STATIONARY SOLUTIONS OF THE AVERAGED THREE-BODY PROBLEM AND SOME PROBLEMS OF PLANET MOTION STABILITY

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Abstract. A general theory of stationary solutions of the averaged N-body problem is briefly described. Numerical results in some particular cases (general three-body problem in the nonresonant case and restricted circular problem both in the nonresonant and resonant ones) are presented. Some applications into the problem of planetary stability are developed.

1. Introduction

At present the significance of the averaged problems of celestial mechanics for investigating planet motions is clearly understood. To a certain degree the stationary solutions of the averaged problems determine the general features of the planet motions and that is why their study is of great importance. If we put aside the problem of convergence, the stationary solutions can be considered as a natural extension of Poincaré's periodic solutions to the case of arbitrary resonances (and to the nonresonant case as well) for the N-planet problem. In recent years a number of works were devoted to the averaged restricted problem of three bodies and its stationary solutions; some important results were obtained by Jefferys and Standish (1966, 1972), Kozai (1962, 1969), Lidov (1962). Other works dealt with applications of the plane stationary solutions to the stability problem for resonant asteroids (Schubart, 1964, 1968; Sinclair, 1969; Marsden, 1970). At the same time the corresponding analysis of the general three-body problem has not yet been developed to such a degree. In this case some results for the problem of the critical inclinations were obtained by von Zeipel (1898, 1901) and Jefferys and Moser (1966). Recently Lieberman (1971) constructed plane stationary solutions by a converging method. For the resonant case some periodic solutions of a type which is called 'trivial' in this paper were constructed by Poincaré (1892–1895), but they by no means exhaust the rich set of the existing periodic solutions. In Krasinsky (1972) an attempt was made to work out a general mathematical theory of the stationary solutions of the averaged N-planet system and to give a unified treatment of all these results which might seem rather heterogeneous. Here we shall use the terminology by Krasinsky (1972). By the averaged system we mean the system obtained after eliminating all short-periodic terms from the original Hamiltonian of the N-planet system by von Zeipel's method. Taking into account arbitrary high (but finite) powers of a small parameter μ (which is of the order of the disturbing masses) we may consider the averaged Hamiltonian as represented by a convergent series. The variant of von Zeipel's method proposed in Krasinsky (1973) enables us to preserve all the invariant properties characterizing

Y. Kozai (ed.), The Stability of the Solar System and of Small Stellar Systems, 95–116.

the Hamiltonian of the original system, at the highest approximations. Namely, at any step the averaged Hamiltonian proves to be invariant under three linear transformations; one of them being a rotation of the reference frame relative to vector **l** of the angular momentum and the others being reflections relative to two orthogonal planes. One of the planes is orthogonal to the vector **l** (invariant Laplace plane) and the other contains **l**. As a result the Hamiltonian H remains independent of time in a reference frame rotating uniformly with the arbitrary angular velocity σ with respect to **l** and differs from the Hamiltonian based on the immovable reference frame by the Coriolis' term $\sigma|\mathbf{l}|$ only. The aim of our investigation is to find stationary solutions of the corresponding system (more exactly, of the system arising after substituting $l_s \to l_s + n_s t$, $s = 1, \ldots, N$, where l_s are the mean longitudes of the averaged system, and n_s are the mean motions). The problem may be reduced to finding extrema of H on the hypersurface determined by the area integral,

$$|\mathbf{l}| = c, \tag{1}$$

under the condition that the invariant plane is chosen as a reference one. The Lagrange factor of this conditional extremum problem coincides with the unknown angular velocity σ. If we restrict ourselves to symmetrical solutions only, we have to seek extrema H relative to nonangular variables, angular variables (the longitudes of the perihelia and nodes and the critical arguments) having to be put equal to certain values according to symmetry conditions. In particular, the critical arguments are equal either to 0 or π. As $\mu \to 0$ the problem reduces to finding the extrema of the averaged perturbation function $[R]$ on the hypersurface (1) relative to the eccentricities and inclinations only. We shall use the scheme of classification of the symmetrical stationary solutions, given in Table I (Krasinsky, 1973).

TABLE I

Classification of the stationary solutions of the N-planet problem

	Solutions of the 1st kind	Solutions of the 2nd and 3rd kind			
		Trivial solutions		Nontrivial solutions (of the 3rd kind)	
		Plane solutions (of the 2nd kind)	Space solutions (of the 3rd kind)	Positive type solutions	Negative type solutions
Averaged values of the eccentricities and inclinations	$e = i = 0$	$e \neq 0$ $i = 0$	$e = 0$ $i \neq 0$	$e \neq 0$ $i \neq 0$	$e \neq 0$ $i \neq 0$
Averaged values of the perihelion arguments	–	–	–	$0, \pi$	$-\frac{1}{2}\pi, \frac{1}{2}\pi$

The extremal treatment of the stationary solutions gives the opportunity to prove easily the existence of 'trivial solutions' (for which the arguments of the perihelia are

https://doi.org/10.1017/S0074180900070443 Published online by Cambridge University Press

undetermined) at any values of the area constant c in (1), i.e. for any values of the eccentricities or inclinations. In fact, the integral (1) which is considered as a function of the eccentricities $e_1, \ldots, e_N$ and inclinations $i_1, \ldots, i_N$, determines a hypersurface which is homeomorphic to a $2 \times N$-dimensional sphere. Hence, existence of at least two trivial plane solutions providing $[R]$ with the minimum and the maximum is evident (if $[R]$ has no singularities on (1)). In the same way existence of two space trivial ('circular') solutions may be established if the resonances are of the odd orders or absent at all. But in this case one of these solutions corresponds to the zero value of σ; it is a well-known solution of the 'first kind' (i.e. plane and 'circular') related to a reference plane which does not coincide with the common plane of the planet orbits.

It is important to investigate the dependence of the trivial solutions on the area constant c. In Krasinsky (1972) two equations were deduced:

$$g^+(c) = 0, \tag{2a}$$

$$g^-(c) = 0, \tag{2b}$$

which determine 'bifurcational' values of c. If any of Equations (2) is fulfilled a nontrivial solution branches from the trivial solution under consideration. If Equation (2a) is fulfilled, the resulting nontrivial solution is of positive type, otherwise this solution is of negative type. Existence of the bifurcational values is intimately connected with stability of the corresponding trivial solution. If the parameter c passes its bifurcational value, then a pair of the characteristic exponents vanishes and the imaginary exponents become real (or on the contrary). Let c_0 be the value of c corresponding to zero eccentricities and inclinations. Then, if c^* is the smallest of the bifurcational values (at $c_0 > c$) the stationary trivial solution under consideration being stable at $c_0 > c > c^*$ becomes unstable as $c < c^*$. The value c^* will be called the critical one. If we consider the space trivial solutions the corresponding inclinations will be called critical. The notion of critical eccentricity may be introduced in the same way.

The main objective of this paper is to present results of the numerical calculations illustrating the general theory for restricted and general three-body problem. For the restricted problem, the stability of stationary values of the critical arguments is investigated as well. In particular, a new type of stationary solutions with librational motion of the critical argument is constructed. For these solutions (which are of the space trivial type) the close approaches of 'asteroid' with 'Jupiter' are impossible, even for commensurability 1:1.

