In any case

$$M \subseteq \bigcap_1^\infty R^n.$$

Finally, if $M$ is both maximal among the set of proper primes of $R$ and a minimal prime of (0) and if $M$ is not contained in $\bigcap_1^\infty R^i$, then the preceding paragraph shows that if $Q_i = M + (x^i)$ where $x \in R - M$, then $Q_i = Q_{i+1}$ for some $i$ and $R/M$ contains an identity. If $M$ were branched, we could find $n > 1$ such that $M^n$ is $M$-primary. Then $R/M^n$ contains an identity so that

$$R/M^n = [R/M^n]^2 = R^2/M^n$$

and $R = R^2$, a contradiction. Hence $M$ is unbranched as asserted.

COROLLARY 3. *If* ($\alpha$) *holds in the ring $R$ where either $R$ is a domain or $R$ is an idempotent ring, then given prime ideals $P_1$ and $P_2$ of $R$ with $P_1 \subset P_2$,*

$$P_1 \subseteq \bigcap_1^\infty P_2^n.$$

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*Proof.* By Theorem 2, we need only examine the case when $P_2 = R$. If $R = R^2$, it is obvious that

$$P_1 \subseteq \bigcap_{1}^{\infty} R^n.$$

If $R$ is a domain and $P_1 \neq (0)$, the statement follows from Theorem 4. For $P_1 = (0)$ it is obvious.

We turn our attention now to the case of a ring $R$ satisfying the ascending chain condition for prime ideals and in which $(\alpha)$ holds. Our principal result is contained in Corollary 4, which shows that an integral domain with identity satisfying these properties is a Prüfer domain. We begin with

THEOREM 5. *Let $R$ be a quasi-local ring in which the ascending chain condition for prime ideals holds. Suppose further that $(\alpha)$ holds in $R$ and that $(0)$ is primary for the ideal $P$. Then given $x, y \in R$, either $x \in (y)$ or $y \in (x)$.*

*Proof.* If $P_0$ is a proper prime of $R$ distinct from $P$, the ascending chain condition for prime ideals implies that there exists a prime ideal $P_1 \subset P_0$ such that there are no prime ideals properly between $P_1$ and $P_0$. Hence if $p_0 \in P_0 - P_1$, $P_0$ is a minimal prime of $P_1 + (p_0)$. Theorem 1 then shows that $P_0 \neq P_0^2$ and Theorem 3 implies that there is a prime ideal $N(P_0) \subset P_0$ such that $N(P_0)$ contains each prime ideal properly contained in $P_0$. It follows that the prime ideals of $R$ are linearly ordered by **(9**, Lemma 3.4**)**.

Now suppose $M$ is the maximal ideal of $R$. If $M = P$, then $M$ is the only prime ideal distinct from $R$. Hence if $x, y \in M$, $(x)$ and $(y)$ are $M$-primary since $\sqrt{(x)} = \sqrt{(y)} = M$. Hence $x \in (y)$ or $y \in (x)$ since $(\alpha)$ holds in $R$. If, say, $x \notin M$, $x$ is a unit in $R$ and $y \in (x) = R$.

If $M \neq P$, then consider $M_1 = N(M)$. The prime ideal $M_1$ has the following property (#):

(#) *If $r, s \in R$, and $r \notin M_1$, then $r \in (s)$ or $s \in (r)$.*

If $r$ or $s$ is a unit, this is clear. If $r, s \in M$ and if $s \in M_1$, then $r \in M - M_1$, $(r)$ is $M$-primary so that $(r) \supset M_1$ by Theorem 3 and $s \in M_1$. If $s \in M - M_1$ also, then $(s)$ is $M$-primary and $(r) \subseteq (s)$ or $(s) \subseteq (r)$ since $(\alpha)$ holds in $R$.

