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Dear Editor,

A note on reliability bounds via conditional inequalities

1. Introduction

In a recent paper, Xie and Lai [3] study the approximation

$$P\left(\bigcap_{i=1}^{n} E_i\right) \approx P(E_1)\prod_{i=2}^{n} P(E_i \mid E_{i-1})$$

for the intersection of the events E_i and give some conditions for when the approximation is an upper bound. An example is given there where the conditions are not satisfied. Here we give an argument using conditions which are satisfied in the example, and in some new settings.

In Section 2 we prove the main result, and in Section 3 we apply it to the linear and circular consecutive k-of-$n : F$ systems, and to the distribution of time until a non-overlapping pattern first appears in a sequence of coin flips.

2. Main result

Here we prove the following theorem and give a useful consequence.

Theorem 2.1. *Given events* $E_1, E_2, \dots, E_i$ *and an integer* $k > 1$, *if*

$$E_i \cup E_{i-j} \supseteq E_{i-1} \qquad \text{for all } j : 2 \le j < i,\ \ j \le k \tag{1}$$

and

$$P\left(E_i \mid E_{i-1} \cap \bigcap_{m=1}^{i-k-1} E_m\right) \le P(E_i \mid E_{i-1}), \tag{2}$$

then

$$P(E_i \mid E_{i-1} \cap \cdots \cap E_1) \le P(E_i \mid E_{i-1}), \tag{3}$$

provided that

$$P(E_{i-1} \cap \cdots \cap E_1) > 0. \tag{4}$$

An immediate useful consequence is that when inequality (3) holds for all i we obtain

$$P\left(\bigcap_{i=1}^{n} E_i\right) \le P(E_1)\prod_{i=2}^{n} P(E_i \mid E_{i-1}). \tag{5}$$

To prove the theorem we need the following lemma from [3]. Below we write BC for $B \cap C$.

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Lemma 2.1. *If $B \subseteq A \cup C$ and $P(BC) \neq 0$ then $P(A \mid BC) \leq P(A \mid B)$.*

Proof. With $a = P(ABC^c)$, $b = P(ABC)$, $c = P(A^cBC)$

$$P(A \mid BC) = \frac{b}{b+c} \leq \frac{b+a}{b+c+a} = \frac{P(AB)}{P(B)} = P(A \mid B),$$

where the inequality follows since $f(x) = (b+x)/(b+c+x)$ is an increasing function.

Proof of theorem. For a given value j such that $2 \leq j \leq k$ let $A = E_i$, $C = E_{i-j}$, and $B = E_{i-1}E_{i-j-1}E_{i-j-2}\cdots E_1$. Note that by assumption (1) of the theorem

$$B \subseteq E_{i-1} \subseteq E_i \cup E_{i-j} = A \cup C,$$

and we can apply the lemma (noting that (4) gives $P(BC) > 0$) to obtain

$$P(E_i \mid E_{i-1}E_{i-j}E_{i-j-1}\cdots E_1) \leq P(E_i \mid E_{i-1}E_{i-j-1}E_{i-j-2}\cdots E_1).$$

Applying this argument for $j = 2, 3, \ldots, k$ gives a chain of inequalities leading to

$$P(E_i \mid E_{i-1}E_{i-2}E_{i-3}\cdots E_1) \leq P(E_i \mid E_{i-1}E_{i-k-1}E_{i-k-2}\cdots E_1) \leq P(E_i \mid E_{i-1}),$$

where the second inequality follows from assumption (2) of the theorem.

3. Some examples

3.1. The consecutive-k-out-of-n : F system

In this section we consider the reliability of the circular and linear consecutive-k-out-of-n system. Such a system has n independent components arranged in either a circle or a row and fails if there are at least k consecutive failed components. The reliability of such systems has been extensively studied (see for example [1] and [2]). Here we apply inequality (5) to get a bound.

