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On embedding closed categories

B.J. Day and M.L. Laplaza

This article contains one method of fully embedding a symmetric closed category into a symmetric monoidal closed category. Such an embedding is very useful in the study of coherence problems. Also we give an example of a non-symmetric closed category for which, under the embedding discussed in this article, the resultant monoidal closed structure has associativity not an isomorphism.

Introduction

General closed categories were introduced and developed by Eilenberg and Kelly [5]. These authors have also considered symmetric closed categories in unpublished work and, recently, the symmetric case was developed formally by de Schipper [9].

In "nature" it is unusual to find examples of symmetric closed categories which are not themselves part of an enveloping monoidal closed structure. The central aim of this article is to show that *all* symmetric closed categories arise this way. Such an embedding can then be used to study coherence problems in abstract closed categories, since Kelly and MacLane [6] have dealt with coherence in a symmetric monoidal closed category.

A modified embedding has been used by Laplaza [8] to show that *all* closed categories arise as full subcategories of monoidal closed categories; however, from the point of view of limit and colimit preserving properties of the embedding (see Day [3] for a *rough* outline with no coherence considerations) it still seems easier to use the method developed

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here in the symmetric case. Nevertheless, this article is entirely complementary to the work of the second author (Laplaza [7] and [8]) which is a continuation of such "transcendental" (that is, non-combinatorial) approaches to coherence problems.

In Section 3 we provide a counterexample to the conjecture that the embedding employed in this article yields an associativity isomorphism when the closed category in question is non-symmetric (see Laplaza [7]). This example is of a topological nature and is of some interest in itself.

Throughout the article we assume some familiarity with the elements of closed category theory as given by Eilenberg and Kelly [5]. Some of the necessary material from de Schipper [9] is collected, for reference, in Section 1. We shall assume the existence of two universes of sets, denoted $\mathcal{E}ns$ and $\mathcal{ENS}$ respectively ($\mathcal{E}ns \in \mathcal{ENS}$) ; these universes are identified with the cartesian closed categories of sets which they define. The statement "A is small" means obj $A \in \mathcal{E}ns$, and all categories are assumed to be locally $\mathcal{E}ns$-small *unless* otherwise indicated. The remaining terminology and notations are mainly those of [1], [5], and [6].

1. Symmetric closed categories and functors

DEFINITION 1.1. A symmetric closed category (SCC) is an ordered 7-tuple $V = (V_0, [-,-], I, i, j, L, s)$ consisting of

(i) a category V_0 (the *underlying category* of V),

(ii) a functor $[-,-] : V_0^{op} \times V_0 \to V_0$ (the *internal*-hom functor),

(iii) an object $I \in V_0$ (the *unit object*),

(iv) a natural isomorphism $i = i_X : X \to [IX]$,

(v) a natural transformation $j = j_X : I \to [XX]$,

(vi) a natural transformation $L = L^X_{YZ} : [YZ] \to [[XY][XZ]]$,

(vii) a natural isomorphism $s = s_{XYZ} : [X[YZ]] \to [Y[XZ]]$.

These data are to satisfy the following axioms:

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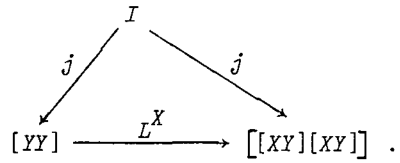
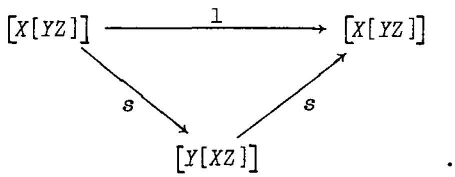
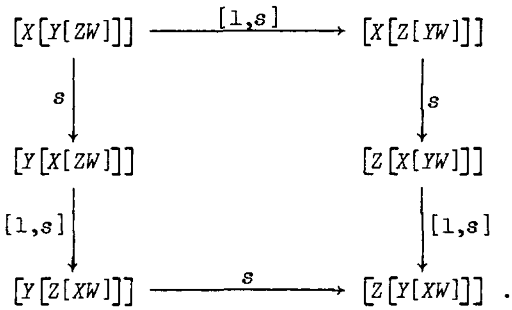
CC1. The following diagram commutes:



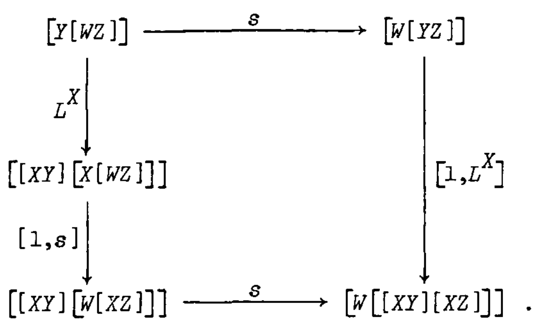
SCC1. The following diagram commutes:



SCC2. The following diagram commutes:



SCC3. The following diagram commutes:



SCC4. The following diagram commutes:

$$\begin{array}{ccc} [XY] & \xrightarrow{L^X} & [[XX][XY]] \\ {\scriptstyle [1,i]}\downarrow & & \downarrow{\scriptstyle [j,1]} \\ [X[IY]] & \xrightarrow{s} & [I[XY]] \end{array} .$$

CC5'. The map $j_* : V_0(XY) \to V_0(I[XY])$ which sends $f \in V_0(XY)$ to the diagonal of

$$\begin{array}{ccc} I & \xrightarrow{j_X} & [XX] \\ {\scriptstyle j_Y}\downarrow & & \downarrow{\scriptstyle [1,f]} \\ [YY] & \xrightarrow[{[f,1]}]{} & [XY] \end{array}$$

is an isomorphism. //

The axioms are written for immediate comparison with those of de Schipper [9], II.2. They differ from de Schipper's axioms in the non-equational part; we have omitted the "basic functor" V from the data, *defining* it by $V = V_0(I-)$, and we have replaced de Schipper's non-equational axioms CC0 and CC5 by the weaker non-equational axiom CC5'.

In order to state relations between the data for an SCC the *first basic situation* (of de Schipper) is defined to consist of:

(i) a category V_0 ,

(ii) a functor $[-, -] : V_0^{op} \times V_0 \to V_0$,

(iii) an object $I \in V_0$,

(iv) a natural isomorphism $\sigma = \sigma_{XYZ} : V_0(X[YZ]) \to V_0(Y[XZ])$ satisfying $\sigma_{YXZ} \cdot \sigma_{XYZ} = 1$.

PROPOSITION 1.2. *Let* V *be an* SCC. *Define a natural isomorphism* $\sigma = \sigma_{XYZ} : V_0(X[YZ]) \to V_0(Y[XZ])$ *by commutativity of*

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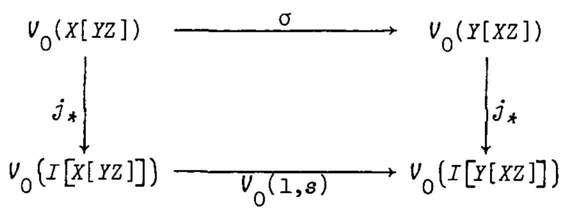
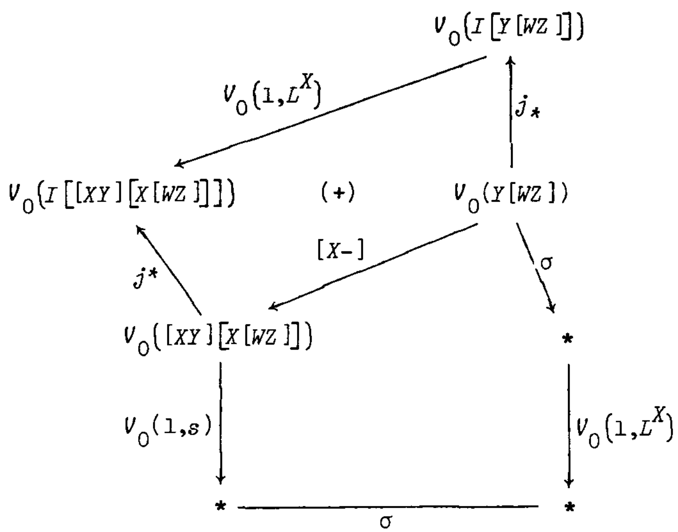