2. The General Three-Body Problem in the Non-Resonant Case. Bifurcational and Critical Values

As it was mentioned above, constructing the stationary solutions may be reduced to finding extrema of the averaged perturbation function $[R]$ on the hypersurface (1). In the nonresonant case $[R]$ depends on the mutual inclination I, the angles, φ_1 and φ_2, of the eccentricities, e_1 and e_2, the arguments of the perihelia, g_1 and g_2 (which are referred to the common line of the orbital plane intersection), and the difference $\Omega_1 - \Omega_2$ of the nodes, Ω_1 and Ω_2 (indices 1 and 2 relating to inner and outer planets, respectively). If Laplace's invariant plane is chosen as the reference one, the area integral (1) may be written in the following form:

$$\beta_1 \sqrt{a_1} \cos\varphi_1 \cos i_1 + \beta_2 \sqrt{a_2} \cos\varphi_2 \cos i_2 = c, \tag{3}$$

where i_1, i_2 are the inclinations, a_1, a_2 are the semimajor axes, m_1, m_2 are the planet masses, $\beta_j = km_0 m_j / \sqrt{(m_0 + m_j)}$, $j = 1, 2$, m_0 is the mass of the central body.

Other two area integrals give

$$\beta_1 \sqrt{a_1} \sin i_1 = \beta_2 \sqrt{a_2} \sin i_2, \tag{4}$$

$$\Omega_1 - \Omega_2 = \pi. \tag{5}$$

As the nodes on the invariant plane coincide, we have

$$I = i_1 + i_2. \tag{6}$$

The function R must be calculated at certain values of the angular variables. Namely, according to formula (5) the difference $\Omega_1 - \Omega_2$ has to be equal to π and for the perihelion arguments it is necessery to set either $g_1 = 0, \pi$; $g_2 = 0, \pi$ (positive type solutions), or $g_1 = -\frac{1}{2}\pi, \frac{1}{2}\pi$; $g_2 = -\frac{1}{2}\pi, \frac{1}{2}\pi$ (negative type solutions). The choice of the values g_1 and g_2 is not arbitrary but determined by a condition of positivity of the extremal values e_1 and e_2. If e_1 (or e_2) proves to be negative we always can add π to the corresponding value of the perihelion argument and thus change the sign e_1 (or e_2). It seems convenient to put $g_1 = g_2 = 0$ (for the positive type solutions) or $g_1 = g_2 = \frac{1}{2}\pi$ for the negative type solutions), and search extrema $[R]$ at $-1 < e_1, e_2 < 1$. Writing down the equations for finding a conditional extremum we have

$$\begin{aligned} -\sin I \frac{\partial [R]}{\partial \cos I} &= -\sigma \beta_1 \sqrt{a_1} \sin i_1 \cos\varphi_1, \\ -\sin I \frac{\partial [R]}{\partial \cos I} &= -\sigma \beta_2 \sqrt{a_2} \sin i_2 \cos\varphi_2; \end{aligned} \tag{7}$$

$$\begin{aligned} \frac{\partial [R]}{\partial \varphi_1} &= -\sigma \beta_1 \sqrt{a_1} \cos i_1 \sin\varphi_1, \\ \frac{\partial [R]}{\partial \varphi_2} &= -\sigma \beta_2 \sqrt{a_2} \cos i_2 \sin\varphi_2, \end{aligned} \tag{8}$$

where σ is a Lagrange factor.

Further, we use nondimensional parameters β and δ:

$$\beta = \frac{\beta_1 \sqrt{a_1}}{\beta_2 \sqrt{a_2}} = \frac{n_2 m_1}{n_1 m_2 \alpha}, \qquad \delta = c/(\beta_1 \sqrt{a_1} + \beta_2 \sqrt{a_2}) \quad (\alpha = a_1/a_2).$$

First we consider conditions of existence of the plane trivial solutions. As it was noticed above there exist at least two such solutions if $[R]$ has no singularity on the surface,

$$\beta_1 \sqrt{a_1} \cos\varphi_1 + \beta_2 \sqrt{a_2} \cos\varphi_2 = c. \tag{9}$$

The singularities may correspond either to the case of the unit value of the eccentricities or the case of intersection of the orbits. Both of these situations cannot take place under conditions

$$a_1(1+e_1) < a_2(1-e_2), \qquad e_1 < 1, \quad e_2 < 1,$$

which are fulfilled if the inequalities,

$$\alpha < \tfrac{1}{2}\left(1 - \sqrt{1 - ((1+\beta)\,\delta - \beta)^2}\right), \qquad \delta > \max(1, \beta)/(1+\beta), \tag{10}$$

hold. It may be proved that, if eccentricities are small (more exactly if $\beta_1 \sqrt{a_1} + \beta_2 \sqrt{a_2} \sim c$), the longitudes of the perihelia for some of these solutions coincide with each other and that for others they differ by π.

In order to construct the space trivial solutions we must put $\varphi_1 = \varphi_2 = 0$ in Equations (7) and (8); then Equations (7) will be satisfied identically, and only one of Equations (8) will remain independent. Finding σ in terms of I from (8) we have

$$\sigma = \frac{\partial [R]}{\partial \cos I} \frac{\sqrt{a_1}\, \beta_1 \cos i_1 + \sqrt{a_2}\, \beta_2 \cos i_2}{\beta_1 \beta_2 \sqrt{a_1 a_2}}. \tag{11}$$

Thus, the space trivial solutions exist for arbitrary mutual inclinations I. Any solutions with small initial eccentricities will be sometimes in a vicinity of the corresponding space trivial solution. That is why finding the critical inclinations (which are the upper bound of the inclinations of the stable stationary 'circular' orbits) is a problem of great interest. As the critical inclinations belong to the set of bifurcational inclinations they can be found by equating the Jacobian of system (8) (relative to φ_1 and φ_2) to zero. Using the notations,

$$[R] = \frac{k^2 m_1 m_2}{a_2} \left[\frac{a_2}{\Delta}\right], \qquad d_{ij} = \frac{\partial^2 [a_2/\Delta]}{\partial \varphi_i \varphi_j}\bigg|_{\substack{\varphi_1 = 0\\ \varphi_2 = 0}}, \qquad v = \frac{\partial [a_2/\Delta]}{\partial \cos I},$$

we have

$$d_{ij} = a_{ij} \cos(g_i - g_j) + b_{ij} \cos(g_i + g_j), \quad i, j = 1, 2$$

(coefficients a_{ij}, b_{ij}, v are found in Krasinsky (1973). Hence, the equations for finding the bifurcational inclinations are expressed as follows:

$$\begin{vmatrix} a_{11} \pm b_{11} - v \cos i_1 (\cos i_2 + \beta \cos i_1), & a_{12} \pm b_{12} \\ (a_{21} \pm b_{21})\, \beta, & \beta (a_{22} \pm b_{22}) - v \cos i_2 (\cos i_2 + \beta \cos i_1) \end{vmatrix} = 0.$$

Expressing the left sides of these equations in terms of $\cos I$ (by means of (4) and (6)) we find

$$\begin{vmatrix} a_{11} \pm b_{11} - v(\beta + \cos I), & a_{12} \pm b_{12} \\ \beta (a_{21} \pm b_{21}), & \beta (a_{22} \pm b_{22}) - v (1 + \beta \cos I) \end{vmatrix} = 0. \tag{12}$$

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Now we consider the bifurcational and critical inclinations for small α. If the ratio of the planet masses is finite as $\alpha \to 0$, according to its definition β tends to zero too. It is