Suppose $P_0 \subseteq M_1$ is a prime ideal of $R$ such that every prime properly containing $P_0$ has Property (#). We show that $P_0$ has Property (#). Thus suppose that $x, y \in R$, $x \notin P_0$. Let $P_1$ be a minimal prime of $(x)$. Because of the linear ordering of the prime ideals of $R$, $P_1 \supset P_0$. Let "$e$" denote extension of ideals of $R$ with respect to the quotient ring $R_{P_1}$. Then $(x)^e$ is $P_1^e$-primary. As shown above, $\bar{y} \in (x)^e$ or $\bar{x} \in (y)^e$, say $\bar{x} \in (y)^e$, so that $xv = uy$ for some $u, v \in R$, $v \notin P_1$. By the hypothesis concerning $P_1$, $u \in (v)$ or $v \in (u)$. If, say, $u = wv$, then $xv = wvy$ and $v(x - wy) = 0$. But $(0)$ is $P$-primary and $v \notin P_1 \supset P_0 \supseteq P$. Thus $x - wy = 0$ and $x \in (y)$. Because the ascending chain condition for prime ideals holds in $R$, the ideal $P$ has Property (#).

Then if $x, y \in R$ and $x \notin P$ or $y \notin P$, then $x \in (y)$ or $y \in (x)$. On the other hand, if $x, y \in P$, then $P$ is a minimal prime of $(x)$ and of $(y)$. Since $(\alpha)$ holds in $R$, $x$ is in the isolated primary component of $(y)$ belonging to $P$ or $y$ is in the isolated primary component of $(x)$ belonging to $P$. If, say, $vx = uy$ where $v \notin P$, then we have $u \in (v)$ or $v \in (u)$ since $P$ has Property (#). If $v = su$, then $u \notin P$ since $v \notin P$ and we have $u(y - sx) = 0$. Since $(0)$ is $P$-primary, $y = sx$ and $y \in (x)$. This completes the proof of the theorem.

COROLLARY 4. *Suppose $(\alpha)$ and the ascending chain condition for prime ideals hold in the domain $D$. If $P$ is a proper prime ideal of $D$, then $D_P$ is a valuation ring. Hence if $D$ contains an identity, $D$ is a Prüfer domain.*

*Note.* Corollary 4 may also be obtained by an application of **(3**, Corollary 2.4**)** and **(9**, Theorem 3.8**)** once we observe that Theorem 3 shows that if $(\alpha)$ holds in the domain $D$ with identity, then each prime ideal $P$ of $D$ is an *S-ideal* according to the terminology of **(3)**.

COROLLARY 5. *If $(\alpha)$ and the ascending chain condition for prime ideals hold in the ring $R$ with identity, then given $P$, a minimal prime of* (0), *there is an integer $k$ such that $P^{k+1} = P^{k+2} = \ldots$ .*

*Proof.* Let $P^k = P^{(k)}$ be the isolated primary component of $P$ belonging to (0). Let $M$ be a maximal ideal containing $P$ and let "$e$" denote extension of ideals with respect to the quotient ring $R_M$. Then $S = R_M/(P^k)^e$ is a ring satisfying the hypothesis of Theorem 5. Hence if $x \in M - P$,

$$P^e/(P^k)^e \subseteq [P^k + (x)]^e/(P^k)^e$$

by Theorem 5. Therefore $P^e \subseteq [P^k + (x)]^e$. Now given $v \in P^k$, there is a $y \notin P$ such that $vy = 0$ by definition of $P^k$. Thus

$$P^e \subseteq [P^k + (y)]^e \quad \text{and} \quad (vP)^e = (v)^e P^e \subseteq [vP^k]^e \subseteq [vP]^e.$$

Hence $(vP)^e = (vP^k)^e$. This holds for each maximal ideal of $R$ so that $vP = vP^k$ for each $v \in P^k$ **(13**, p. 94**)**. In particular, $P^{k+1} \subseteq P^{2k}$ so that $P^{k+1} = P^{k+2} = \ldots$ .

