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ON SCALAR DEPENDENT ALGEBRAS

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1. Introduction. The intent of this paper is to study a class of algebras which do not necessarily obey the association law but instead obey a law which bears a marked resemblance to associativity. For lack of a better name we call this class the class of scalar dependent algebras. Specifically, an algebra A over a field F is called *scalar dependent* if there is a map $g: A \times A \times A \to F$ such that $(xy)z = g(x, y, z)x(yz)$, for all x, y, z in A. To obtain our results we shall assume throughout that A is a scalar dependent algebra with an identity element e over a field of characteristic not 2 satisfying

$$(x, x, x) = 0. \tag{I}$$

As usual, the associator (x, y, z) is defined by $(x, y, z) = (xy)z - x(yz)$. An example is given to show that (I) is not implied by scalar dependency.

The main result of the paper is that if A has an idempotent other than e then A is associative. If A has no such idempotent then the same result can be obtained if the assumptions of finite dimensionality and semisimplicity are added. Finally an example is given to show that not every scalar dependent algebra is associative even if it is both power-associative and finite dimensional. These results generalize the authors' earlier work [**3**; **6**] where it was shown that a finite dimensional semisimple scalar dependent algebra satisfying (I) and the additional property that $g(x_1, x_2, x_3) = g(x_{1\pi}, x_{2\pi}, x_{3\pi})$ for all x_1, x_2, x_3 in A and all π in S_3, is associative.

2. Power-associativity. An algebra is called *power-associative* if every subalgebra generated by a single element is associative.

THEOREM 1. *If A is a scalar dependent algebra with identity element e satisfying* (I) *over a field F of characteristic $\neq 2$, then A is power-associative.*

Proof. We show first that A is fourth power-associative. Let x be in A, $s = g(x, x, x^2)$, and $t = g(x + x^2, x, x)$. Then $x^2x^2 = sxx^3$. If $s = 1$ for all x in A, then A is fourth power-associative and we are done. Suppose $s \neq 1$ for some x in A. Then $((x + x^2)x)x = t(x + x^2)x^2$ or $(1 - t)x^3 = tx^2x^2 - x^3x = tsxx^3 - x^3x$. Since characteristic $F \neq 2$, however, linearization of (I) implies that $xx^3 = x^3x$. Therefore $(1 - t)x^3 = (ts - 1)xx^3$. If $ts = 1$ then $t \neq 1$ since $s \neq 1$. Thus $x^3 = 0$ which implies that $xx^3 = 0$ and, from $x^2x^2 = xx^3$, that $x^2x^2 = 0$. Therefore, if $ts = 1$, $x^2x^2 = xx^3$ and A is fourth power-associative.

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If $ts \neq 1$ then $xx^3 = \alpha x^3$ for $\alpha = (1 - t)/(ts - 1)$. Hence we have $x^3x = xx^3 = \alpha x^3$ or $x^2x^2 = \alpha s x^3$.