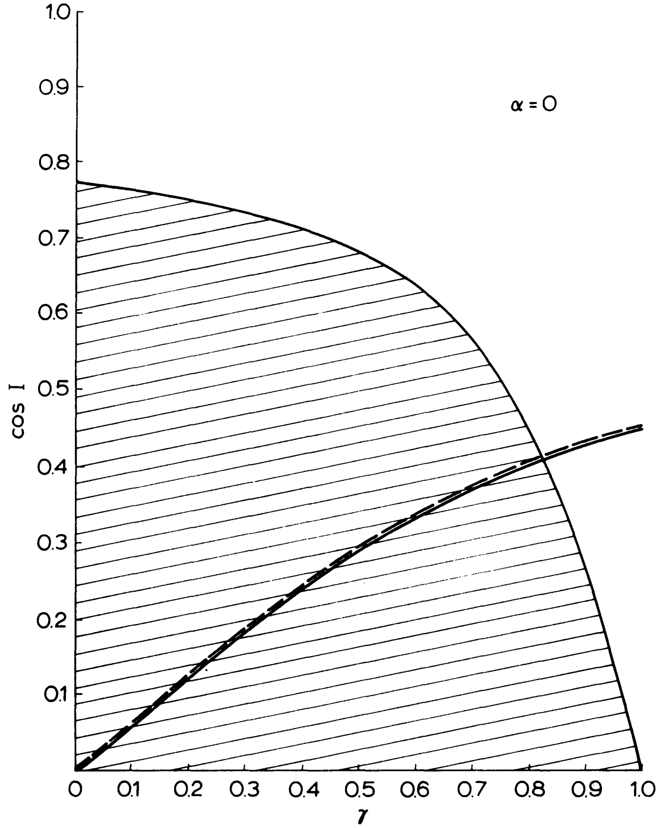

Fig. 1. Stable and unstable regions for circular motion of $\alpha = 0.0$. Unstable regions are hatched.

more convenient to consider α and β as independent parameters for any α. Writing down only the lowest terms in α for the coefficients a_{ij}, b_{ij} and ν we have

$$a_{11} = a_{22} = \tfrac{1}{2}\alpha^2(-1 + 3\cos^2 I),$$
$$b_{11} = b_{22} = \tfrac{15}{8}\alpha^2 \sin^2 I, \qquad \nu = -\tfrac{3}{4}\alpha^2 \cos^2 I,$$
$$a_{12} = a_{21} = b_{12} = b_{21} = 0.$$

Thus, as $\alpha \to 0$ the equation determining the bifurcational inclinations for the negative type solution becomes

$$(5\cos^2 I - 3 + \beta \cos I)(5\beta \cos^2 I - \beta + 2\cos I) = 0.$$

For the positive type solutions we have

$$(1 + 2\beta \cos I)(5\beta \cos^2 I - \beta + 2\cos I) = 0.$$

Roots of these equations in the interval $(-1, 1)$ are the following (negative $\cos I$ corresponding to the retrograde orbites):

$$\begin{aligned}
\cos I_1^- &= \cos I_1^+ = \sqrt{\frac{1}{5}+\frac{1}{25\beta^2}} - \frac{1}{5\beta},\\
\cos I_2^+ &= \cos I_2^- = -\sqrt{\frac{1}{5}+\frac{1}{25\beta^2}} - \frac{1}{5\beta}\\
\cos I_3^+ &= -\frac{1}{2\beta}, \quad 2 \leqslant \beta \leqslant \infty,\\
\cos I_3^- &= \sqrt{\frac{3}{5}+\frac{\beta^2}{100}} - \frac{\beta}{10}, \quad 0 \leqslant \beta \leqslant \infty,\\
\cos I_4^- &= -\sqrt{\frac{3}{5}+\frac{\beta^2}{100}} - \frac{\beta}{10}, \quad 0 \leqslant \beta \leqslant 2.
\end{aligned} \tag{13}$$

Here plus and minus sign mark the roots for which bifurcation into the positive

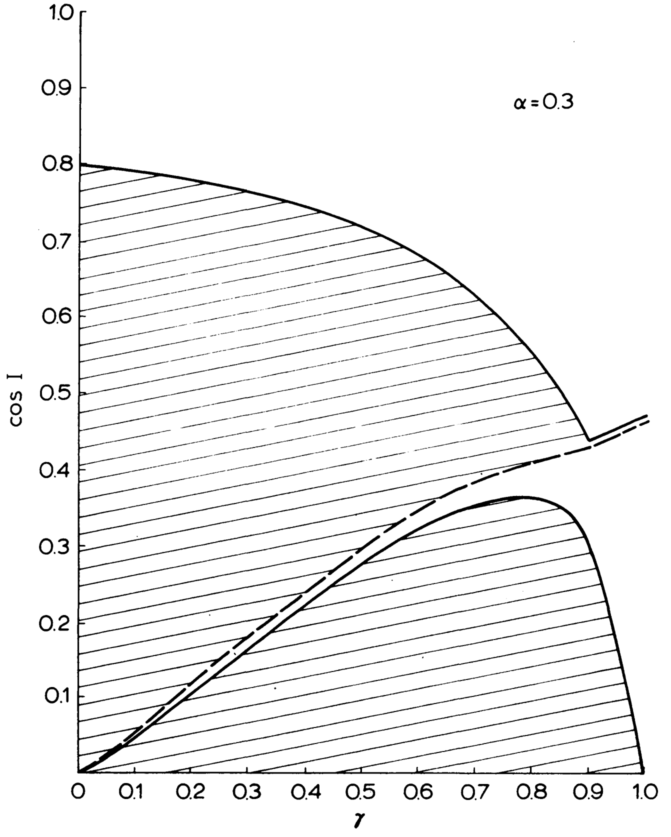

Fig. 2. Stable and unstable regions for circular motion of $\alpha = 0.3$.

or the negative type solutions takes place. The roots $I_1^- = I_1^+$, $I_2^- = I_2^+$ are double ones of the equation for the characteristic exponents and though a pair of these exponents becomes equal to zero at $I = I_1^-$ or $I = I_2^-$ the stability of the space trivial solution is

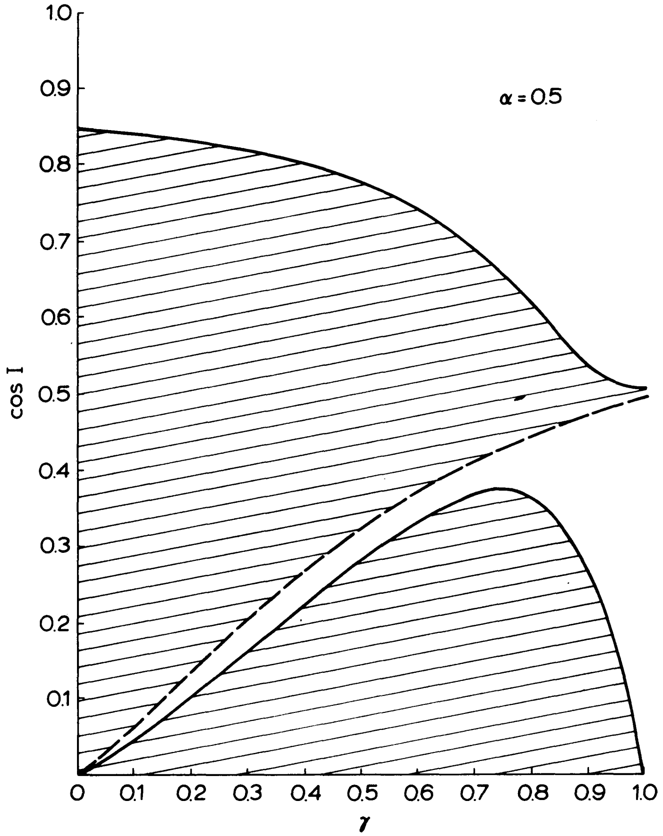

Fig. 3. Stable and unstable regions for circular motion of $\alpha = 0.5$.

preserved. For the graphic demonstration it seems convenient to use the parameter $\gamma = \beta/(1+\beta)$ instead of β. The subdivision of the phase plane $(\cos I, \gamma)$ into domains of stability and unstability of the circular motion is given in Figures 1–5 for several α's. The unstable domain is hatched. The boundary between stable and unstable domains is drawn by a broken or solid line if it corresponds to the points of bifurcation of the positive or negative type. If $\alpha \neq 0$ the structure of the phase plane undergoes a qualitative change because the coinciding (at $\alpha = 0$) lines now become divergent and there arises a new narrow region of stability. These results were obtained by solving Equation (12) numerically.

