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GROUND STATE SOLUTIONS FOR *p*-SUPERLINEAR *p*-LAPLACIAN EQUATIONS

YI CHEN and X. H. TANG✉

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Abstract

In this paper, we deduce new conditions for the existence of ground state solutions for the p-Laplacian equation

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) + V(x)|u|^{p-2}u = f(x,u), & x \in \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N), \end{cases}$$

which weaken the Ambrosetti–Rabinowitz type condition and the monotonicity condition for the function $t \mapsto f(x,t)/|t|^{p-1}$. In particular, both $tf(x,t)$ and $tf(x,t) - pF(x,t)$ are allowed to be sign-changing in our assumptions.

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1. Introduction

Consider the following p-Laplacian equation:

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) + V(x)|u|^{p-2}u = f(x,u), & x \in \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $p > 1$, $V : \mathbb{R}^N \to \mathbb{R}$ and $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$. For $p = 2$, (1.1) turns into the following semilinear Schrödinger equation:

$$\begin{cases} -\triangle u + V(x)u = f(x,u), & x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N). \end{cases} \tag{1.2}$$

The Schrödinger equation has found a great deal of interest in recent years because not only is it important in applications but also it provides a good model for developing mathematical methods. Many authors have studied the existence of entire solutions of

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Schrödinger equations under various stipulations (cf. for example, [2, 3, 5, 6, 8–10, 13, 21, 23, 25, 28, 29] and the references quoted in them).

For the p-Laplacian equation (1.1), we assume that the potential $V(x)$ and the nonlinearity $f(x,u)$ satisfy the following conditions:

(V1) $V \in C(\mathbb{R}^N, \mathbb{R})$ is 1-periodic in each of $x_1, x_2, \ldots, x_N$ and $0 < \inf_{\mathbb{R}^N} V \le \sup_{\mathbb{R}^N} V < +\infty$;

(S1) $f \in C(\mathbb{R}^{N+1}, \mathbb{R})$ is 1-periodic in each of $x_1, x_2, \ldots, x_N$ and

$$\lim_{|u|\to\infty} \frac{|f(x,u)|}{|u|^{p^*-1}} = 0, \quad \text{uniformly in } x \in \mathbb{R}^N, \tag{1.3}$$

where $p^* = pN/(N-p)$ if $N > p$ and $p^* \in (p, +\infty)$ if $N \le p$;

(S2) $f(x,t) = o(|t|^{p-1})$, as $|t| \to 0$, uniformly in $x \in \mathbb{R}^N$.

Let $F(x,u) = \int_0^u f(x,s)\,ds$. By (S1) and (S2), we deduce that there exists a constant $C > 0$ such that

$$|F(x,u)| \le C(|u|^p + |u|^{p^*}), \quad \forall (x,u) \in \mathbb{R}^N \times \mathbb{R}.$$

Consequently, the energy functional $\Phi : W^{1,p}(\mathbb{R}^N) \to \mathbb{R}$,

$$\Phi(u) = \frac{1}{p} \int_{\mathbb{R}^N} (|\nabla u|^p + V(x)|u|^p)\,dx - \int_{\mathbb{R}^N} F(x,u)\,dx, \tag{1.4}$$

is of class C^1. Moreover, the critical points of Φ are weak solutions of (1.1).

In the present paper, we are concerned with the existence of ground state solutions, that is, solutions corresponding to the least positive critical value of Φ.

In many studies of p-superlinear elliptic equations, the (AR) condition due to Ambrosetti–Rabinowitz (see [1, 4, 7]) or the Nehari condition (Ne) is commonly assumed (see [11, 16, 18, 21, 27]).

(AR) There exists $\mu > p$ such that

$$0 < \mu F(x,t) \le t f(x,t), \quad \forall (x,t) \in \mathbb{R}^N \times (\mathbb{R} \setminus 0);$$

(Ne) $f(x,t)/|t|^{p-1}$ is strictly increasing in t on $\mathbb{R} \setminus \{0\}$, for every $x \in \mathbb{R}^N$.

In a recent paper [19], instead of (AR), Liu used the following two conditions:

(S3) $\lim_{|t|\to\infty} (F(x,t)/|t|^p) = \infty$, uniformly in $x \in \mathbb{R}^N$;

(S4) there exists $\theta \ge 1$ such that

$$\theta \mathcal{F}(x,t) \ge \mathcal{F}(x,st), \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R},\ s \in [0,1],$$

where $\mathcal{F}(x,t) = (1/p) f(x,t)t - F(x,t)$.

Specifically, the author established the following theorem in [19].

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THEOREM 1.1 [19]. *Assume that V and f satisfy (V1), (S1), (S2), (S3) and (S4). Then (1.1) has a nontrivial solution $u \in E$ such that $\Phi(u) = \inf_{\mathcal{M}} \Phi \geq 0$, where*

$$E = \left\{ u \in W^{1,p}(\mathbb{R}^N) \ : \ \int_{\mathbb{R}^N} (|\nabla u|^p + V(x)|u|^p)\, dx < +\infty \right\} \tag{1.5}$$

and

$$\mathcal{M} = \{u \in E : \Phi'(u) = 0, \ u \neq 0\}.$$

Note that for the semilinear case $p = 2$, (1.3) and (S3) were first introduced by Liu and Wang [21] and then were used in [16]. The condition (S4) is due to Jeanjean [12], which is weaker than (Ne), see [20]. To overcome the difficulty that the Palais–Smale (PS) sequences of Φ may be unbounded, he established a variant of the mountain pass lemma, which asserts that a sequence of perturbed functionals possesses bounded (PS) sequences. In [14], this condition is also used with a Cerami-type argument in singularly perturbed elliptic problems in $\mathbb{R}^N$ with autonomous nonlinearity. For quasilinear elliptic problems on a bounded domain, (S4) is also used in [22] to obtain infinitely many solutions and in [15] to compute the critical groups of Φ at infinity and obtain nontrivial solutions via Morse theory.