using axiom CC5'. *Then we obtain the first basic situation,* L *and* s *being related by* [9], II (3.6), *and* i *and* j *being related by* [9], II (3.15).

Proof. This is immediate from SCC1 and CC5'. //

We derive [9] II (3.6) by considering a diagram of the form



where, for example, (+) commutes by CC1, CC5', and naturality of L^X. The relation [9], II (3.15) is similarly obtained from SCC4. We then have [9] (4.2), (4.3), (4.4), (4.5), and (4.7).

PROPOSITION 1.3 (Kelly). $i_I = j_I : I \to [II]$.

Proof. First we note that any SCC is a closed category with CC5' holding (see de Schipper [9], II, Theorem 4.2). Then the assertion is a simple consequence of the closed category axioms and naturality. //

PROPOSITION 1.4. *If a natural transformation* $h : [A_1[DC]] \to [[A_2[A_3D]][A_1[A_2[A_3C]]]]$ *is defined by the composite*

(1.1)
$$\begin{array}{ccc} (0) & \xrightarrow{\ h\ } & (1) \\ {\scriptstyle [1,L^{A_3}]}\downarrow & & \uparrow{\scriptstyle s} \\ (3) & \underset{[1,L^{A_2}]}{\longrightarrow} & (2) \end{array} ,$$

(2) $[A_1[[A_2[A_3D]][A_2[A_3C]]]]$,

(3) $[A_1[[A_3D][A_3C]]]$,

then the following diagrams commute:

(1.2)
$$\begin{array}{ccc} (0) & \xrightarrow{\ h\ } & (1) \\ {\scriptstyle R^{[A_2[A_3C]]}}\downarrow & & \uparrow{\scriptstyle [R^{[A_3C]},1]} \\ (4) & \underset{[[L^{A_3},1],1]}{\longrightarrow} & (5) \end{array} ,$$

(4) $[[[DC][A_2[A_3C]]], [A_1[A_2[A_3C]]]]$,

(5) $[[[[A_3D][A_3C]][A_2[A_3C]]], [A_1[A_2[A_3C]]]]$,

(1.3)
$$\begin{array}{ccc} (0) & \xrightarrow{\ h\ } & (1) \\ {\scriptstyle s}\downarrow & & \uparrow{\scriptstyle [1,s]} \\ (6) & & (9) \\ {\scriptstyle L^{A_3}}\downarrow & & \uparrow{\scriptstyle [1,[1,s]]} \\ (7) & \underset{L^{A_2}}{\longrightarrow} & (8) \end{array} ,$$