3. The Averaged Restricted Circular Three-Body Problem in the Nonresonant Case

In the nonresonant case the averaged restricted circular problem is integrable and the stationary solutions determine the topological structure of the phase plane (e, g) at the different values of the 'area integral',

$$\cos\varphi \cos I = c, \tag{14}$$

(which is an integral only for the averaged system). In as much as the averaged

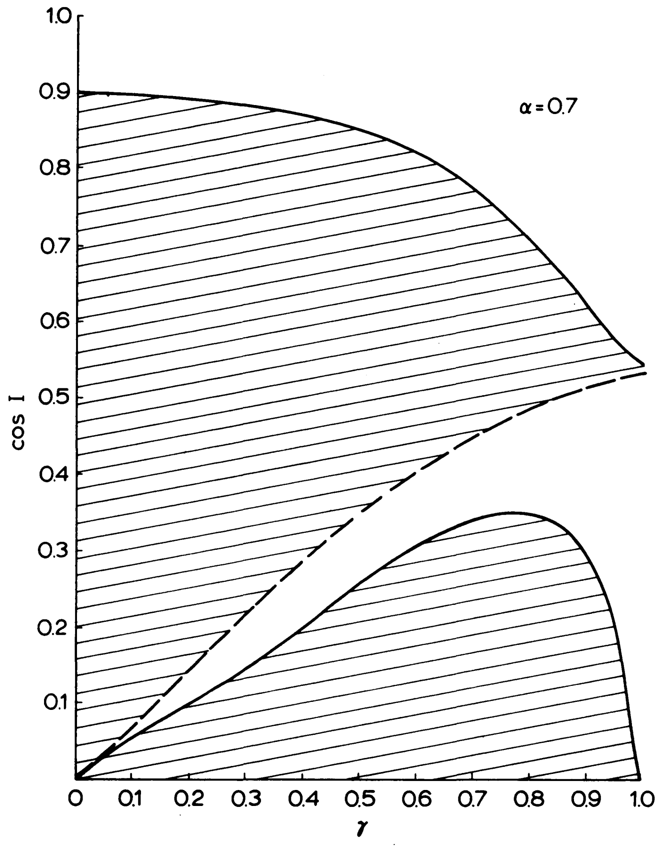

Fig. 4. Stable and unstable regions for circular motion of $\alpha = 0.7$.

perturbation function depends on $\cos^2 I$ the region of the direct and retrograde orbits are similar to each other and we can consider orbits with the direct motion only (for which $\cos I \geqslant 0$). The equations to find the bifurcational points for the inner variant of the problem may be deduced from (12) as $\beta \to 0$:

$$a_{11} \pm b_{11} - \nu \cos I = 0.$$

For the outer variant $\beta \to \infty$ we have the equation

$$a_{22} \pm b_{22} - v \cos I = 0.$$

In accordance with (13) we find for the inner case the following values of the bifurcational values (they are critical ones too) as $\alpha \to 0$:

$$\cos I = \sqrt{\tfrac{3}{5}}.$$

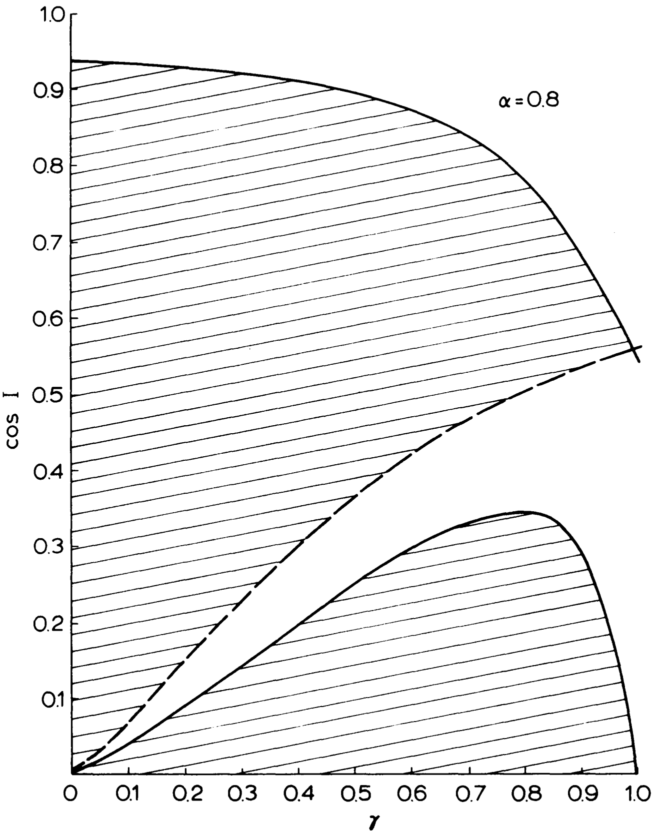

Fig. 5. Stable and unstable regions for circular motion of $\alpha = 0.8$.

For the outer case we have

$$\cos I = \sqrt{\tfrac{1}{5}}.$$

Existence of the critical inclinations in the inner variant may be proved at the arbitrary α (Krasinsky, 1972).

In this case the critical inclinations $I^*(\alpha)$ decrease with increasing α (see Table II).

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Moreover, as it is proved in Krasinsky (1972) $\lim_{\alpha\to 1} I(\alpha)=0$, and the following asymptotic formula takes place:

$$\cos I^* = 1 - \tfrac{1}{4}(1-\alpha) + 0(|1-\alpha|^{3/2}\ln(1-\alpha)|).$$

For the outer problem calculations show (at $\alpha<0.95$) the existence of points of bifurcation in the positive ($I=I^+(\alpha)$) as well in the negative ($I=I^-(\alpha)$) type solutions.