THEOREM 6. *Let $P$ be a proper prime ideal of a valuation ring $R$.*

(a) *In order that $P$ be unbranched it is necessary and sufficient that $P$ be the union of a chain of prime ideals properly contained in $P$. If $P$ is unbranched, $P$ is idempotent.*

(b) *If $P$ is branched, then the intersection $M$ of all $P$-primary ideals is a prime ideal containing each prime ideal properly contained in $P$.*

(c) *If $P$ is branched, then each $P$-primary ideal is a power of $P$ if and only if $P \neq P^2$.*

*Proof.* (a) follows from **(3**, Lemmas 1.6, 3.4**)**, and (b) follows from **(9**, Lemma 2.12**)**.

To prove (c), note that if $P \supset P^2$, then given $Q$ primary for $P$, $Q$ contains a power of $P$ by **(3**, Lemma 1.6**)**. If, say, $Q$ contains $P^{n+1}$ but not $P^n$, then choose

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$x \in P^n - Q$. We have $Q \subset (x)$ so that $Q = xQ_1$ for some ideal $Q_1$ of $R$. Since $Q$ is primary and $x \notin Q$, we must have $Q_1 \subseteq P$. Thus

$$Q = xQ_1 \subseteq P^n \cdot P = P^{n+1} \quad \text{and} \quad Q = P^{n+1}.$$

Hence if $P \neq P^2$, $P$-primary ideals are prime powers. The converse is evident since $P$ is branched.

Now suppose $R_v$ is the valuation ring of a valuation $v$ with value group $G$. If $P$ is a branched prime ideal of $R$ and if $M$ is the intersection of all $P$-primary ideals, we consider the isolated subgroups $\Delta_2$ and $\Delta_1$ of $G$ corresponding to $M$ and $P$, respectively; see **(13**, p. 40**)**. Since there are no prime ideals properly between $M$ and $P$, there are no isolated subgroups properly between $\Delta_1$ and $\Delta_2$ so that $\Delta_2/\Delta_1$ has rank one and its elements may be considered to be real numbers **(13**, p. 45**)**. Let $H$ denote the set of positive elements of $\Delta_2/\Delta_1$. Part (c) of Theorem 6 shows that each $P$-primary ideal is a power of $P$ if and only if $P \neq P^2$. Because $H - (H + H)$, where $H + H = \{g + h | g, h \in H\}$, is the set of positive elements of $\Delta_2/\Delta_1$ corresponding to $P^2$, we have: each $P$-primary ideal is a power of $P$ if and only if $H \supset H + H$. Because $\Delta_2/\Delta_1$ has rank one, this is equivalent to the assertion that $\Delta_2/\Delta_1 \simeq Z$, the additive group of integers; see **(11**, p. 239**)**. In summary we can say: $(\alpha)$ holds in $R_v$ if and only if given $H_1 \subset H_2$ consecutive isolated subgroups of $G$, $H_2/H_1 \simeq Z$. In accordance with terminology used in case $R_v$ has finite rank, we shall call such a valuation ring *discrete* **(13**, p. 48**)**. In terms of its ideal theory, $R_v$ is discrete if and only if every idempotent prime in $R_v$ is unbranched. Equivalently, $R_v$ is discrete if and only if the only idempotent ideals in $R_v$ are unbranched prime ideals **(3**, Corollary 1.4**)**. To summarize we state

THEOREM 7. *Suppose the ascending chain condition for prime ideals holds in the integral domain $D$ with identity. Then $(\alpha)$ holds in $D$ if and only if $D_P$ is a discrete valuation ring for each proper prime ideal $P$ of $D$.*

The proof is immediate once we observe:

LEMMA 1. *If $D$ is an integral domain with identity such that $(\alpha)$ holds in $D_P$ for each proper prime $P$ of $D$, then $(\alpha)$ holds in $D$.*