The role of (AR) is to ensure the boundedness of the PS sequences of the functional Φ. This is crucial in applying the critical point theory. There are many functions, for example the p-superlinear function

$$f(x,t) = |t|^{p-2} t \ln(1 + |t|), \tag{1.6}$$

which satisfy (S3) and (S4), but do not satisfy (AR) for any $\mu > p$. However, (AR) does not imply (S4). For example, let

$$f(x,t) = 3|t|t \int_0^t |s|^{1+\sin s} s\, ds + |t|^{4+\sin t} t;$$

then

$$F(x,t) = |t|^3 \int_0^t |s|^{1+\sin s} s\, ds.$$

It is easy to see that $f(x,t)$ satisfies (AR) for $p = 2$ and $\mu = 3$, but it does not satisfy (S4), see [26]. Therefore, (S3) and (S4) are complement conditions with (AR).

In the present paper, we will deduce some new conditions which weaken both the weaker version (WAR) of the (AR) condition and the weaker version (WN) of the Nehari condition (Ne).

(WAR) There exists $\mu > p$ such that

$$0 \leq \mu F(x,t) \leq t f(x,t), \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R};$$

(WN) $f(x,t)/|t|^{p-1}$ is increasing in t on $\mathbb{R} \setminus \{0\}$ for every $x \in \mathbb{R}^N$.

Before presenting our theorems, we introduce the following assumptions, where, and in the sequel, $\gamma_s = \sup_{u \in E,\ \|u\|=1} \|u\|_s$, for $p \leq s \leq p^*$.

(S2′) $\lim_{|t|\to 0} |f(x,t)|/|t|^{p-1} < \gamma_p^{-p}$, uniformly in $x \in \mathbb{R}^N$, and $tf(x,t) - pF(x,t) = o(|t|^p)$, as $|t| \to 0$, uniformly in $x \in \mathbb{R}^N$;

(S3′) $\lim_{|t|\to\infty} |F(x,t)|/|t|^p = \infty$, for almost every $x \in \mathbb{R}^N$, and there exists $r_0 \geq 0$ such that $F(x,t) \geq 0$, for $|t| \geq r_0$;

(S5) there exist $\theta_0 \in (0,1)$ and $K \geq 1$ such that

$$\frac{1-\theta^p}{p} tf(x,t) \geq F(x,t) - KF(x,\theta t), \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R},\ \theta \in [0,\theta_0];$$

(S6) there exists a $\theta_0 \in (0,1)$ such that

$$\frac{1-\theta^p}{p} tf(x,t) \geq \int_{\theta t}^{t} f(x,s)\,ds, \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R},\ \theta \in [0,\theta_0];$$

(S7) there exists a $\mu > p$ such that

$$\mu F(x,t) \leq tf(x,t), \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}.$$

Now we are in a position to state the main results of this paper.

Theorem 1.2. *Assume that V and f satisfy (V1), (S1), (S2), (S3′) and (S5). Then (1.1) has a nontrivial solution $u \in E$ such that $\Phi(u) = \inf_{\mathcal{M}} \Phi \geq 0$.*

Theorem 1.3. *Assume that V and f satisfy (V1), (S1), (S2′), (S3′) and (S6). Then (1.1) has a nontrivial solution $u \in E$ such that $\Phi(u) = \inf_{\mathcal{M}} \Phi \geq 0$.*

Theorem 1.4. *Assume that V and f satisfy (V1), (S1), (S2′), (S3′) and (S7). Then (1.1) has a nontrivial solution $u \in E$ such that $\Phi(u) = \inf_{\mathcal{M}} \Phi \geq 0$.*

Remark 1.5. If the (WAR) condition is satisfied, then we let $\theta_0 = [(\mu - p)/\mu]^{1/p}$. Hence,

$$\frac{1-\theta^p}{p} tf(x,t) \geq \frac{1}{\mu} tf(x,t) \geq F(x,t) \geq F(x,t) - F(x,\theta t), \quad \forall \theta \in [0,\theta_0].$$

This shows that (S6) holds. In addition, it is easy to verify that $f(x,u)$ defined by (1.6) satisfies (S6). Therefore, (S3′) and (S6) weaken (WAR) considerably. Furthermore, it is easy to check that (WN) implies (S6).

Example 1.6. It is easy to check that the following function:

$$f(x,t) = a|t|^{p-2}t\ln(\tfrac{1}{2} + |t|)$$

satisfies (S2′), (S3′) and (S6) with $p > 2$ and $0 < a < 1/(\gamma_p^p \ln 2)$, while

$$F(x,t) = b_1|t|^{p+2} - b_2|t|^{p+1}$$

satisfies (S2′), (S3′) and (S7) with $b_1, b_2 > 0$ and $\mu = p+1$. However, these functions do not satisfy (WAR) and (S4).