TABLE II

Bifurcational inclinations for the nonresonant case

α	$\cos I$		
	Inner case		Outer case
	Bifurcations into the negative type solutions		Bifurcations into the positive type solutions
0.0	0.774595	0.447213	0.447213
0.1	0.777588	0.449883	0.449221
0.2	0.786443	0.457716	0.455186
0.3	0.800822	0.470189	0.464942
0.4	0.820211	0.486396	0.478234
0.5	0.843981	0.505978	0.494743
0.6	0.871435	0.523958	0.514116
0.7	0.901835	0.540466	0.535987
0.8	0.934368	0.550547	0.5560
0.9	0.9680	0.45013	0.5858

The critical inclination I^* for the direct orbits are determined by the relation $I^*=\min(I^+, I^-)$. Also we set $I^{**}=\max(I^+, I^-)$. The dependence of the phase plane (e, g) on the integral constant c is given in Figures 6–10. If $c>c^*=\cos I^*$, then for the inner as well as the outer problem the phase plane has a simple structure (Figure 6) from which follows the stability of the 'circular' orbits. For the inner problem at $c<c^*$, and for the outer problem at $c^*>c>c^{**}=\cos I^{**}$, the topological structure of the phase plane is the same, and it is characterized by a single stationary point. The corresponding nontrivial stationary solution is of the negative type for the inner case as well as for the outer case at $\alpha<\alpha_0=0.75\ldots$, otherwise (at $\alpha>\alpha_0$) in the last case the stationary solution is of the positive type (Figures 7 and 8). Finally, if $c<c^{**}$ then in the outer case there exist two nontrivial solutions. One of them is stable and another unstable; if $\alpha<\alpha_0$, the stable solution is of the negative type, and if $\alpha>\alpha_0$, it is of the positive type (Figures 9 and 10). It is worth mentioning that, as $c<c^{**}$, the space trivial ('circular') solution becomes stable again. A similar situation was investigated by Izsak (1962) for the satellite motion in the gravitational field of a nonspherical planet. In Figure 11 the subdivision of the phase plane (α, $\cos I$) into stable and unstable regions is given for the outer case. Broken and solid lines refer to the points of

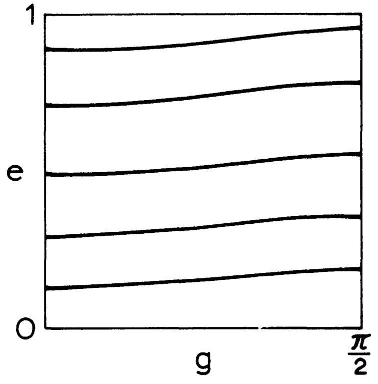

Fig. 6.

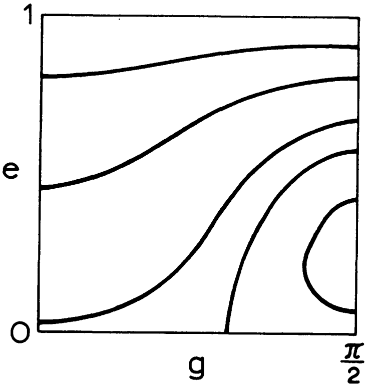

Fig. 7.

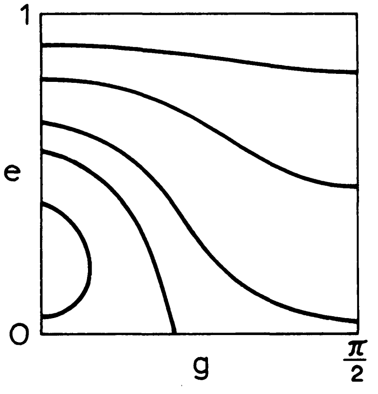

Fig. 8.

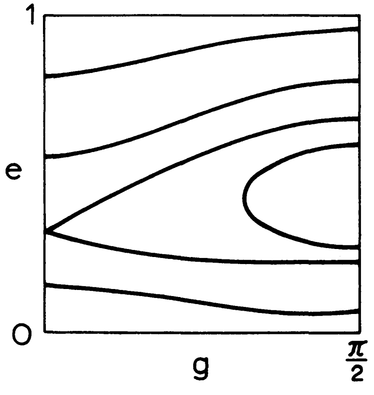

Fig. 9.

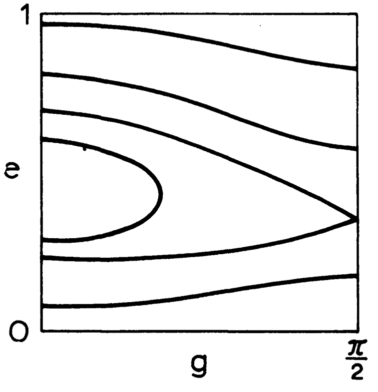

Fig. 10.

Figs. 6–10. Trajectories on (e, g)-plane for different values of c.

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bifurcation into positive and negative type solutions. The unstable region is hatched.

Now we consider the inner problem again. Let $e^* = \sin\varphi^*$ and I^* be the eccentricity and inclination of the nontrivial stationary solution generated by the bifurcational

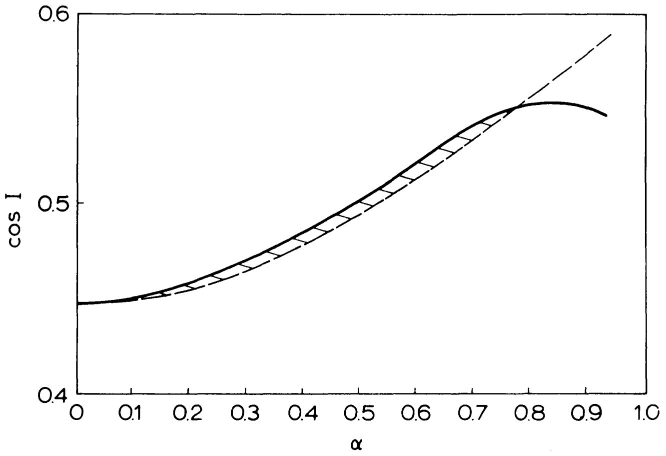

Fig. 11. Stable and unstable (hatched) regions for outer circular motion.

point and let c be the corresponding integral constant in (14). Considering the qualitative structure of the phase plane (e, g) (Figure 7) we can see that for any initial values of e_0 and I_0 (at condition $\cos\varphi_0 \cos I_0 = c$ and $\sin\varphi_0 = e_0$) there exists a moment of time t for which $e(t, e_0) \geqslant e^*$, $(e(0, e_0) = e_0)$. In particular, if $I_0 \sim \tilde{I}$, where $\cos\tilde{I} = c$, the initial value e_0 may be chosen arbitrarily small. Thus,

$$e^* = \min_{e_0} \max_{t} e(t, e_0)$$

at the condition $\cos\varphi_0 \cos I_0 = c$. That is why the problem of finding e, I on the non-trivial stationary solution is of great importance. As $\alpha \to 0$ the following relation between e, I on the stationary solution was obtained by Kozai (1962):

$$5 \cos^2 I - 3 \cos^2 \varphi = 0. \tag{15}$$

Hence, $\varphi = \frac{1}{2}\pi$, i.e. the eccentricity of the orbit, which was circular at the initial moment and whose orbital plane was perpendicular to the orbital plane of the disturbing body, tends to unit (Lidov, 1962). It is easy to understand that this result holds true for arbitrary α (at least if $\alpha < 0.5$). And indeed, the equation determining extrema of $[R]$ on the surface (14) has the form

$$\cos\varphi \frac{\partial [R]}{\partial \cos\varphi} - \cos I \frac{\partial [R]}{\partial \cos I} = 0, \tag{16}$$

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and connects e with I. If $\alpha<0.5$ the aphelion distance is less than the orbital radius of the disturbing body and $[R]$ has no singularities on the surface (14) (even for $\cos\varphi= =0$; see Krasinsky (1973)). Hence, if $\varphi=I=\frac{1}{2}\pi$, Equation (16) will be satisfied. We investigated Equation (16) numerically for several values of α. In Figure 12 the