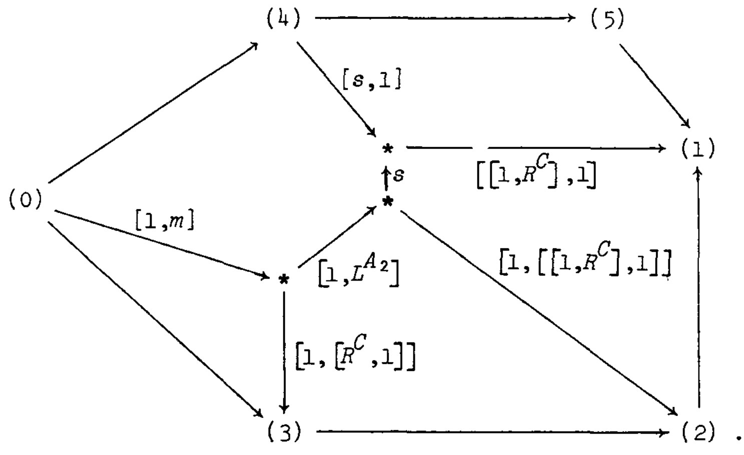
(6) $[D[A_1C]]$,

(7) $[[A_3D], [A_3[A_1C]]]$,

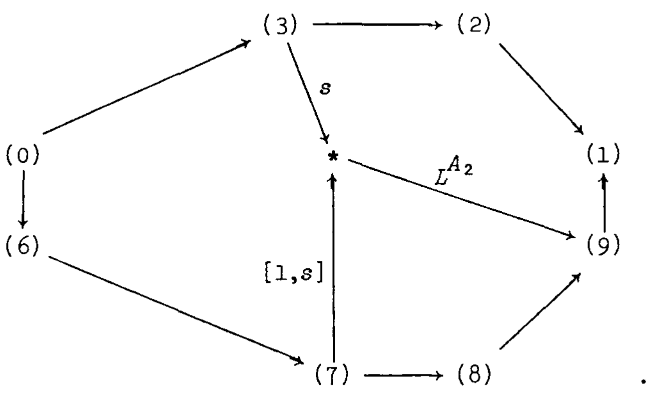
(8) $[[A_2[A_3D]], [A_2[A_3[A_1C]]]]$,

(9) $[[A_2[A_3D]], [A_2[A_1[A_3C]]]]$.

Proof. Expand diagram (1.2) to



Then we have: (451*) by [9], II 4.4; (**12) because s is natural; (04***) by definition of R and naturality of s and L; (**23) because L is natural; (*03) because $L = \sigma(R)$. Expand diagram (1.3) to



Then we have (3067*) by SCC3; (*789) because L is natural; (123*9) by SCC3. //

COROLLARY 1.5. *The transformation*

$$\int^D V_0(A_1[DC]) \times V_0(A_2[A_3D]) \to V_0(A_1[A_2[A_3C]])$$

which maps (f, g) *to*

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$$\begin{array}{ccc} A_1 & \longrightarrow & [A_2[A_3C]] \\ {\scriptstyle f}\downarrow & & \uparrow{\scriptstyle [g,1]} \\ [DC] & \xrightarrow{\ L\ } & [[A_3D],[A_3C]] \end{array}$$

coincides with the natural isomorphism

$$\begin{aligned} \int^D V_0(A_1,[DC]) \times V_0(A_2[A_3D]) &\cong \int^D V_0(D[A_1C]) \times V_0(A_2[A_3D]) \\ &\cong V_0(A_2[A_3[A_1C]]) \\ &\cong V_0(A_1[A_2[A_3C]]) \ . \end{aligned}$$

Proof. The first transformation is an evaluated form of (1.2) while the second transformation is an evaluated form of (1.3). //

DEFINITION 1.6. A *symmetric closed functor* $\Phi : V \to V'$ between two symmetric closed categories is an ordered triple $\Phi = (\phi, \hat{\phi}, \phi^0)$ consisting of:

(i) a functor $\phi : V_0 \to V'_0$,

(ii) a natural transformation $\hat{\phi} = \hat{\phi}_{XY} : \phi[XY] \to [\phi X, \phi Y]$ in V'_0 ,

(iii) a morphism $\phi^0 : I' \to \phi I$ in V'_0 .