Next let $m = g(e + x^2, e + x, e + x)$. Then $[(e + x^2)(e + x)](e + x) = m(e + x^2)(e + x)^2$ or $e + 2x + 2x^2 + 2x^3 + x^3x = m(e + 2x + 2x^2 + 2x^3 + x^2x^2)$. If $m = 1$ then $x^3x = x^2x^2$ and we are done. Otherwise $(1 - m)(e + 2x + 2x^2 + 2x^3) = mx^2x^2 - x^3x$. However, $x^3x = \alpha x^3$ and $x^2x^2 = \alpha s x^3$. Therefore $(1 - m)(e + 2x + 2x^2 + 2x^3) = \alpha(ms - 1)x^3$ and $(1 - m)(e + 2x + 2x^2) = \gamma x^3$ with $\gamma = \alpha m s - \alpha - 2(1 - m)$. If $\gamma = 0$ then e, x, x^2 are linearly dependent and $x^2 = \alpha e + \beta x$ for α, β in F. Therefore $(x^2, x, x) = (\alpha e + \beta x, x, x) = (x, x, x) = 0$ by (I) and A is fourth power-associative. If $\gamma \neq 0$ then $x^3 = \beta e + 2\beta x + 2\beta x^2$ for β in F. Therefore $xx^3 = \beta x + 2\beta x^2 + 2\beta x^3$ or $\alpha x^3 = \beta x + 2\beta x^2 + 2\beta x^3$. Hence $(\alpha - 2\beta)x^3 = \beta x + 2\beta x^2$. But $x^3 = \beta e + 2\beta x + 2\beta x^2$. Therefore $(\alpha - 2\beta)x^3 = (\alpha - 2\beta)\beta e + 2(\alpha - 2\beta)\beta x + 2(\alpha - 2\beta)\beta x^2$ or $\beta x + 2\beta x^2 = (\alpha - 2\beta)\beta e + 2(\alpha - 2\beta)\beta x + 2(\alpha - 2\beta)\beta x^2$. Now if e, x, x^2 are linearly dependent we have fourth power-associativity as before. Therefore e, x, x^2 are linearly independent. Equating coefficients in the last identity we have, $(\alpha - 2\beta)\beta = 0$ and $(\alpha - 2\beta)\beta = \beta$. Therefore $\beta = 0$ and from $x^3 = \beta + 2\beta x + 2\beta x^2$ we get $x^3 = 0$. Therefore $xx^3 = 0$ and $x^2x^2 = sxx^3 = 0$ and we conclude that A is fourth power-associative.

By [**4**, Theorem 1] third and fourth power-associativity and $xx^n = x^nx$ for all n is sufficient to guarantee power-associativity if the characteristic of F is not equal to 2,3, or 5. We next show that $xx^n = x^nx$ for all positive integers n by induction on the power of x. First note that the attached algebra A^+ is power-associative; (A^+ is the same vector space as A with multiplication in A^+ defined by $x \cdot y = (xy + yx)/2$ with xy the multiplication in A) for A^+ is commutative so that fourth power-associativity implies power-associativity [**1**].

Assume now that $xx^m = x^mx$ for all $m < n$. By repeated application of scalar dependency it follows that $x^nx = rxx^n$ for some r in F. If $r = 1$ we are done. Otherwise let $s = g(x + x^{n-1}, x, x)$ and let $x^{n-1}x^2 = txx^n$. Then $((x + x^{n-1})x)x = s(x + x^{n-1})x^2$ or $x^3 + x^nx = s(x^3 + txx^n)$. Therefore $(1 - s)x^3 = (st - r)xx^n$. First, if $s = 1$ then $(t - r)xx^n = 0$. Therefore either $xx^n = x^nx = 0$ or $t = r$. If $t = r$, however, then $x^nx = rxx^n$ and $x^{n-1}x^2 = rxx^n$. Thus $x^nx = x^{n-1}x^2$. Since A^+ is power-associative $x^n \cdot x = x^{n-1} \cdot x^2$ or $x^nx + xx^n = x^{n-1}x^2 + x^2x^{n-1}$. Therefore $xx^n = x^2x^{n-1}$. A linearization of $(x, x, x) = 0$ gives $(x, x^{n-1}, x) + (x, x, x^{n-1}) + (x^{n-1}, x, x) = 0$ which reduces to $2x^nx - 2xx^n = x^{n-1}x^2 - x^2x^{n-1}$. But $x^nx = x^{n-1}x^2$ and $xx^n = x^2x^{n-1}$. Therefore $xx^n = x^nx$ if $s = 1$. Assume then that $s \neq 1$. If $st = r$ then $x^3 = 0$ which implies that $x^n = 0$ so that $xx^n = x^nx = 0$. Finally if $s \neq 1$ and $st \neq r$ then $xx^n = \alpha x^3$ with $\alpha = (1 - s)/(st - r) \neq 0$ and $x^nx = \alpha r x^3$.