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REMARK 1.7. When $p = 2$, the (AR) condition has also been weakened by Ding and Lee [8] (see (N_3) and (N_4) in [8]) under an additional assumption that $|f(x,u)| \le C_0(1 + |u|^{q-1})$, $\forall (x,u) \in \mathbb{R}^N \times \mathbb{R}$ for some $C_0 > 0$ and $q \in (2, 2^*)$. However, it is assumed, in (N_4), that $F(x,t) > 0$ and $tf(x,t) - pF(x,t) > 0$ for all $x \in \mathbb{R}^N$ and $t \neq 0$. In contrast, the functions $F(x,t)$ and $tf(x,t) - pF(x,t)$ are allowed to be sign-changing in our assumptions.

2. Proofs of the main results

Let E be the space defined by (1.5) equipped with the norm

$$\|u\| = \left\{ \int_{\mathbb{R}^N} (|\nabla u|^p + V(x)|u|^p)\, dx \right\}^{1/p}, \quad u \in E.$$

Then E is a Banach space. Under (V1), (S1) and (S2′), the functional Φ defined by (1.4) is of class $C^1(E, \mathbb{R})$. Moreover,

$$\Phi(u) = \frac{1}{p}\|u\|^p - \int_{\mathbb{R}^N} F(x,u)\, dx, \quad \forall u \in E \tag{2.1}$$

and, for all $u, v \in E$,

$$\langle \Phi'(u), v \rangle = \int_{\mathbb{R}^N} (|\nabla u|^{p-2} \nabla u \nabla v + V(x)|u|^{p-2}uv)\, dx - \int_{\mathbb{R}^N} f(x,u)v\, dx. \tag{2.2}$$

LEMMA 2.1. *Let X be a Banach space. Let M_0 be a closed subspace of the metric space M and $\Gamma_0 \subset C(M_0, X)$. Define*

$$\Gamma = \{\gamma \in C(M, X) : \gamma|_{M_0} \in \Gamma_0\}.$$

If $\Psi \in C^1(X, \mathbb{R})$ satisfies

$$\infty > c := \inf_{\gamma \in \Gamma} \sup_{t \in M} \Psi(\gamma(t)) > a := \sup_{\gamma_0 \in \Gamma_0} \sup_{t \in M_0} \Psi(\gamma_0(t)), \tag{2.3}$$

then there exists a sequence $\{u_n\} \subset X$ satisfying

$$\Psi(u_n) \to c, \quad \|\Psi'(u_n)\|(1 + \|u_n\|) \to 0.$$

PROOF. For any $\gamma \in \Gamma$, define the set $K_\gamma = \{\gamma(t) : t \in M\}$ in X and the collection $\mathcal{K} = \{K_\gamma : \gamma \in \Gamma\}$. Let $A = \{\gamma_0(t) : \gamma_0 \in \Gamma_0,\ t \in M_0\}$,

$$\Lambda = \{\varphi \in C(X, X) : \varphi^{-1} \in C(X, X),\ \text{both } \varphi \text{ and } \varphi^{-1} \text{ are bounded on bounded sets}\}$$

and

$$\Lambda(A) = \{\varphi \in \Lambda : \varphi(u) = u, u \in A\}.$$

For any $\gamma \in \Gamma$ and $\varphi \in \Lambda(A)$, let $\gamma_0 = \gamma|_{M_0}$ and $\tilde{\gamma}(t) = \varphi(\gamma(t))$, $t \in M$. Then $\gamma_0 \in \Gamma_0$ and $\tilde{\gamma} \in C(M, X)$. Hence,

$$\tilde{\gamma}(t) = \varphi(\gamma_0(t)) = \gamma_0(t), \quad \forall t \in M_0,$$

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that is, $\tilde{\gamma}|_{M_0} = \gamma_0 \in \Gamma_0$. Therefore,

$$\varphi(K) \in \mathcal{K}, \quad \forall \varphi \in \Lambda(A), \quad K \in \mathcal{K}.$$

These show that the collection $\mathcal{K}$ is a minimax system for A. Since (2.3) implies

$$\infty > c := \inf_{K \in \mathcal{K}} \sup_K \Psi > a := \sup_A \Psi,$$

it follows from [24, Theorem 2.4] that the result is true. □

Lemma 2.2. *Under (V1), (S1), (S2′) and (S5),*

$$\Phi(u) \geq \Phi(tu) - (K-1) \int_{\mathbb{R}^N} F(x, tu)\, dx + \frac{1-t^p}{p} \langle \Phi'(u), u \rangle, \quad \forall u \in E,\ t \in [0, \theta_0]. \tag{2.4}$$

Proof. Note that

$$\langle \Phi'(u), u \rangle = \|u\|^p - \int_{\mathbb{R}^N} f(x, u) u\, dx, \quad \forall u \in E. \tag{2.5}$$

Thus, by (2.1), (2.5) and (S5),

$$\begin{aligned}
\Phi(u) - \Phi(tu) &= \frac{1-t^p}{p} \|u\|^p + \int_{\mathbb{R}^N} [F(x, tu) - F(x, u)]\, dx \\
&= \frac{1-t^p}{p} \langle \Phi'(u), u \rangle + \int_{\mathbb{R}^N} \Big[\frac{1-t^p}{p} f(x, u) u + F(x, tu) - F(x, u) \Big]\, dx \\
&= \frac{1-t^p}{p} \langle \Phi'(u), u \rangle + \int_{\mathbb{R}^N} \Big[\frac{1-t^p}{p} f(x, u) u + K F(x, tu) - F(x, u) \Big]\, dx \\
&\quad - (K-1) \int_{\mathbb{R}^N} F(x, tu)\, dx \\
&\geq -(K-1) \int_{\mathbb{R}^N} F(x, tu)\, dx + \frac{1-t^p}{p} \langle \Phi'(u), u \rangle, \quad t \in [0, \theta_0].
\end{aligned}$$

This shows that (2.4) holds. □

Since (S6) implies (S5) with $K = 1$, we have the following corollary immediately.