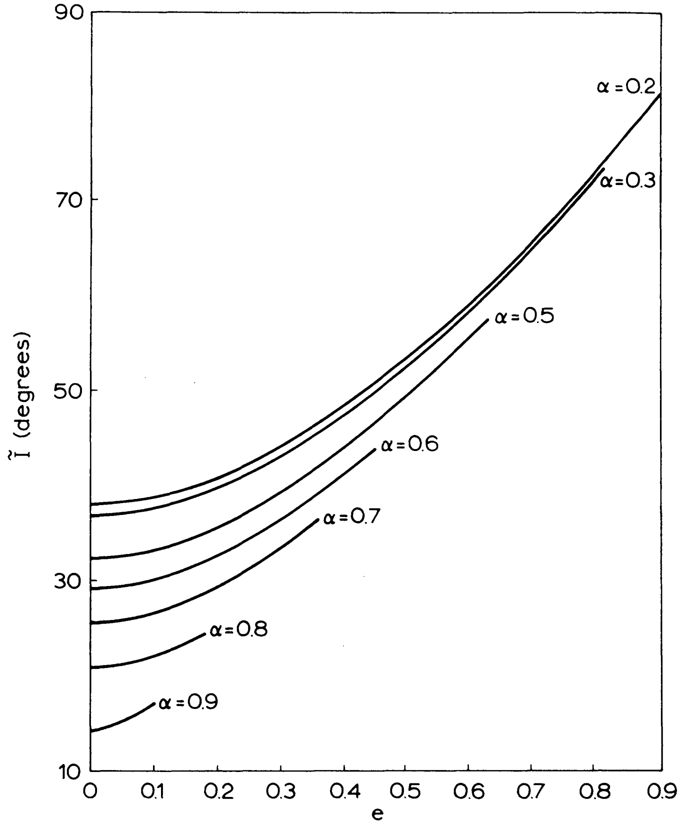

Fig. 12. Relations between the eccentricity and $\tilde{I}=\arccos c$.

dependence of e on $\tilde{I}$ ($\tilde{I}=\arccos c$) is presented. The figure shows that for $\alpha\sim0.5$ and 0.6 (these are characteristic values for the minor planets) e increases rapidly with $\tilde{I}$. Perhaps this fact may explain the well-known peculiarity in the distributions of eccentricities and inclinations of the minor planets: large inclinations are commonly accompanied by large eccentricities. In order to illustrate this fact we plotted (Figure 13) the inclinations I and semiaxes a of all minor planets whose eccentricities are less than 0.1; the curve of the critical inclinations is drawn too. As it may be expected, all points representing the pairs (a, I) are beneath of this curve (excluding the commensurability 1:1 for which the analysis is not applicable). All inclinations are referred to the ecliptic, the mutual inclinations of the Earth and Jupiter being neglected.

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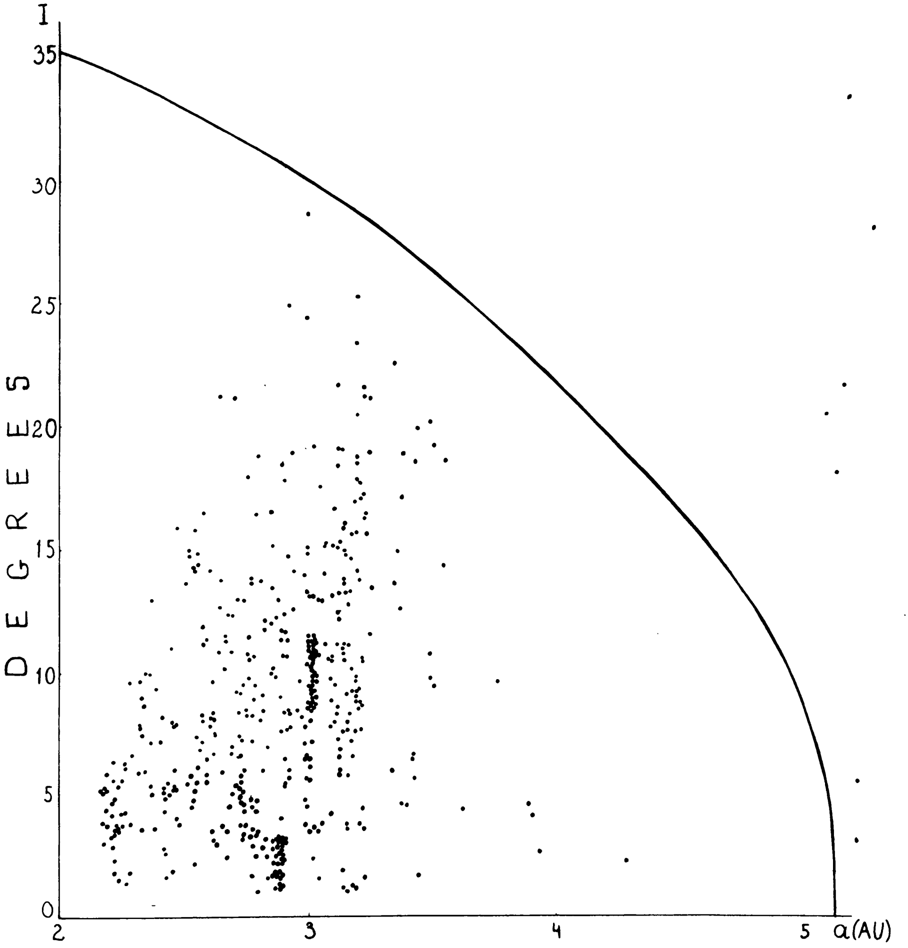

Fig. 13. Distribution of the inclinations of asteroids with respect to the semimajor axes.

In Figure 14 the dependence of e on I for the outer problem is presented for the nontrivial solutions of both positive and negative type. As $\alpha \to 0$, instead of the Kozai's relation (15), we have $\cos^2 I = \frac{1}{5}$ for any e. For $\alpha \neq 0$ we considered only moderate eccentricities; the case of large eccentricities might be useful for comet astronomy and needs further investigations.

4. The Averaged Restricted Circular Three-Body Problem in the Resonant Case

In this section the mean motions n and n' of disturbing and disturbed planets are supposed to be connected by a resonant relation

$$pn + qn' = 0,$$

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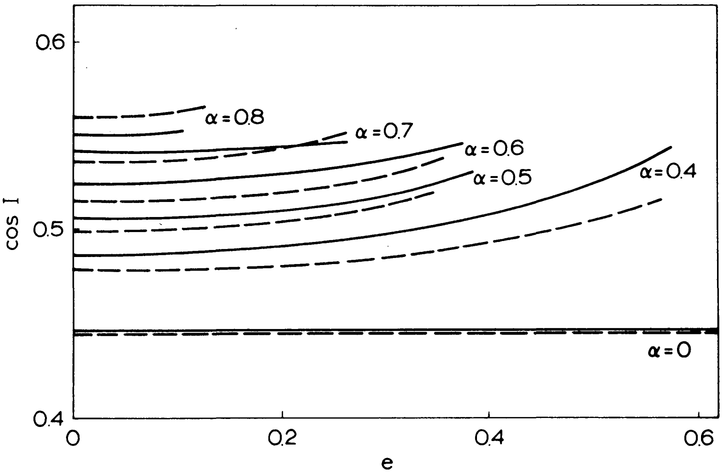

Fig. 14. Relations between e and I for outer cases.