*Proof.* Let $Q$ be primary in $D$ and let $P = \sqrt{Q}$. We show that $Q$ is a power of $P$. We need consider only the case when $Q \subset P$. Then $QD_P = P^k D_P$ for some $k$ since $(\alpha)$ holds in $D_P$. Now if $M$ is a maximal ideal of $D$ containing $P$, then Corollary 2 shows that $P^k D_M$ is primary in $D_M$. But $QD_M$ is also primary in $D_M$ so that

$$\begin{aligned} P^k D_M = (P^k D_M) D_P \cap D_M = P^k D_P \cap D_M &= QD_P \cap D_M \\ &= (QD_M) D_P \cap D_M = QD_M. \end{aligned}$$

Since this equality holds for each maximal ideal $M$ containing $Q$, this implies that $Q = P^k$ **(13**, p. 94**)** and $(\alpha)$ holds in $D$ as asserted.

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*Remarks.* If $E$ denotes the ring of even integers, then the ascending chain condition for prime ideals holds in $E$, and for each proper prime $P$ of $E$, $E_P$ is a discrete rank-one valuation ring; see (**7**, Lemma 3). Yet ($\alpha$) does not hold in $E$; (18) is (6)-primary but is not a power of (6).

In view of Theorem 6 one can easily see that the ring $S$ of (**9**, Section 5) is a domain in which ($\alpha$) holds. Yet $S$ is not integrally closed, and is therefore not a Prüfer domain. Hence Corollary 1 is false if the ascending chain condition for prime ideals is dropped from the hypothesis.

## 3. Rings with Property ($\delta$).

In this section we obtain a complete classification of rings satisfying Property ($\delta$). Theorems 11, 13, 14 contain these classifications.

THEOREM 8. *If* ($\delta$) *holds in the ring* $R$, *then* ($\alpha$) *holds in* $R$.

*Proof.* Let $Q$ be a primary ideal of $R$ and let $P = \sqrt{Q}$. Let

$$Q = \bigcap_{i=1}^{n} P_i^{e_i}$$

be a representation of $Q$ as an intersection of powers of distinct prime ideals. We have $P \subseteq P_i$ for each $i$. But since $P$ is prime and $P \supseteq \bigcap P_i^{e_i}$, we must have $P \supseteq P_i$ for some $i$ and therefore, say, $P = P_1$. Then $P_1 \subset P_i$ for $i \geqslant 2$, so $\bigcap_{i=1}^{n} P_i^{e_i}$ is not contained in $P_1$. (If $n = 1$, in particular if $P = R$, then we have $Q = P_1^{e_1}$ and $Q$ is a prime power). Hence since

$$P_1^{e_1}\Big(\bigcap_{i=2}^{n} P_i^{e_i}\Big) \subseteq Q$$

and $Q$ is $P_1$-primary, $P_1^{e_1} \subseteq Q$. It follows that $Q = P_1^{e_1}$ and $Q$ is a prime power.

THEOREM 9. *Let $R$ be a ring in which* ($\delta$) *holds and in which each prime ideal $P$ distinct from $R$ is contained in $\bigcap_{n=1}^{\infty} R^n$. Then an ideal of $R$ with prime radical is a prime power.*

*Proof.* Suppose $A$ is an ideal of $R$ with radical $P$, a prime ideal. If $P = R$, $A$ is $R$-primary and $A = R^k$ for some $k$ by Theorem 8. If $P \subset R$, then the hypothesis concerning $R$ implies that in at least one representation of $A$,

$$A = \bigcap_{i=1}^{n} P_i^{e_i}$$

as an intersection of powers of distinct prime ideals, each $P_i \neq R$. Then as in the proof of Theorem 8, we may assume that $P = P_1$ and $P \subset P_i$ for $i \geqslant 2$. Theorem 2 then shows that $P_i^{e_i} \supseteq P_1^{e_1}$ for $i \geqslant 2$ so that $A = P_1^{e_1}$ and our proof is complete.