Corollary 2.3. *Under (V1), (S1), (S2′) and (S6),*

$$\Phi(u) \geq \Phi(tu) + \frac{1-t^p}{p} \langle \Phi'(u), u \rangle, \quad \forall u \in E,\ t \in [0, \theta_0].$$

Now, we define

$$\Gamma = \{\gamma \in C([0, 1], E) : \gamma(0) = 0,\ \Phi(\gamma(1)) < 0\}$$

and

$$c := \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} \Phi(\gamma(t)).$$

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LEMMA 2.4. *Under (V1), (S1) and (S2′), $c > 0$ and there exists a sequence $\{u_n\} \subset E$ satisfying*

$$\Phi(u_n) \to c, \quad \|\Phi'(u_n)\|(1 + \|u_n\|) \to 0. \tag{2.6}$$

PROOF. In order to prove Lemma 2.4, we apply Lemma 2.1 with $M = [0, 1]$, $M_0 = \{0, 1\}$ and

$$\Gamma_0 = \{\gamma_0 : \{0, 1\} \to E : \gamma_0(0) = 0, \Phi(\gamma_0(1)) < 0\}.$$

By (S2′), there exist $\varepsilon_0 > 0$ and $r_1 > 0$ such that

$$|f(x,t)| \leq \frac{1}{\gamma_p^p + \varepsilon_0}|t|^{p-1}, \quad |t| \leq r_1. \tag{2.7}$$

Combining (2.7) with (S1),

$$|f(x,t)| \leq \frac{1}{\gamma_p^p + \varepsilon_0}|t|^{p-1} + C_{\varepsilon_0}|t|^{p^*-1}, \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}.$$

Then

$$|F(x,t)| \leq \frac{1}{p(\gamma_p^p + \varepsilon_0)}|t|^p + \frac{C_{\varepsilon_0}}{p^*}|t|^{p^*}, \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}. \tag{2.8}$$

Hence, it follows from (1.4) and (2.8) that

$$\begin{aligned}
\Phi(u) &= \frac{1}{p}\|u\|^p - \int_{\mathbb{R}^N} F(x,u)\,dx \\
&\geq \frac{1}{p}\|u\|^p - \frac{1}{p(\gamma_p^p + \varepsilon_0)}\|u\|_p^p - \frac{C_{\varepsilon_0}}{p^*}\|u\|_{p^*}^{p^*} \\
&\geq \frac{\varepsilon_0}{p(\gamma_p^p + \varepsilon_0)}\|u\|^p - \frac{\gamma_{p^*}^{p^*} C_{\varepsilon_0}}{p^*}\|u\|^{p^*},
\end{aligned}$$

which implies that there exists $r > 0$ such that

$$\min_{\|u\| \leq r} \Phi(u) = 0, \quad \inf_{\|u\| = r} \Phi(u) > 0.$$

Hence, we obtain

$$c \geq \inf_{\|u\|=r} \Phi(u) > 0 = \sup_{\gamma_0 \in \Gamma_0} \sup_{t \in M_0} \Phi(\gamma_0(t)).$$

These show that all assumptions of Lemma 2.1 are satisfied. Therefore, there exists a sequence $(u_n) \subset E$ satisfying (2.6). □

LEMMA 2.5. *Under (V1), (S1), (S2), (S3′) and (S5), any sequence $\{u_n\} \subset E$ satisfying*

$$\Phi(u_n) \to c, \quad \langle \Phi'(u_n), u_n \rangle \to 0 \tag{2.9}$$

is bounded in E.

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PROOF. To prove the boundedness of $\{u_n\}$, arguing by contradiction, suppose that $\|u_n\| \to \infty$. Let $v_n = u_n/\|u_n\|$. Then $\|v_n\| = 1$. Passing to a subsequence, we may assume that $v_n \rightharpoonup v$ in E, $v_n \to v$ in $L^s_{\mathrm{loc}}(\mathbb{R}^N)$, $p \le s < p^*$ and $v_n \to v$ almost everywhere on $\mathbb{R}^N$.

If

$$\delta := \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |v_n|^p \, dx = 0,$$

then by Lions' concentration compactness principle [17] or [27, Lemma 1.21], $v_n \to 0$ in $L^s(\mathbb{R}^N)$ for $p < s < p^*$. Fix $q \in (p, p^*)$ and $R > [2p(c+1)]^{1/p}$. By (S1) and (S2), for $\varepsilon = 1/[2pK(\gamma_p^p + \gamma_{p^*}^{p^*} R^{p^*-p})] > 0$ there exists $C_\varepsilon > 0$ such that

$$|f(x,t)| \le \varepsilon(|t|^{p-1} + |t|^{p^*-1}) + C_\varepsilon |t|^{q-1}, \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}.$$

Then

$$|F(x,t)| \le \varepsilon(|t|^p + |t|^{p^*}) + C_\varepsilon |t|^q, \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}.$$

It follows that

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} F(x, Rv_n)\, dx \le \varepsilon[(R\gamma_p)^p + (R\gamma_{p^*})^{p^*}] + R^q C_\varepsilon \lim_{n\to\infty} \|v_n\|_q^q = \frac{R^p}{2Kp}. \tag{2.10}$$