where p and q are mutual prime integer numbers. The value $|p+q|$ will be called the order of the resonance. The averaged perturbation function in this case depends on the eccentricity e, the inclination I, the perihelion argument g and the critical argument β, whose definition is:

$$\beta = \begin{cases} pl + ql' - (p+q)\,\Omega, & \text{if} \quad I \neq 0, \\ pl + ql' - (p+q)\,\tilde{\omega}, & \text{if} \quad I = 0 \end{cases} \tag{17}$$

(l and l' being the mean longitudes, Ω and ϖ the longitudes of the node and perihelion). For the stationary solution the critical argument is equal either to 0 or to π and the perihelion argument either to 0, π (the positive type solution) or to $-\frac{1}{2}\pi$, $\frac{1}{2}\pi$ (the negative type solution). The problem of constructing the stationary solutions has some peculiarities in this case because the averaged system does not possess the area integral and the averaged Hamiltonian even in a rotating coordinate system still depends on time. This dependence vanishes if we seek a stationary solution for which the corresponding mean motion is equal to $n(1+\varrho) = -n'(1+\varrho)\,q/p$, where $\varrho = -\sigma/n'$, and σ is the angular velocity of the rotating coordinate frame. In fact, we obtain the Poincaré–Schwarzschield's periodic orbits with a variable period. The equations to find the stationary solutions coincide with the conditions of periodicity and may be treated as the equations determining extrema of the averaged perturbation function $[R]$ on the surface (14) (which is not an integral one in this case). Excluding the Lagrange factor σ we again obtain Equation (15) connecting the eccentricity and the inclination on the stationary solution. This equation was investigated numerically by Jefferys and Standish (1966, 1072) and Kozai (1969). Equation (15) determines as a rule several different curves, the axis $I = 0$ corresponding to the plane trivial solutions and the axis $e = 0$ (for the even order $|p+q|$ of the resonance) corresponding to the space trivial solutions. According to the general theory, if the stable trivial solution passes a bifurcational point (i.e. the point of the intersection of a curve determined by Equation (15) with a coordinate axis) it becomes unstable. In the non-resonant case there exist only intersections with the axis I (the critical and bifurcational inclinations). In the resonant case the existence of intersections with axis e is proved in Jefferys and Standish (1966, 1972) and Kozai (1969). Thus, in these papers the first examples of the

TABLE III

Characteristics of the symmetric trivial periodic solutions in the restricted circular three-body problem (commensurability $pn+qn'=0$)

Oddness or evenness p, q	Type of conjunctions or oppositions	Mean values of the critical argument	Stability or instability
	Plane solutions		
$p+q$ odd	conjunction at perihelion Opposition at aphelion	0	Stability
q odd	Conjunction at aphelion Opposition at perihelion	π	Instability
$p+q$ odd	Conjunction at perihelion Opposition at perihelion	0	Stability
q even	Conjunction at aphelion Opposition at aphelion	π	Instability
$p+q$ even	Conjunction at perihelion Conjunction at aphelion	0	Instability
$p+q$ even	Opposition at perihelion Opposition at aphelion	π	Stability
	Space solutions		
$p+q$ even	Conjunction at the ascending node Conjunction at the descending node	0	Instability
	Opposition at the ascending node Opposition at the descending node	π	Stability

critical and bifurcational eccentricities are constructed. Unlike the problem of the critical inclinations the calculation of the critical eccentricities gives great difficulties because large eccentricities have to be considered. This is why there are discrepancies between corresponding numerical results by Jefferys and Standish, and by Kozai (1969).

All the nontrivial solutions may be subdivided into four groups which differ by the type (positive or negative) and values of the critical argument. This subdivision does not coincide with that by Kozai because the different definitions of the critical argument are used. For the trivial solutions the type is undetermined and they may be distinguished only by the values of the critical argument. The solutions belonging to the different groups have different types of symmetry which we are going to describe briefly. It is easy to prove the existence of two moments of time, T_1 and T_2, ($T_2 - T_1 = \frac{1}{2}T$, where $T = 2\pi|p|/(n' - \sigma)$ is the period of this periodic solution) for which the mean longitudes of the disturbed and disturbing bodies are either equal to each other or differ by π. Following Poincaré we refer to the first case as 'conjunction', to the second case as 'opposition'. At these moments the critical argument coincides with its mean value (0 or π) and the bodies are either on the line of the apsides (for the nontrivial and plane trivial solutions) or on the nodal line (for the space trivial solutions). If the non-trivial solution is the positive type, the lines of the nodes and the apsides coincide; if it is the netative type these lines are orthogonal (at the moments T_1 and T_2). In the last case the plane P containing the three bodies is orthogonal to the plane S of the disturbing body, the velocity of the disturbed body being perpendicular to P and parallel to S. This kind of symmetry was studied first by Jefferys (1965). In Table III for all virtual cases mutual positions of the bodies at T_1 and T_2 are given for the trivial solutions (depending on the evenness or oddness of p and q and the values of the critical argument). In the last column we marked whether the stationary values of the critical arguments are stable or not (for small eccentricities and inclinations), in other words whether the critical argument has a bifurcational or circular motion. The corresponding analytical proof is given in Krasinsky (1973).

In recent years the theory of the stable stationary solutions was applied to the problem of resonant asteroids. For instance, learning the commensurability 3:2 (Hilda group) we see from Table IV that for the stable plane stationary solution (which corresponds to the value π of the critical argument) the conjunctions with Jupiter take place only when the asteroid is in the perihelion. If the eccentricity is not small, the close approach of the asteroid to Jupiter does not occur. This situation was

TABLE IV

Bifurcational inclinations for resonant cases

$n:n'$	$\cos I$	
	Bifurcations into the positive type solutions	Bifurcations into the positive type solutions
3:1	0.322742	0.590852
5:1	–	0.787049
7:1	–	0.796043
9:1	–	0.790342
11:1	–	0.786699
5:3	0.585475	0.438795
7:3	–	0.802517
11:3	–	0.824638
7:5	0.668089	0.112050
9:5	–	0.787081
11:5	–	0.861418
9:7	0.111854	–
11:7	–	0.737298

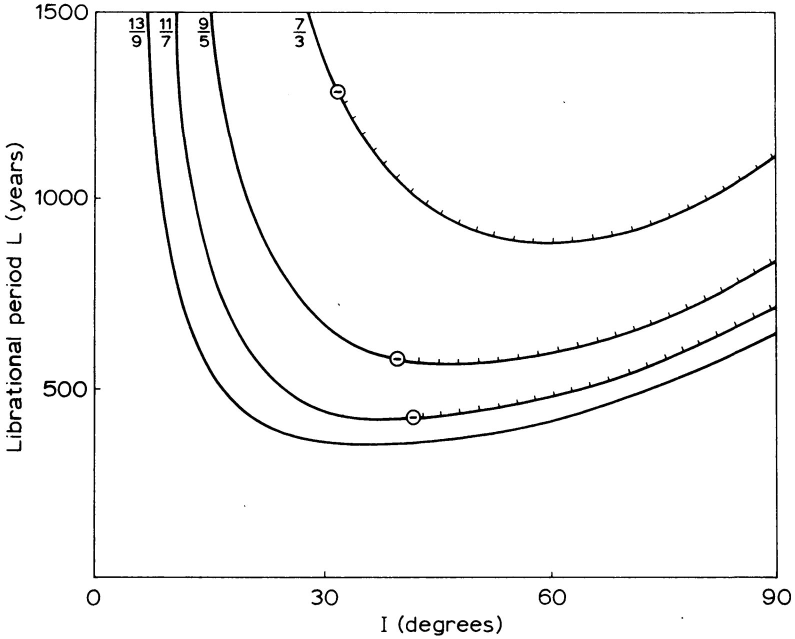

Fig. 15.