THEOREM 10. *Let $D$ be a domain in which* ($\delta$) *holds. If $D$ is idempotent, $D$ is Dedekind. If $D$ is not idempotent, then each non-zero ideal of $D$ is a power of $D$.*

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*Proof.* Since ($\delta$) holds in $D$, ($\alpha$) also holds in $D$. Corollary 3 then implies that if $P$ is a proper prime ideal of $D$, $P \subseteq \bigcap D^n$. By Theorem 9, an ideal of $D$ with prime radical is a prime power. By **(8**, Lemma 6**)**, proper prime ideals of $D$ are maximal; see also **(1**, Lemma 1.1**)**.

Now if $D \supset D^2$, Corollary 1 shows that (0) is the only prime ideal of $D$ distinct from $D$. Hence each non-zero ideal of $D$ has radical $D$; it is therefore a power of $D$, and our conclusion holds.

If $D = D^2$, $D$ is Noetherian, as we shall presently see. Thus suppose $A$ and $B$ are proper ideals of $D$ and $A \subset B$. Let $A = M_1^{e_{11}} \cap \ldots \cap M_k^{e_{1k}}$ where each $M_j$ is a proper prime ideal and $M_i \neq M_j$ for $i \neq j$. If $B = P_1^{e_{21}} \cap \ldots \cap P_s^{e_{2s}}$, with the same requirements on the $P_i$'s, then each $P_i$ contains some $M_j$ and is therefore equal to $M_j$ since $M_j$ is maximal. Further, $P_i^{e_{2i}}$ is primary by Theorem 3 and

$$\prod_{v \neq j} M_v^{e_{1v}} \text{ is not contained in } P_i$$

so that $M_j^{e_{1j}} \subseteq P_i^{e_{2i}}$. It follows that $B$ is of the form $M_1^{e_{21}} \cap \ldots \cap M_k^{e_{2k}}$ for some set of non-negative integers $e_{21}, \ldots, e_{2k}$ where $e_{21} \leqslant e_{1i}$ for each $i$ and for at least one $i$, $e_{2i} < e_{1i}$. This shows that any ascending chain of ideals of $D$ whose first element is $A$ is finite. Because $A$ is arbitrary, $D$ is Noetherian. Hence $D$ contains an identity by **(4**, Corollary 2**)**. That each ideal of $D$ is a product of prime ideals then follows easily from the intersection representation and the fact that proper prime ideals are maximal.

*Note.* In **(6)** it is shown that an integral domain $D$, each non-zero ideal of which is a power of $D$, is characterized as the unique maximal ideal of a valuation ring $R$ such that $R = GF(p) + D$ for some prime number $p$.

Before proving Theorem 11, we shall need

LEMMA 2. *Let $S$ be a ring such that $SA = A$ for each ideal $A$ of $S$. If $A$ and $B$ are comaximal ideals of $S$, $A \cap B = AB$. If $A$ is comaximal with each of $B$ and $C$, $A$ is comaximal with $BC$.*

*Proof.* The proof is analogous to that given when $S$ contains an identity **(12**, p. 177**)**.

THEOREM 11. *Suppose $R$ is an idempotent ring in which ($\delta$) holds. Then each ideal of $R$ is a product of prime ideals. Consequently, $R$ is Noetherian, contains an identity, and is a* Z.P.I. *ring* **(10)**.

*Proof.* Let $P$ be prime in $R$, $P \neq R$. Then $R/P$ is a domain satisfying ($\delta$). By Theorem 10, $R/P$ has dimension $\leqslant 1$.

Next we note that if $P_1$ and $P_2$ are prime ideals of $R$ neither of which contains the other, then $P_1 + P_2 = R$. For if $P_1 + P_2 \subset R$, then $P_1 + P_2 \subseteq M \subset R$ since $P_1 + P_2$ is an intersection of prime power ideals and $R$ is idempotent. Then since $R$ has dimension $\leqslant 1$, $M$ is a minimal prime of $P_1 + (x)$ for any

$x \in M - P_1$ and a minimal prime of $P_2 + (y)$ for any $y \in M - P_2$. Then Theorems 1 and 3 show that

$$P_1 = \bigcap_1^\infty M^n = P_2,$$

a contradiction. Hence $P_1 + P_2 = R$.