Since $\|u_n\| \to \infty$, $R/\|u_n\| \in [0, \theta_0]$ for large $n \in \mathbb{N}$. Hence, by using (2.9), (2.10) and Lemma 2.2,

$$\begin{aligned}
c + o(1) &= \Phi(u_n) \\
&\ge \Phi(Rv_n) - (K-1)\int_{\mathbb{R}^N} F(x, Rv_n)\, dx + \Big(\frac{1}{p} - \frac{R^p}{p\|u_n\|^p}\Big)\langle \Phi'(u_n), u_n\rangle \\
&= \frac{R^p}{p} - K\int_{\mathbb{R}^N} F(x, Rv_n)\, dx + \Big(\frac{1}{p} - \frac{R^p}{p\|u_n\|^p}\Big)\langle \Phi'(u_n), u_n\rangle \\
&\ge \frac{R^p}{2p} + o(1) > 1 + c + o(1),
\end{aligned}$$

which is a contradiction. Thus, $\delta > 0$.

Going if necessary to a subsequence, we may assume the existence of $k_n \in \mathbb{Z}^N$ such that $\int_{B_{1+\sqrt{N}}(k_n)} |v_n|^p \, dx > (\delta/2)$. Let $w_n(x) = v_n(x + k_n)$; then

$$\int_{B_{1+\sqrt{N}}(0)} |w_n|^p \, dx > \frac{\delta}{2}. \tag{2.11}$$

Now we define $\tilde{u}_n(x) = u_n(x + k_n)$; then $\|\tilde{u}_n\| = \|u_n\|$ and $\tilde{u}_n/\|u_n\| = w_n$. Passing to a subsequence, we have $w_n \rightharpoonup w$ in E, $w_n \to w$ in $L^s_{\mathrm{loc}}(\mathbb{R}^N)$, $p \le s < p^*$ and $w_n \to w$ almost everywhere on $\mathbb{R}^N$. Thus, (2.11) implies that $w \ne 0$.

By (S1) and (S2), there exists $C_3 > 0$ such that

$$|f(x,t)| \le \frac{1}{\gamma_p^p}|t|^{p-1} + C_3 |t|^{p^*-1}, \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R},$$

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which implies that

$$|F(x,t)| \le \frac{1}{p\gamma_p^p}|t|^p + C_3|t|^{p^*}, \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}. \tag{2.12}$$

For $0 \le a < b$, let

$$\Omega_n(a,b) = \{x \in \mathbb{R}^N : a \le |\tilde{u}_n(x)| < b\}.$$

Set $A := \{x \in \mathbb{R}^N : w(x) \neq 0\}$; then $\operatorname{meas}(A) > 0$. For almost every $x \in A$, we have $\lim_{n\to\infty} |\tilde{u}_n(x)| = \infty$. Hence, $A \subset \Omega_n(r_0, \infty)$ for large $n \in \mathbb{N}$; it follows from (2.1), (2.9), (2.12), (S3′) and Fatou's lemma that

$$\begin{aligned}
0 &= \lim_{n\to\infty} \frac{c+o(1)}{\|u_n\|^p} = \lim_{n\to\infty} \frac{\Phi(u_n)}{\|u_n\|^p} \\
&= \lim_{n\to\infty}\Big[\frac{1}{p} - \int_{\mathbb{R}^N} \frac{F(x,\tilde{u}_n)}{|\tilde{u}_n|^p}|w_n|^p\,dx\Big] \\
&= \lim_{n\to\infty}\Big[\frac{1}{p} - \int_{\Omega_n(0,r_0)} \frac{F(x,\tilde{u}_n)}{|\tilde{u}_n|^p}|w_n|^p\,dx - \int_{\Omega_n(r_0,\infty)} \frac{F(x,\tilde{u}_n)}{|\tilde{u}_n|^p}|w_n|^p\,dx\Big] \\
&\le \limsup_{n\to\infty}\Big[\frac{1}{p} + \Big(\frac{1}{p\gamma_p^p} + C_3 r_0^{p^*-p}\Big)\int_{\mathbb{R}^N}|w_n|^p\,dx - \int_{\Omega_n(r_0,\infty)} \frac{F(x,\tilde{u}_n)}{|\tilde{u}_n|^p}|w_n|^p\,dx\Big] \\
&\le \frac{1}{p} + \Big(\frac{1}{p\gamma_p^p} + C_3 r_0^{p^*-p}\Big)\gamma_p^p - \liminf_{n\to\infty}\int_{\Omega_n(r_0,\infty)} \frac{F(x,\tilde{u}_n)}{|\tilde{u}_n|^p}|w_n|^p\,dx \\
&= \frac{1}{p} + \Big(\frac{1}{p\gamma_p^p} + C_3 r_0^{p^*-p}\Big)\gamma_p^p - \liminf_{n\to\infty}\int_{\mathbb{R}^N} \frac{|F(x,\tilde{u}_n)|}{|\tilde{u}_n|^p}[\chi_{\Omega_n(r_0,\infty)}(x)]|w_n|^p\,dx \\
&\le \frac{1}{p} + \Big(\frac{1}{p\gamma_p^p} + C_3 r_0^{p^*-p}\Big)\gamma_p^p - \int_{\mathbb{R}^N} \liminf_{n\to\infty} \frac{F(x,\tilde{u}_n)}{|\tilde{u}_n|^p}[\chi_{\Omega_n(r_0,\infty)}(x)]|w_n|^p\,dx \\
&= -\infty,
\end{aligned}$$

which is a contradiction. Thus, $\{u_n\}$ is bounded in E. □

Lemma 2.6. *Under (V1), (S1), (S2′), (S3′) and (S6), any sequence $\{u_n\} \subset E$ satisfying (2.9) is bounded in E.*

Proof. To prove the boundedness of $\{u_n\}$, arguing by contradiction, suppose that $\|u_n\| \to \infty$. Let $v_n = u_n/\|u_n\|$. Then $\|v_n\| = 1$. Passing to a subsequence, we may assume that $v_n \rightharpoonup v$ in E, $v_n \to v$ in $L^s_{\mathrm{loc}}(\mathbb{R}^N)$, $p \le s < p^*$ and $v_n \to v$ almost everywhere on $\mathbb{R}^N$.