These data are to satisfy the following two axioms:

CF1. The following diagram commutes:

$$\begin{array}{ccc} I' & \xrightarrow{\ j'\ } & [\phi X,\phi X] \\ {\scriptstyle \phi^0}\downarrow & & \uparrow{\scriptstyle \hat{\phi}} \\ \phi I & \xrightarrow{\ \phi j\ } & \phi[XX] \ . \end{array}$$

SCF3. The following diagram commutes:

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$$\begin{array}{ccc}
\phi[X[YZ]] & \xrightarrow{\phi s} & \phi[Y[XZ]] \\
\downarrow{\scriptstyle\hat{\phi}} & & \downarrow{\scriptstyle\hat{\phi}} \\
[\phi X, \phi[YZ]] & & [\phi Y, \phi[XZ]] \\
\downarrow{\scriptstyle[1,\hat{\phi}]} & & \downarrow{\scriptstyle[1,\hat{\phi}]} \\
[\phi X[\phi Y, \phi Z]] & \xrightarrow{s'} & [\phi Y[\phi X, \phi Z]]
\end{array} \quad . \qquad //$$

These axioms are the same as given by de Schipper.

DEFINITION 1.7. A symmetric closed functor $\Phi : V \to V'$ is called a *symmetric closed full embedding* if:

(i) $\phi : V_0 \to V'_0$ is a full embedding,

(ii) $\hat{\phi}_{XY}$ is an isomorphism for all $X, Y \in V_0$,

(iii) ϕ^0 is an isomorphism. //

2. The embedding theorem

We recall from Day [1] that a monoidal biclosed structure on a functor category of the form $[A, Ens]$ (with A small) is determined to within an isomorphism by its "structure constants". These "constants" comprise what is called a *promonoidal structure* on A.

DEFINITION 2.1. A promonoidal category $A = (A_0, P, J, \alpha, \lambda, \rho)$ is said to be *monoidal* if the functors $P(AB-) : A_0 \to Ens$ and $J : A_0 \to Ens$ are representable for all $A, B \in A_0$. //

Such categories correspond to monoidal categories.

DEFINITION 2.2. A promonoidal category $A = (A_0, P, J, \alpha, \lambda, \rho)$ is said to be *closed* if the functors $P(-AB) : A_0^{op} \to Ens$ and $J : A_0 \to Ens$ are representable for all $A, B \in A_0$.

We warn the reader that such categories do *not* correspond to closed categories but rather to equationally defined "associative" closed categories (see Day [3], Example 3.2).

When A is monoidal the convolution structure on $[A, Ens]$ is given by the formulas:

$$F*G = \int^{AB} FA \times GB \times A_0(A \otimes B, -) ,$$

$$F\backslash G = \int_{AB} [FA \times A_0(A \otimes -, B), GB]$$

$$\cong \int_A [FA, G(A \otimes -)] ,$$

$$G/F = \int_{AB} [FA \times A_0(- \otimes A, B), GB]$$

$$\cong \int_A [FA, G(- \otimes A)] .$$

When A is a closed promonoidal category the convolution structure on $[A, Ens]$ is given by:

$$F*G = \int^{A} F[A-] \times GA ,$$

$$F\backslash G = \int_{AB} [FA \times A_0(A[-B]), GB]$$

$$\cong \int_A [F[-A], GA] ,$$

$$G/F = \int_{AB} [FA \times A_0(-[AB]), GB] .$$

If A is *symmetric* as a promonoidal category then $[A, Ens]$ is symmetric monoidal closed.

PROPOSITION 2.3. *Each (small) symmetric closed category* V *has a canonical symmetric closed promonoidal structure.*

Proof. We define $P : V_0^{op} \times V_0^{op} \times V_0 \to Ens$ by $P(X, Y, Z) = V_0(X[YZ])$ and $J : V_0 \to Ens$ by $JX = V_0(IX)$. Thus there are natural transformations:

$$\alpha : \int^X V_0(A_1[XB]) \times V_0(A_2[A_3X]) \to \int^X V_0(X[A_3B]) \times V_0(A_1[A_2X])$$
$$\cong V_0(A_1[A_2[A_3B]]) ,$$