Now note: If ($\delta$) holds in the idempotent ring $R$, Theorem 9 shows that each ideal of $R$ with prime radical is a prime power. Clearly $R$ is a $u$-ring also. Then (**8**, Theorem 15) shows that an ideal of $R$ with prime radical is primary. Because $R$ has dimension $\leqslant 1$, $Rx = (x)$ for each $x \in R$ (**8**, Theorem 5), and hence $RA = A$ for each ideal $A$ or $R$.

Having observed all these facts, let $A$ be an ideal of $R$ and let

$$A = P_1^{e_1} \cap \ldots \cap P_s^{e_s}$$

where $P_i \neq P_j$ for $i \neq j$ and where each $P_i$ is prime. In view of Corollary 3, we may suppose that $P_i$ does not contain $P_j$ for $i \neq j$. Then our previous observations and Lemma 2 show that $A = P_1^{e_1} \ldots P_s^{e_s}$. Hence each ideal of $R$ is a product of prime ideals and is therefore Noetherian (**10**, Satz 11). Since $R = R^2$, it then follows that $R$ contains an identity (**4**, Corollary 2).

*Remark.* Theorem 13 will show that in a ring without identity in which ($\delta$) holds, it need not be true that each ideal is a product of prime ideals.

THEOREM 12. *Suppose the ring $R$ has Property ($\delta$) and $R \neq R^2$. Then $R$ is Noetherian and* $\dim R \leqslant 0$.

*Proof.* We have previously observed that $\dim R \leqslant 1$. Suppose $\dim R = 1$ and let $P \subset M \subset R$ be a chain of prime ideals of $R$. Now ($\delta$) holds in the domain $R/P$. Theorem 10 then shows that $R/P$ is a Dedekind domain since $M/P$ is not a power of $R/P$. In particular,

$$R/P = [R/P]^2 = R^2 + P/P = R^2 + M^2 + P/P = R^2 + M^2/P = R^2/P,$$

the equality $R^2 + M^2 + P = R^2 + M^2$ following from Theorem 2. Thus $R = R^2$, a contradiction. It follows that $R$ has dimension $\leqslant 0$.

Now let $(0) = M_1^{e_1} \cap \ldots \cap M_k^{e_k} \cap R^e$ where the $M_i$ are distinct prime ideals properly contained in $R$. Then $\{M_1, \ldots, M_k, R\}$ is the set of prime ideals of $R$. Now $R$ is not contained in $\cup M_i$ by (**12**, p. 215), so if we choose $r \in R - \cup M_i$, then $(r) = R^t$ for some $t$. Note then that if $s \in R - R^2$, then $R = R^2 + (s)$. Then $R^2 = R^3 + sR$ so that

$$R = R^3 + Rs + (s) \subseteq R^3 + (s) \quad \text{and} \quad R = R^3 + (s).$$

Continuing we find that $R = R^t + (s) = (r, s)$ so that $R$ is finitely generated. Now consider any $M_i$, say $M_1$. Since $M_1$ is not contained in $\cup_{j \neq 1} M_j$, we may

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choose $a \in M_1 - \bigcup_{j \neq 1} M_j$. If $M_1 = M_1^2$, let $b$ be any element of $M_1$. If $M_1 \supset M_1^2$, let $b \in M_1 - M_1^2$. Then $(a, b) = M_1 \cap R^e$. If

$$M_1 \subseteq \bigcap_{n=1}^{\infty} R^n,$$

$(a, b) = M_1$. If $M_1$ is contained in $R^v$ but not in $R^{v+1}$, choose $c \in M_1 - R^{v+1}$. Then $(a, b, c) = M_1 \cap R^u$ where $u < v + 1$ so that $M_1 \subseteq R^u$ and $(a, b, c) = M_1$. In any case, $M_1$ is finitely generated. That $R$ is Noetherian now follows from the following lemma.