If

$$\delta := \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |v_n|^p\,dx = 0,$$

then by Lions' concentration compactness principle [17] or [27, Lemma 1.21], $v_n \to 0$ in $L^s(\mathbb{R}^N)$ for $p < s < p^*$. Fix $q \in (p, p^*)$ and $R > [p(c+1)(\gamma_p^p + \varepsilon_0)/\varepsilon_0]^{1/p}$, where ε_0 is the same as in (2.7). By (S1) and (2.7), for $\varepsilon = p^*/[4(R\gamma_{p^*})^{p^*}] > 0$ there exists $C_\varepsilon > 0$ such that

$$|f(x,t)| \le \frac{1}{\gamma_p^p + \varepsilon_0}|t|^{p-1} + \varepsilon|t|^{p^*-1} + C_\varepsilon|t|^{q-1}, \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}.$$

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Then

$$|F(x,t)| \le \frac{1}{p(\gamma_p^p+\varepsilon_0)}|t|^p + \frac{\varepsilon}{p^*}|t|^{p^*} + \frac{C_\varepsilon}{q}|t|^q, \quad \forall (x,t)\in\mathbb{R}^N\times\mathbb{R}.$$

It follows that

$$\begin{aligned}\limsup_{n\to\infty}\int_{\mathbb{R}^N} F(x,Rv_n)\,dx &\le \frac{(R\gamma_p)^p}{p(\gamma_p^p+\varepsilon_0)} + \frac{\varepsilon(R\gamma_{p^*})^{p^*}}{p^*} + \frac{R^qC_\varepsilon}{q}\lim_{n\to\infty}\|v_n\|_q^q \\ &= \frac{(R\gamma_p)^p}{p(\gamma_p^p+\varepsilon_0)} + \frac{1}{4}. \end{aligned} \tag{2.13}$$

Since $\|u_n\|\to\infty$, $R/\|u_n\|\in[0,\theta_0]$ for large $n\in\mathbb{N}$. Hence, using (2.9), (2.13) and Corollary 2.3,

$$\begin{aligned} c+o(1) = \Phi(u_n) &\ge \Phi(Rv_n) + \Big(\frac{1}{p} - \frac{R^p}{p\|u_n\|^p}\Big)\langle\Phi'(u_n),u_n\rangle \\ &= \frac{R^p}{p} - \int_{\mathbb{R}^N} F(x,Rv_n)\,dx + \Big(\frac{1}{p} - \frac{R^p}{p\|u_n\|^p}\Big)\langle\Phi'(u_n),u_n\rangle \\ &\ge \frac{\varepsilon_0R^p}{p(\gamma_p^p+\varepsilon_0)} - \frac{1}{4} + o(1) > \frac{3}{4} + c + o(1), \end{aligned}$$

which is a contradiction. Thus, $\delta>0$. The rest of the proof is the same as that of Lemma 2.5. □

Lemma 2.7. *Under (V1), (S1), (S2′) and (S7), any sequence $\{u_n\}\subset E$ satisfying (2.9) is bounded in E.*

Proof. By (2.1), (2.2), (2.9) and (S7),

$$\begin{aligned} c+1 &\ge \Phi(u_n) - \frac{1}{\mu}\langle\Phi'(u_n),u_n\rangle \\ &= \frac{\mu-p}{p\mu}\|u_n\|^p + \int_{\mathbb{R}^N}\Big[\frac{1}{\mu}f(x,u_n)u_n - F(x,u_n)\Big]\,dx \\ &\ge \frac{\mu-p}{p\mu}\|u_n\|^p \quad \text{for large } n\in\mathbb{N}, \end{aligned}$$

which implies that $\{u_n\}$ is bounded in E. □

Lemma 2.8. *Under (V1), (S1), (S2), (S3′) and (S5), (1.1) has a nontrivial solution, that is, $\mathcal{M}\ne\phi$.*

Proof. Lemma 2.4 implies the existence of a sequence $\{u_n\}\subset E$ satisfying (2.6) and so (2.9). By Lemma 2.5, $\{u_n\}$ is bounded in E. Thus, there exists a constant $C_4>0$ such that

$$\|u_n\|_p + \|u_n\|_{p^*} \le C_4.$$

If

$$\delta := \limsup_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B_1(y)} |u_n|^p\,dx = 0,$$

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then by Lions' concentration compactness principle [17] or [27, Lemma 1.21], $u_n \to 0$ in $L^s(\mathbb{R}^N)$ for $p < s < p^*$. Fix $q \in (p, p^*)$. By (S1) and (S2), for $\varepsilon = c/(C_4^p + C_4^{p^*}) > 0$ there exists $C_\varepsilon > 0$ such that

$$|tf(x,t) - pF(x,t)| \le \varepsilon(|t|^p + |t|^{p^*}) + C_\varepsilon |t|^q, \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}.$$

Thus,

$$\begin{aligned}\limsup_{n\to\infty} \int_{\mathbb{R}^N} \Big[\frac{1}{p} f(x,u_n)u_n - F(x,u_n)\Big]\,dx &\le \frac{\varepsilon}{p}(C_4^p + C_4^{p^*}) + \frac{C_\varepsilon}{p} \lim_{n\to\infty} \|u_n\|_q^q \\ &= \frac{c}{p},\end{aligned}$$

which, together with (2.9), implies that

$$\begin{aligned} c &= \Phi(u_n) - \frac{1}{p}\langle \Phi'(u_n), u_n\rangle + o(1) \\ &= \int_{\mathbb{R}^N} \Big[\frac{1}{p} f(x,u_n)u_n - F(x,u_n)\Big]\,dx + o(1) \le \frac{c}{p} + o(1).\end{aligned}$$

This contradiction shows that $\delta > 0$.