Let $g(e + x, e + x^{n-1}, e + x) = k$. Then

$$((e + x)(e + x^{n-1}))(e + x) = k(e + x)((e + x^{n-1})(e + x))$$

or

$$e + 2x + x^2 + x^{n-1} + 2x^n + x^nx = k(e + 2x + x^2 + x^{n-1} + 2x^n + xx^n).$$

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If $k = 1$ then $x^n x = xx^n$ and we are done. Otherwise

$$(1-k)e + 2(1-k)x + (1-k)x^2 + (1-k)x^{n-1} + 2(1-k)x^n$$
$$= kxx^n - x^n x = (k-r)xx^n = (k-r)\alpha x^3.$$

Therefore x^n is a linear combination of e, x, x^2, x^3, and x^{n-1}. Now by the induction hypothesis these elements commute with x. Therefore $xx^n = x^n x$.

For characteristic $F = 3, 5$ one must in addition show that $x^4 x = x^3 x^2$ and $x^5 x = x^4 x^2$, respectively [**4**, Theorems 2 and 3]. The same methods as those employed in the general case yield these results so that if F is not of characteristic 2 then A is power-associative.

Not every scalar dependent algebra is third power-associative as shown by the following example. Let A have basis a, b, c, d, e, f over a field F whose characteristic is not 2 with multiplication given by $ab = c$, $cd = e$, $bd = \alpha f$, $af = e$, and all other products zero. Here α, denotes a fixed non zero scalar in F. It is easily verified that A is scalar dependent with $g(x, y, z) = \alpha^{-1}$ for all x, y, z in A. However $(a + b + d)^2(a + b + d) = (c + \alpha f)(a + b + d) = e$ and $(a + b + d)(a + b + d)^2 = (a + b + d)(c + \alpha f) = \alpha e$. Thus if α is chosen so that $\alpha \neq 1$ then A is not third power-associative.

3. The Peirce decomposition. As before, A denotes a scalar dependent algebra with identity e satisfying (I) throughout.

LEMMA 1. *If u is an idempotent of A then $(u, u, a) = (u, a, u) = (a, u, u) = 0$ for all a in A.*

Proof. Let $\alpha = g(a, u - e, u)$. Then

$$(a, u, u) = (au)u - au = (au - a)u = [a(u - e)]u = \alpha a(u^2 - u) = 0.$$

Now let $\beta = g(u, a - au, a)$. Then

$$(u, a, u)u = [(ua)u]u - [u(au)]u = (ua)u - [u(au)]u = (ua - u(au))u$$
$$= [u(a - au)]u = \beta u[(a - au)u].$$

But $au = (au)u$. Therefore $(u, a, u)u = 0$ or $((ua)u)u = (u(au))u$. Again

$$0 = (ua, u, u) = ((ua)u)u - (ua)u = (u(au))u - (ua)u$$
$$= (u(au))u - (u, a, u) - u(au) = -(u, a, u) + (u(au))(u - e)$$
$$= -(u, a, u) + \gamma u((au)(u - e))$$

where $\gamma = g(u, au, u - e)$. However

$$(au)(u - e) = (au)u - au = (a, u, u) = 0.$$

Therefore $(u, a, u) = 0$. From $(a, u, u) = (u, a, u) = 0$ and (I) we conclude that $(u, u, a) = 0$ which completes the proof.

An immediate consequence of Lemma 1 is:

LEMMA 2. *Relative to any idempotent u, A has a vector space decomposition*

$$A = A_{11}(u) + A_{10}(u) + A_{01}(u) + A_{00}(u)$$

where

$$A_{ij}(u) = \{x | ux = ix, xu = jx\}$$

for $i, j = 0, 1$.

We will write A_{ij} for $A_{ij}(u)$ whenever there is no danger of confusion as to which idempotent we are dealing with.