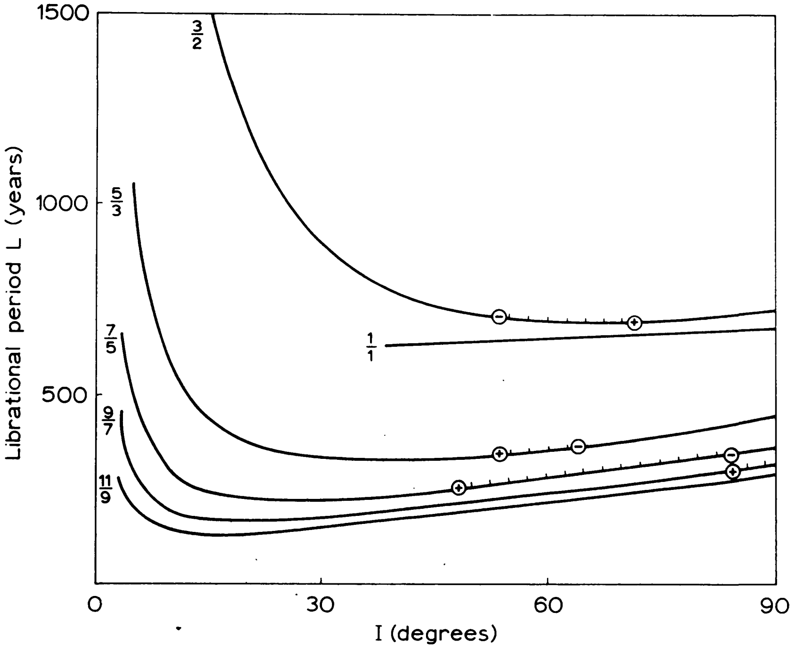

Fig. 16.

Figs. 15–16. Relations between libration periods and the inclinations for commensurable asteroids. Broken lines correspond to unstable points.

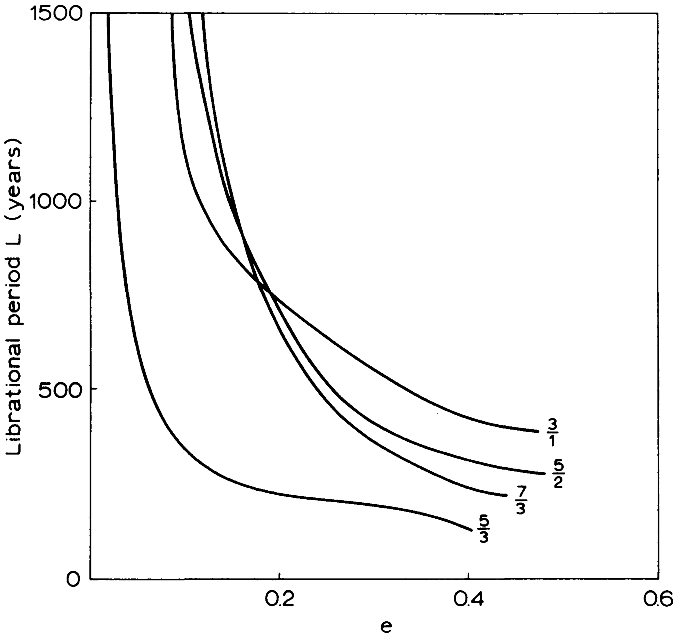

Fig. 17.

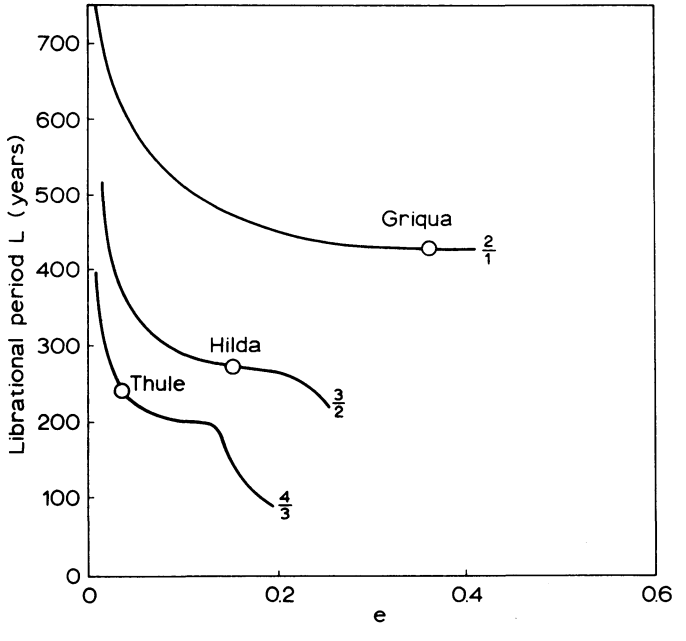

Fig. 18.

Figs. 17–18. Relations between libration periods and the eccentricities for commensurable asteroids.

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studied in detail by Schubart (1964, 1968), who established that all of the asteroids of the Hilda group are in the vicinity of the stable periodic orbit (excluding two asteroids with small eccentricities). This seems to explain the stability of the Hilda group (as well as of some other groups). The space trivial solution gives another example of the libration of the critical argument and the absence of close approaches with Jupiter. The corresponding stable periodic solution (existing only if the order $|p+q|$ of the resonance is an even number) for which the mean value of the critical argument is equal to π, has no conjunctions with Jupiter at all. In the most dangerous points of the orbits (in the nodes) the longitudes l and l' of the asteroid and Jupiter differ by π. The case of commensurability 1:1 is of particularly great interest. In this case the value $l-l'$ differs from π only by small short-periodic terms. This solution is symmetrical unlike the well-known periodic solutions which correspond to the librational points L_4 and L_5. For the commensurability 1:1 the plane symmetrical solution with non-zero eccentricity exists too, but it cannot be constructed by means of the analytical method used. In Figures 15–18 we present the results of numerical calculations of the 'librational period' for the stable trivial solution in the system Jupiter–asteroid. The parts of the curves in Figures 15 and 16 marked by the broken line correspond to the region of instability of the eccentricity. The small circles with the plus or minus sign inside mark the bifurcational points corresponding to the solution of the positive or the negative type which branch from these points. The values of the bifurcational and critical inclinations are also given in Table IV. (The critical eccentricities were not considered in this work.) The numerical results may be compared in some cases with those by Kozai (1969) and Jefferys and Standish (1966, 1972). The coincidence is good enough (it is necessary to keep in mind the differences between the definitions of the critical argument).

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DISCUSSION

F. Nahon: What is the meaning of the word 'stationary' in your title?

G. A. Krasinsky: The stationary solutions provide the averaged Hamiltonian with the extremal value. For the immovable reference frame we have no extrema of the averaged Hamiltonian excluding the case $e = i = 0$, however, they exist for the rotating reference frame. By a similar way we found the conditional extrema for the immovable reference frame. This approach was initiated by Poincaré for the periodical case. We extended this approach to the case of arbitrary resonances as well as to the nonresonant case.

J. Moser: Do you have an explanation for the distribution of the orbital elements for asteroids according to your study?

G. A. Krasinsky: I have no explanation for the distribution of the perihelia, however, I can offer some explanations for the distributions of the eccentricities and the inclinations.

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