Going if necessary to a subsequence, we may assume the existence of $k_n \in \mathbb{Z}^N$ such that $\int_{B_{1+\sqrt{N}}(k_n)} |u_n|^p\,dx > \delta/2$. Let us define $v_n(x) = u_n(x + k_n)$ so that

$$\int_{B_{1+\sqrt{N}}(0)} |v_n|^p\,dx > \frac{\delta}{2}. \tag{2.14}$$

Since $V(x)$ and $f(x,u)$ are periodic, we have $\|v_n\| = \|u_n\|$ and

$$\Phi(v_n) \to c, \quad \|\Phi'(v_n)\|(1 + \|v_n\|) \to 0.$$

Passing to a subsequence, we have $v_n \rightharpoonup v$ in E, $v_n \to v$ in $L^s_{\mathrm{loc}}(\mathbb{R}^N)$, $p \le s < p^*$ and $v_n \to v$ almost everywhere on $\mathbb{R}^N$. Thus, (2.14) implies that $v \ne 0$. For every $w \in C_0^\infty(\mathbb{R}^N)$,

$$\langle \Phi'(v), w\rangle = \lim_{n\to\infty} \langle \Phi'(v_n), w\rangle = 0.$$

Hence, $\Phi'(v) = 0$. This shows that $v \in \mathcal{M}$ is a nontrivial solution of (1.1). □

Lemma 2.9. *Under (V1), (S1), (S2′), (S3′) and (S6) or (S7), (1.1) has a nontrivial solution, that is, $\mathcal{M} \ne \phi$.*

The proof is similar to that of Lemma 2.8, so we omit it. □

The proof of Theorem 1.2 is rather similar to that of Theorem 1.3, so we only give the proof of Theorem 1.3.

Proof of Theorem 1.3. Lemma 2.9 shows that $\mathcal{M}$ is not an empty set. Let $c_0 = \inf_{\mathcal{M}} \Phi$. By Corollary 2.3, one has $\Phi(u) \ge \Phi(0) = 0$ for all $u \in \mathcal{M}$. Thus, $c_0 \ge 0$.

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Let $\{u_n\} \subset \mathcal{M}$ be such that $\Phi(u_n) \to c_0$. Then $\langle \Phi'(u_n), u_n\rangle = 0$. In view of the proof of Lemma 2.6 ($c > 0$ is not necessary), $\{u_n\}$ is bounded in E, and

$$\|u_n\|^p = \int_{\mathbb{R}^N} f(x, u_n)u_n\,dx.$$

Let $\inf_{n\in\mathbb{N}} \|u_n\| = \delta_0$. If $\delta_0 = 0$, going if necessary to a subsequence, we may assume that $\|u_n\| \to 0$. Fix $q \in (p, p^*)$; by (S1) and (S2′), there exist $\varepsilon_0 > 0$ and $C_5 > 0$ such that

$$|f(x,t)| \le \frac{1}{\gamma_p^p + \varepsilon_0}|t|^{p-1} + |t|^{p^*-1} + C_5|t|^{q-1}, \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}.$$

Thus,

$$\begin{aligned}
\|u_n\|^p &= \int_{\mathbb{R}^N} f(x,u_n)u_n\,dx \\
&\le \frac{1}{\gamma_p^p + \varepsilon_0}\|u_n\|_p^p + \|u_n\|_{p^*}^{p^*} + C_5\|u_n\|_q^q \\
&\le \frac{\gamma_p^p}{\gamma_p^p + \varepsilon_0}\|u_n\|^p + \gamma_{p^*}^{p^*}\|u_n\|^{p^*} + C_5\gamma_q^q\|u_n\|^q,
\end{aligned}$$

which implies that

$$\frac{\varepsilon_0}{\gamma_p^p + \varepsilon_0} \le \gamma_{p^*}^{p^*}\|u_n\|^{p^*-p} + C_5\gamma_q^q\|u_n\|^{q-p} = o(1).$$

This contradiction shows that $\inf_{n\in\mathbb{N}} \|u_n\| = \delta_0 > 0$. Choose a constant $C_6 > 0$ such that $\|u_n\|_{p^*} \le C_6$. If

$$\delta := \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |u_n|^p\,dx = 0,$$

then by Lions' concentration compactness principle [27, Lemma 1.21], $u_n \to 0$ in $L^s(\mathbb{R}^N)$ for $p < s < p^*$. Fix $q \in (p, p^*)$. By (S1) and (2.7), for $\varepsilon = \varepsilon_0\delta_0^p/[2(\gamma_p^p + \varepsilon_0)C_6^{p^*}] > 0$, where ε_0 is given by (2.7), there exists $C_\varepsilon > 0$ such that

$$|f(x,t)| \le \frac{1}{\gamma_p^p + \varepsilon_0}|t|^{p-1} + \varepsilon|t|^{p^*-1} + C_\varepsilon|t|^{q-1}, \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}.$$