LEMMA 3. *A_{11} and A_{00} are subalgebras of A.*

Proof. Let x, y be in $A_{11}(u)$ and let $\alpha = g(u, x, y)$. Then $xy = (ux)y = \alpha u(xy)$. Now if $xy = a_{11} + a_{10} + a_{01} + a_{00}$ then we have that

$$a_{11} + a_{10} + a_{01} + a_{00} = \alpha a_{11} + \alpha a_{10}.$$

Therefore $a_{01} + a_{00} = 0$, and $xy \in A_{11} + A_{10}$. Thus we have

$$A_{11}^2 \subseteq A_{11} + A_{10} \tag{1}$$

Similarly, let w, z be in $A_{00}(u)$. Then $(wz)u = g(w, z, u)w(zu) = 0$. Therefore

$$A_{00}^2 \subseteq A_{00} + A_{10} \tag{2}$$

It is clear that $A_{ij}(u) = A_{ji}(e - u)$. Therefore x, y are in $A_{00}(e - u)$. Now from (1) $xy \in A_{00}(e - u) + A_{01}(e - u)$ and from (2) $xy \in A_{00}(e - u) + A_{10}(e - u)$. Therefore $xy \in A_{00}(e - u) = A_{11}(u)$ and A_{11} is a subalgebra. Since $A_{00}(u) = A_{11}(e - u)$, it follows that A_{00} is also a subalgebra.

LEMMA 4. $(A_{11} + A_{01})(A_{00} + A_{01}) = 0$.

Proof. Let $x \in A_{11} + A_{01}$, $y \in A_{00} + A_{01}$ and $\alpha = g(x, u, y)$. Then

$$xy = (xu)y = \alpha x(uy) = 0.$$

LEMMA 5. $A_{11}A_{10} \subseteq A_{10}$, $A_{01}A_{11} \subseteq A_{01}$, *and* $A_{10}A_{01} \subseteq A_{11}$.

Proof. For the first let $x \in A_{11}$, $y \in A_{10}$, and $\alpha = g(x, y, u)$. Then $(xy)u = \alpha x(yu) = 0$ or $xy \in A_{00} + A_{10}$. Let $xy = a_{00} + a_{10}$ and let $\beta = g(u, x, y)$. Then $xy = (ux)y = \beta u(xy)$ or $a_{00} + a_{10} = \beta a_{10}$. Therefore $a_{00} = 0$ and $xy \in A_{10}$. For the second inclusion let $x \in A_{10}(u)$, $y \in A_{00}(u)$. Then $(xy)u = g(x, y, u)x(yu) = 0$ so that $xy \in A_{00}(u) + A_{10}(u)$. Therefore

$$A_{10}A_{00} \subseteq A_{00} + A_{10} \tag{3}$$

On the other hand $xy = (ux)y = g(u, x, y)u(xy)$ so that $xy \in A_{10}(u) + A_{11}(u)$. Therefore

$$A_{10}A_{00} \subseteq A_{10} + A_{11} \tag{4}$$

Combining (3) and (4), $A_{10}A_{00} \subseteq (A_{00} + A_{10}) \cap (A_{11} + A_{10})$. Therefore $A_{10}A_{00} \subseteq A_{10}$. Since $A_{ij}(u) = A_{ji}(e - u)$ it follows that $A_{01}(u)A_{11}(u) \subseteq$

$A_{01}(u)$ proving the second inclusion. For the last one let $x \in A_{10}$, $y \in A_{01}$. Then $xy = (ux)y = g(u, x, y)u(xy)$. Therefore $xy \in A_{11} + A_{10}$. Again $x \in A_{01}(e-u)$, $y \in A_{10}(e-u)$. Then $(xy)(e-u) = g(x, y, e-u)x(y-yu) = 0$. Therefore $xy \in A_{00}(e - u) + A_{10}(e - u) = A_{11}(u) + A_{01}(u)$. Thus $xy \in (A_{11} + A_{10}) \cap (A_{11} + A_{01}) \subseteq A_{11}$. Hence $A_{10}A_{01} \subseteq A_{11}$.