For the linear system, number the components from left to right and let E_i be the event that the k consecutive components starting with component i are not all failed. Note that

$$P(\text{system works}) = P\left(\bigcap_{i=1}^{n-k} E_i\right). \tag{6}$$

The following result mentioned in [3] holds:

Proposition 3.1. *If each component fails independently with probability q,*

$$P(\textit{linear system works}) \leq P(E_1)\prod_{i=2}^{n-k+1} P(E_i \mid E_{i-1}) = (1-q^k)\left(\frac{1-2q^k+q^{k+1}}{1-q^k}\right)^{n-k}.$$

Proof. Consider two adjacent or overlapping runs of length k, and a third run starting between the two. Since the components of the third run are completely contained in the other two runs, if the components in the third run are not all failed it ensures that the components in at least one of the other two runs are not all failed. This is exactly condition (1) of the theorem. Also, since non-overlapping runs are independent, condition (2) holds. The result follows from inequality (5) applied to Equation (6) since $P(E_1) = 1 - q^k$ and calculation gives $P(E_i \mid E_{i-1}) = (1 - 2q^k + q^{k+1})/(1 - q^k)$.

Note. In [3, p. 106] the theorem presented had as its assumption that

$$E_{i-1} \subseteq E_i \cup E_{i-j} \quad \text{for all } j \geq 2 \text{ and for all } i \geq 1.$$

Though in [3, p. 111] it was asserted to hold, in the example here this condition does not hold since for a 4-of-13 system if the first four and last four components are the only failed components, then E_9 occurs but neither E_{10} nor E_1 occur.

3.2. Non-overlapping patterns

In this section we consider the problem of waiting for a non-overlapping pattern of heads and tails of length k in a sequence of coin flips. If E_i is the event that the pattern does not appear as a run in flips $i, i+1, \ldots, i+k-1$, a non-overlapping pattern is one where $E_i^c \cap E_{i-j}^c = \varnothing$ if $0 < j < k$.

Proposition 3.2. *Letting T be the number of flips required until the pattern first appears as a run,*

$$P(T > n) \leq \prod_{i=1}^{n-k+1} P(E_i) = (P(E_1))^{n-k+1}.$$

Proof. Since non-overlapping implies $(E_i \cup E_{i-j})^c = \varnothing$, $j < k$, we immediately have $\Omega \subseteq E_i \cup E_{i-j}$, where Ω is the whole sample space. Thus condition (1) holds for the events $(E_1, \ldots, E_{i-1}, \Omega, E_i)$, and (2) also holds since events separated by at least k flips are independent. Applying Theorem 2.1 we get

$$P(E_i \mid E_1 \cdots E_{i-1}) \leq P(E_i \mid \Omega) = P(E_i).$$

The proposition follows using inequality (5) and noting that events $E_{n-k+2}, \ldots, E_n$ all trivially occur.

3.3. The circular consecutive k-out-of-$n : F$ system

For the circular system, number the components clockwise and, using the same definitions as the previous section, let E_i be the event that there is a clockwise run of k failed components starting with component i. The following holds:

Proposition 3.3.

$$P(\textit{circular system works}) \leq P(E_1) \prod_{i=2}^{n-k+1} P(E_i \mid E_{i-1}E_1).$$

Proof. For a fixed value of i define events

$$(F_{i-1}, \ldots, F_1) \equiv (E_i, E_{i-1} \cap E_1, E_{i-k}, E_{i-k+1}, \ldots, E_{i-2}, E_2, E_3, \ldots, E_{i-k-1})$$

and let $k^* = k+n-i$. Note that conditions (1) and (2) apply and we can then apply Theorem 2.1 using these events and the integer k^* to obtain

$$P(F_{i-1} \mid F_{i-2} \cdots F_1) \leq P(F_{i-1} \mid F_{i-2}),$$

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or equivalently

$$P(E_i \mid E_{i-1} \cdots E_1) \leq P(E_i \mid E_{i-1} E_1).$$

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