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$$\lambda : \int^X V_0(IX) \times V_0(X[AB]) \to V_0(AB) ,$$

$$\rho : \int^X V_0(IX) \times V_0(A[XB]) \to V_0(AB) ,$$

where α is an isomorphism by Corollary 1.5 and the representation theorem, whilst λ and ρ are isomorphisms by the closed-category axioms together with the representation theorem. The axioms for a symmetric promonoidal category structure on V follow easily from the representation theorem and the closed-category axioms for V . //

THEOREM 2.4. *Each small symmetric closed category V admits a symmetric closed full embedding into a symmetric monoidal closed category* $[A, \mathit{Ens}]$.

Proof. Give V the canonical symmetric promonoidal structure (Proposition 2.3) and form the convolution $[V, \mathit{Ens}]$. Let A consist of all the finite $*$-paths of representable functors in $[V, \mathit{Ens}]$ together with J (the "zero" path). Then $V^{\mathrm{op}} \subset A \subset [V, \mathit{Ens}]$ where A is a small symmetric monoidal category. We again form the convolution $[A, \mathit{Ens}]$ and obtain $V \subset A^{\mathrm{op}} \subset [A, \mathit{Ens}]$. The composite embedding is a symmetric closed functor because each factor is a symmetric promonoidal functor (see Day [4]). To establish that the embedding is a symmetric closed full embedding we refer to Day [3], Example 3.2. //

REMARK 2.5. Several refinements of the theorem are available (Day [3], Example 3.2) and they show that the embedding can be modified to preserve certain limits and colimits. //

3. A counterexample

The aim of this section is to provide several concrete examples of closed categories which are not monoidal, not symmetric and, moreover, are not associative in the sense that the convolution structure which they generate under the embedding of Section 2 has an associativity transformation which is not always an isomorphism. The examples are obtained from categories of quasi-topological spaces, although the constructions can be generalised (see [2], Section 3).

Let *Top* be the category of all topological spaces and continuous maps

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and choose $A \subset Top$ such that A is closed under finite products in Top and contains all the finite discrete spaces. The category $\overline{A}$ of quasi-topological bases on A is formed in the usual manner. A quasi-topological base is a set X together with, for each $A \in A$, a set of "selected" morphisms from A to X denoted $\mathrm{sel}(A, X)$. These sets are subject to two axioms:

Q1. $\mathrm{sel}(A, X)$ contains all constant maps;

Q2. if $f \in A(A, B)$ and $g \in \mathrm{sel}(B, X)$ then $gf \in \mathrm{sel}(A, X)$.

A morphism $f : X \to Y$ of bases is a set map such that $g \in \mathrm{sel}(A, X)$ implies $fg \in \mathrm{sel}(A, Y)$; such a map is said to be "continuous". The category A is canonically embedded in $\overline{A}$ by the choices $\mathrm{sel}(A, B) = A(A, B)$.

There exist two well-known symmetric monoidal closed structures on $\overline{A}$. The first is the cartesian closed structure where the monoidal product is given by the cartesian product in $\overline{A}$ and the closed structure $[X, Y]_c$ is the set $\overline{A}(X, Y)$ with $f : A \to [X, Y]_c$ selected if the associated morphism $f' : A \times X \to Y$ is continuous.

The second canonical monoidal closed structure is the "pointwise" structure $[X, Y]_p$ for which $f : A \to [X, Y]_p$ is selected if, for each $x \in X$, $f(-)(x) : A \to Y$ is selected. The associated monoidal structure $X \otimes Y$ is defined by selecting $f : A \to X \times Y$ if and only if f is of the form $\{x\} \times g$ or $g \times \{y\}$ for some selected g.