Lemmas 3-5 and the duality relationship between A_{ij} and A_{ji} yield

THEOREM 2. *The subspaces $A_{ij} = A_{ij}(u)$ enjoy the multiplicative properties*

$$A_{ij}A_{ks} = 0 \text{ if } j \neq k, \quad i, j, k, s = 0, 1$$

$$A_{ij}A_{js} \subseteq A_{is} \text{ for } i, j, s = 0, 1.$$

4. Classification of the algebras. In what follows we let (A_{ij}, A_{kt}, A_{rs}) denote $\{(a_{ij}, a_{kt}, a_{rs}) | a_{ij} \in A_{ij}, a_{kt} \in A_{kt}, a_{rs} \in A_{rs}\}$

LEMMA 6. *Relative to any idempotent $u \neq e$, $(A_{11}, A_{1j}, A_{jk}) = 0$ for any j, k.*

Proof. Let $a_{11} \in A_{11}$, $a_{1j} \in A_{1j}$, and $a_{jk} \in A_{jk}$. From scalar dependency

$$[(e + a_{11})(e + a_{1j})](e + a_{jk}) = \alpha(e + a_{11})[(e + a_{1j})(e + a_{jk}]$$

with $\alpha = g(e + a_{11}, e + a_{1j}, e + a_{jk})$. Therefore we have:

$$\begin{aligned}(5) \quad & e + a_{11} + a_{1j} + a_{jk} + a_{11}a_{1j} + a_{11}a_{jk} + a_{1j}a_{jk} + (a_{11}a_{1j})a_{jk} \\ & = \alpha(e + a_{11} + a_{1j} + a_{jk} + a_{11}a_{1j} + a_{11}a_{jk} + a_{1j}a_{jk} + a_{11}(a_{1j}a_{jk})).\end{aligned}$$

Now $e = u + (e - u) \in A_{11} + A_{00}$. First, if $j \neq 0$ or $k \neq 0$ then by Theorem 2 none of the a_{rs} appearing in (5) are in A_{00}. Therefore, equating the elements of A_{00} in (5) we have $e - u = \alpha(e - u)$. Since $u \neq e$ it follows that $\alpha = 1$ so that $(a_{11}a_{1j})a_{jk} = a_{11}(a_{1j}a_{jk})$. Next, if $j = k = 0$ we get

$$(e - u) + a_{00} = \alpha(e - u + a_{00})$$

or $(1 - \alpha)[(e - u) + a_{00}] = 0$. Thus, either $\alpha = 1$ or $a_{00} = u - e$. If $\alpha = 1$ then (5) gives $(a_{11}a_{1j})a_{jk} = a_{11}(a_{1j}a_{jk})$ and we are done. On the other hand if $a_{00} = u - e$ then $(a_{11}a_{10})(u - e) = -a_{11}a_{10} = a_{11}(a_{10}(u - e))$. Therefore the lemma holds.

LEMMA 7. *Relative to any idempotent u, $(A_{10}, A_{0j}, A_{jk}) = 0$.*

Proof. Let $a_{10} \in A_{10}$, $a_{0j} \in A_{0j}$, and $a_{jk} \in A_{jk}$. As in the previous lemma we get

$$\begin{aligned}(6) \quad & e + a_{10} + a_{0j} + a_{jk} + a_{10}a_{0j} + a_{10}a_{jk} + a_{0j}a_{jk} + (a_{10}a_{0j})a_{jk} \\ & = \alpha(e + a_{10} + a_{0j} + a_{jk} + a_{10}a_{0j} + a_{10}a_{jk} + a_{0j}a_{jk} + a_{10}(a_{0j}a_{jk}))\end{aligned}$$

with $\alpha = g(e + a_{10}, e + a_{0j}, e + a_{jk})$. If $k \neq 0$ then as in the previous lemma we obtain $e - u = \alpha(e - u)$ by equating the elements of A_{00} in (6). Therefore $\alpha = 1$ and $(a_{10}a_{0j})a_{jk} = a_{10}(a_{0j}a_{jk})$. If $j = k = 0$ we equate the elements of (6) which are in A_{11} to obtain $u = \alpha u$ so that $\alpha = 1$ and $(a_{10}a_{0j})a_{jk} = a_{10}(a_{0j}a_{jk})$.