LEMMA 3. *If each prime ideal of the ring $R$ is finitely generated, then $R$ is Noetherian.*

*Proof.* For rings with identity, the lemma was first proved by Cohen (**2**, p. 29). For arbitrary $R$, let $S$ be a ring of characteristic zero obtained by adjoining an identity to $R$. $R$ is Noetherian if and only if $S$ is Noetherian, and if $A$ is an ideal of $S$ such that $A \cap R$ is finitely generated in $R$, then $A$ is finitely generated in $S$ by (**4**, Theorem 1). Thus, in our case, if $P$ is a prime ideal of $S$, $P \cap R$ is prime in $R$ and is therefore finitely generated in $R$. Hence $P$ is finitely generated in $S$. By Cohen's theorem, $S$ is Noetherian and therefore $R$ is Noetherian.

COROLLARY 6. *$(\delta)$ holds in the ring $R$ if and only if $R$ is Noetherian and $(\alpha)$ holds in $R$.*

*Proof.* By Theorems 8, 11, and 12 the conditions are necessary. That they are sufficient follows from the primary representation theorem in Noetherian rings.

THEOREM 13. *If $(\delta)$ holds in the ring $R$ where $R \neq R^2$ and if there exists a prime ideal $M$ such that*

$$M \subseteq \bigcap_{n=1}^{\infty} R^n,$$

*then $R = F_1 \oplus \ldots \oplus F_k \oplus D$ where $F_i$ is a field and $D$ is a non-zero domain, not a field, such that each non-zero ideal of $D$ is a power of $D$.*

*Conversely, if $\{F_i\}_1^k$ and $D$ are as just described and if*

$$S = F_1 \oplus \ldots \oplus F_k \oplus D,$$

*then $(\delta)$ holds in $S$ and $F_1 + \ldots + F_k$ is a prime ideal of $S$ contained in $\bigcap_{n=1}^{\infty} S^n$.*

*Proof.* Since $(\delta)$ holds in $R/M$, Theorems 10 and 12 show that $M = \bigcap_1^{\infty} R^n$. Then if $M_0$ is a prime ideal of $R$ distinct from $M$ and $R$, a repetition of the idea used in the proof of Theorem 4 shows that $R/M_0$ is a field (i.e. if $M_0$ is not contained in $\bigcap_1^{\infty} R^n$ and if $x \in R - M_0$, then $M_0 + (x^n) = M_0 + (x^{n+1})$ for some $n$. This equality implies that $R/M_0$ contains an identity and $\bar{x}$ is a unit in $R/M_0$).

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Now let $(0) = M_1^{e_1} \cap \ldots \cap M_k^{e_k} \cap M^e$ be an irredundant primary representation of (0). (By Theorem 12, $R$ is Noetherian, and by Theorem 8, $(\alpha)$ holds in $R$, so that such a representation exists. $R$ is not a prime belonging to (0) since $M \subseteq \cap_1^\infty R^n$, the intersection of all $R$-primary ideals.) For each $i$, $M_i \subseteq \cap_1^\infty R^n$, $M_i$ is maximal, and $M_i$ is a minimal prime of (0) by Theorem 12. Hence $e_i = 1$ for each $i$ by Theorem 4. Thus

$$R/(M_1 \cap \ldots \cap M_k) \simeq R/M_1 \oplus \ldots \oplus R/M_k$$

(**12**, p. 178). Therefore

$$\begin{aligned} R/(M_1 \cap \ldots \cap M_k) &= [R/M_1 \cap \ldots \cap M_k]^2 \\ &= \{R^2 + (M_1 \cap \ldots \cap M_k)\}/(M_1 \cap \ldots \cap M_k). \end{aligned}$$

Since $R \supset R^2$, $R^2$ cannot contain $M_1 \cap \ldots \cap M_k$.