Thus,

$$\|u_n\|^p = \int_{\mathbb{R}^N} f(x,u_n)u_n\,dx \le \frac{\gamma_p^p}{\gamma_p^p + \varepsilon_0}\|u_n\|^p + \varepsilon\|u_n\|_{p^*}^{p^*} + C_\varepsilon\|u_n\|_q^q,$$

which yields that

$$\frac{\varepsilon_0\delta_0^p}{\gamma_p^p + \varepsilon_0} \le \frac{\varepsilon_0}{\gamma_p^p + \varepsilon_0}\|u\|^p \le \varepsilon\|u\|_{p^*}^{p^*} + C_\varepsilon\|u\|_q^q \le \varepsilon C_6^{p^*} + o(1) = \frac{\varepsilon_0\delta_0^p}{2(\gamma_p^p + \varepsilon_0)} + o(1).$$

This contradiction shows that $\delta > 0$.

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Going if necessary to a subsequence, we may assume the existence of $k_n \in \mathbb{Z}^N$ such that $\int_{B_{1+\sqrt{N}}(k_n)} |u_n|^p\,dx > (\delta/2)$. Let us define $v_n(x) = u_n(x + k_n)$ so that

$$\int_{B_{1+\sqrt{N}}(0)} |v_n|^p\,dx > \frac{\delta}{2}. \tag{2.15}$$

Since $V(x)$ and $f(x, u)$ are periodic, we have $\|v_n\| = \|u_n\|$ and

$$\Phi(v_n) \to c_0, \quad \Phi'(v_n) = 0.$$

Passing to a subsequence, we have $v_n \rightharpoonup v_0$ in E, $v_n \to v_0$ in $L^s_{\mathrm{loc}}(\mathbb{R}^N)$, $p \le s < p^*$ and $v_n \to v_0$ almost everywhere on $\mathbb{R}^N$. Thus, (2.15) implies that $v_0 \neq 0$. For every $w \in C_0^\infty(\mathbb{R}^N)$,

$$\langle \Phi'(v_0), w\rangle = \lim_{n\to\infty} \langle \Phi'(v_n), w\rangle = 0.$$

Hence $\Phi'(v_0) = 0$. This shows that $v_0 \in \mathcal{M}$ and so $\Phi(v_0) \ge c_0$. On the other hand, by using (S6) and Fatou's lemma,

$$\begin{aligned}
c_0 &= \lim_{n\to\infty}\Big[\Phi(v_n) - \frac{1}{p}\langle \Phi'(v_n), v_n\rangle\Big] \\
&= \lim_{n\to\infty} \int_{\mathbb{R}^N} \Big[\frac{1}{p} f(x, v_n)v_n - F(x, v_n)\Big]\,dx \\
&\ge \int_{\mathbb{R}^N} \lim_{n\to\infty} \Big[\frac{1}{p} f(x, v_n)v_n - F(x, v_n)\Big]\,dx \\
&= \int_{\mathbb{R}^N} \Big[\frac{1}{p} f(x, v_0)v_0 - F(x, v_0)\Big]\,dx \\
&= \Phi(v_0) - \frac{1}{p}\langle \Phi'(v_0), v_0\rangle = \Phi(v_0).
\end{aligned}$$

This shows that $\Phi(v_0) \le c_0$ and so $\Phi(v_0) = c_0 = \inf_{\mathcal{M}} \Phi$. □

Proof of Theorem 1.4. By the same proof of Theorem 1.3 using Lemma 2.7 instead of Lemma 2.6, there exists $v_0 \in E \setminus \{0\}$ such that $\Phi'(v_0) = 0$. This shows that $v_0 \in \mathcal{M}$ and so $\Phi(v_0) \ge c_0$. On the other hand, by using (S7) and Fatou's lemma,

$$\begin{aligned}
c_0 &= \lim_{n\to\infty}\Big[\Phi(v_n) - \frac{1}{\mu}\langle \Phi'(v_n), v_n\rangle\Big] \\
&= \lim_{n\to\infty}\Big\{\frac{\mu - p}{p\mu}\|v_n\|^p + \int_{\mathbb{R}^N} \Big[\frac{1}{\mu} f(x, v_n)v_n - F(x, v_n)\Big]\,dx\Big\} \\
&\ge \frac{\mu - p}{p\mu} \liminf_{n\to\infty} \|v_n\|^p + \int_{\mathbb{R}^N} \liminf_{n\to\infty} \Big[\frac{1}{\mu} f(x, v_n)v_n - F(x, v_n)\Big]\,dx \\
&\ge \frac{\mu - p}{p\mu}\|v_0\|^p + \int_{\mathbb{R}^N} \Big[\frac{1}{\mu} f(x, v_0)v_0 - F(x, v_0)\Big]\,dx \\
&= \Phi(v_0) - \frac{1}{\mu}\langle \Phi'(v_0), v_0\rangle = \Phi(v_0).
\end{aligned}$$

This shows that $\Phi(v_0) \le c_0$ and so $\Phi(v_0) = c_0 = \inf_{\mathcal{M}} \Phi$. □

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YI CHEN, School of Mathematics and Statistics,
Central South University, Changsha, Hunan 410083, PR China
e-mail: mathcyt@163.com

X. H. TANG, School of Mathematics and Statistics,
Central South University, Changsha, Hunan 410083, PR China
e-mail: tangxh@csu.edu.cn