Finally, if $a_{0j} = a_{01} \in A_{01}$ and $a_{jk} = b_{10} \in A_{10}$ we consider

$$[(e + a_{10})a_{01}](e + b_{10}) = \beta(e + a_{10})[a_{01}(e + b_{10})]$$

with $\beta = g(e + a_{10}, a_{01}, e + b_{10})$ to get:

$$(7) \qquad a_{01} + a_{01}b_{10} + a_{10}a_{01} + (a_{10}a_{01})b_{10} = \beta(a_{01} + a_{01}b_{10} + a_{10}a_{01} + a_{10}(a_{01}b_{10}))$$

If $a_{01} = 0$ the lemma follows trivially. Otherwise equate the elements of (7) which are in A_{01} to get $a_{01} = \beta a_{01}$. Therefore $\beta = 1$ and $(a_{10}a_{01})b_{10} = a_{10}(a_{01}b_{10})$. Thus the lemma holds in all cases.

Lemmas 6 and 7 together with Theorem 2 state that an associator whose first component is either in A_{11} or in A_{10} is zero. By duality the same result holds for associators whose first component is in A_{00} or in A_{01}. We have proven

THEOREM 3. *If A has an idempotent $u \neq e$ then A is associative.*

If e is the only idempotent of A we are not able to prove that A is associative in general. However, the following theorem can be applied. Since A is power-associative it has a nil radical. By semisimple we shall mean semisimple with respect to the nil radical.

THEOREM 4. *If A is finite dimensional and semisimple then A is associative.*

Proof. If A has an idempotent $u \neq e$ then Theorem 3 applies. Therefore assume that e is the only idempotent of A. By a well known argument of Albert [**2**], $A = Fe + N$ where N is the set of nilpotent elements of A. By Albert [**2**] and Oehmke [**5**], N is a subspace of A. Let x, y be elements of N and assume that $xy \notin N$. Then $xy = \alpha e + n$ with $\alpha \neq 0$ and n in N. Therefore xy has an inverse $(xy)^{-1}$ in A. Since y is in N there is a positive integer k such that $y^{k-1} \neq 0$ but $y^k = 0$. By scalar dependency $y^{k-1} = [(xy)^{-1}(xy)]y^{k-1} = \beta(xy)^{-1}[(xy)y^{k-1}] = \beta\gamma(xy)^{-1}[xy^k] = 0$ where $\beta = g((xy)^{-1}, xy, y^{k-1})$ and $\gamma = g(x, y, y^{k-1})$. But this contradicts $y^{k-1} \neq 0$. Therefore $xy \in N$ and N is a subalgebra of A. Hence N is a nil ideal of A. By semisimplicity it follows that $N = 0$, and $A = Fe$. Hence A is associative.

Example. The following example illustrates that not every power-associative scalar dependent algebra is associaive. Let A be the 8-dimensional algebra over F of characteristic $\neq 2$ with basis a, b, c, d, e, f, g, h with multiplication given by $ab = c$, $cd = e$, $bd = f$, $af = 2e$, $db = g$, $ga = -e$, $ba = h$, $dh = -2e$ and all other products equal to zero. Then

$$[(A_1a + B_1b + \ldots + H_1h)(A_2a + \ldots + H_2h)](A_3a + \ldots + H_3h) = (A_1B_2D_3 - D_1B_2A_3)e$$

(here capital letters denote scalars). On the other hand

$$(A_1a + \ldots + H_1h)[(A_2a + \ldots + H_2h)(A_3a + \ldots + H_3h)] = 2(A_1B_2D_3 - D_1B_2A_3)e.$$

Thus if we let $g(x, y, z) = \frac{1}{2}$ for all x, y, z in A then A is a scalar dependent algebra which is not associative. Notice that if $A_1 = A_3$ and $D_1 = D_3$ then both products are zero. In particular $x^2x = xx^2$ for all x in A. Since products of more than three elements are all zero, A is power associative.

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