This implies that $(M_1 \cap \ldots \cap M_k) + M^e = R$, for no prime distinct from $R$ contains both $M^e$ and $M_1 \cap \ldots \cap M_k$ (assuming $k \geqslant 1$. It will be shown at once that $e = 1$ so that if $k = 0$, $R$ is a domain in which each non-zero ideal is a power of $R$ and Theorem 13 holds) so that $M^e + (M_1 \cap \ldots \cap M_k) = R^s$ for some $s$. But $R^2$ does not contain $M_1 \cap \ldots \cap M_k$ so that $s = 1$. Consequently,

$$R \simeq [R/(M_1 \cap \ldots \cap M_k)] \oplus R/M^e \simeq R/M_1 \oplus \ldots \oplus R/M_k \oplus R/M^e.$$

Our proof will be complete as soon as we prove that $e = 1$. Since $(\delta)$ holds in $R/M^e$, this follows from

LEMMA 4. *Let $S$ be a ring in which $(\delta)$ holds. Suppose $S$ has a unique prime $P \subset S$, that (0) is $P$-primary, and that $P = \cap_{n=1}^\infty S^n$. Then $P = (0)$ and $S$ is a domain.*

*Proof.* If $r \in S - S^2$, $(r) = S$. Hence $(r^2) = S^2 \supset P$ so that $P = [P:(r)](r)$ and therefore $P = P(r)$.

Since $S$ is Noetherian, there exists $b \in S$ such that $m = bm$ for each $m \in P$ (**4**, Corollary 1). Now there exists $v \in S$ such that $v - bv \notin P$ since $S/P$ does not contain an identity $(S/P \supset S^2/P)$. Then if $m \in P$ we have

$$0 = v(m - bm) = m(v - bv),$$

$v - bv \notin P$, so that $m \in (0)$ since (0) is $P$-primary. Hence $P = (0)$.

We proceed to prove the converse. If $R = F_1 \oplus \ldots \oplus F_k \oplus D$, $R$ is Noetherian. Corollary 6 shows that $(\delta)$ holds in $R$ if and only if $(\alpha)$ holds in $R$. But $(\alpha)$ does hold since the primary ideals of $R$ are of the form

$$F_1 + \ldots + (0) + \ldots + F_k + D, \quad F_1 + \ldots + F_k,$$

or

$$F_1 + \ldots + F_k + D^q.$$

It is likewise clear that

$$F_1 + \ldots + F_k = \bigcap_{n=1}^{\infty} R^n.$$

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THEOREM 14. *If* ($\delta$) *holds in* $R$ *where* $R \neq R^2$ *and if there exists no prime ideal* $P$ *such that* $P \subseteq \bigcap_{n=1}^{\infty} R^n$, *then* $R = F_1 \oplus \ldots \oplus F_k \oplus S$ *where each* $F_i$ *is a field and* $S$ *is a non-zero ring, every ideal of which is a power of* $S$. *The converse also holds.*

*Proof.* The hypothesis concerning $R$, Theorem 12, and the proof of Theorem 4 show that if $P$ is a prime ideal distinct from $R$, $R/P$ contains an identity. Now $R$ is Noetherian by Theorem 12 and ($\alpha$) holds in $R$ by Theorem 8. Hence (0) has a primary representation

$$(0) = M_1^{e_1} \cap \ldots \cap M_k^{e_k} \cap R^e.$$

Since each $M_i^{e_i}$ is $M$-primary, $R/M_i^{e_i}$ has an identity for each $i$. Thus

$$R/(M_1^{e_1} \cap \ldots \cap M_k^{e_k}) \simeq R/M_1^{e_1} \oplus \ldots \oplus R/M_k^{e_k}$$

has an identity also and therefore $M_1^{e_1} \cap \ldots \cap M_k^{e_k}$ is not contained in $R^2$. By Theorem 4, each $e_i = 1$. Also $e \neq 0$ since $R$ does not contain an identity. It follows that $(M_1 \cap \ldots \cap M_k) + R^e = R$ so that

$$R \simeq R/M_1 \oplus \ldots \oplus R/M_k \oplus R/R^e.$$

This completes the proof of Theorem 14. The proof of the converse is similar to that of Theorem 13.

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