A third closed structure which is not monoidal is defined as follows. Define $[X, Y]$ to be $\overline{A}(X, Y)$ with $f : A \to [X, Y]$ selected if and only if either f is constant or, for all $g \in \mathcal{E}ns(B, X)$ with $B \in A$, the composite

$$e(f \times g) : A \times B \to \overline{A}(X, Y) \times X \to Y$$

is selected. The verification that this definition provides a closed structure on $\overline{A}$ is left to the reader (see [6], Proposition I.2.11). We note that the bijections

$$[X, Y] \to [X, Y]_c \to [X, Y]_p$$

are continuous and are not isomorphisms in general.

It is readily seen that in the case where A is chosen to be the category Fin of (discrete) finite sets the above construction reduces to the third closed structure discussed by Eilenberg and Kelly in [5], IV, Section 7.

The inclusion $\phi : \mathit{Fin} \subset A$ induces an adjunction $\overline{\phi} \dashv \phi^* : \overline{A} \to \overline{\mathit{Fin}}$ (see [2], Example 3.3). This gives the following:

PROPOSITION 3.1. *The functor* $[X, -] : \overline{A} \to \overline{A}$ *does not in general preserve powers.*

Proof. Let s^n denote the set of $n+1$ points with the "trivial" quasi-topological base structure; that is, $\mathrm{sel}(A, s^n) = \mathit{Ens}(A, s^n)$. Let ms^n be the m-fold coproduct $s^n + \ldots + s^n$ in $\overline{A}$. Then

$$\phi^*[ms^0, ns^k] = ns^{(k+1)^m-1} + (n^m-n)(k+1)^m s^0 .$$

In particular

$$\phi^*[2s^0, 2s^1 \times 2s^1] \cong \phi^*[2s^0, 4s^3] \cong 4s^{15} + 192s^0 ,$$

while

$$\phi^*([2s^0, 2s^1] \times [2s^0, 2s^1]) \cong \phi^*[2s^0, 2s^1] \times \phi^*[2s^0, 2s^1] \cong$$
$$\cong (2s^3+8s^2) \times (2s^3+8s^0) \cong 4s^{15} + 32s^3 + 64s^0 .$$

Thus $[2s^0, -] : \overline{A} \to \overline{A}$ does not preserve powers. //

REMARK 3.2. One may similarly deduce that $[-, X] : \overline{A}^{op} \to \overline{A}$ does not preserve limits in general, so $[X, Y]$ is not symmetric.

PROPOSITION 3.3. *The closed structure* $[X, Y]$ *on* $\overline{A}$ *is not associative.*

Proof. Suppose that the convolution structure generated by $[X, Y]$ on $[\overline{A}, \mathit{ENS}]$ is associative. This is easily seen to be equivalent to the statement that the canonical transformation

$$\int^{XY} F[X, Z] \times (G[Y, X] \times HY) \to \int^{XY} (F[X, [Y, Z]] \times GX) \times HY$$

is an isomorphism for all $F, G, H \in [\overline{A}, \mathit{ENS}]$. By the representation theorem this implies that the following transformation is an isomorphism

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for all $U, V, W \in \bar{A}$:

$$\int^X \bar{A}(U, [X, Z]) \times \bar{A}(W, [V, X]) \to \bar{A}(U, [W, [V, Z]]) \ .$$

Let $U = ns^0$. Then

$$\int^X \bar{A}(ns^0, [X, Z]) \times \bar{A}(W, [V, X]) \cong \int^X \bar{A}(X, Z)^n \times \bar{A}(W, [V, X])$$

$$\cong \int^X \bar{A}(X, Z^n) \times \bar{A}(W, [V, X])$$

$$\cong \bar{A}(W, [V, Z^n])$$

by the representation theorem. Also

$$\bar{A}(ns^0, [W, [V, Z]]) \cong \bar{A}(W, [V, Z]^n) \ .$$

Thus, by the representation theorem, the morphism

$$[V, Z^n] \to [V, Z]^n$$

is an isomorphism; this contradicts Proposition 3.1. //

From this we may deduce that the closed structure $[X, Y]$ on $\bar{A}$ is not symmetric nor is it part of an associative biclosed structure on $\bar{A}$.

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