2 Configuration Space

A robot is mechanically constructed by connecting a set of bodies, called **links**, to each other using various types of **joints**. **Actuators**, such as electric motors, deliver forces or torques that cause the robot's links to move. Usually an **end-effector**, such as a gripper or hand for grasping and manipulating objects, is attached to a specific link. All the robots considered in this book have links that can be modeled as rigid bodies.

Perhaps the most fundamental question one can ask about a robot is, where is it? The answer is given by the robot's **configuration**: a specification of the positions of all points of the robot. Since the robot's links are rigid and of a known shape,[1] only a few numbers are needed to represent its configuration. For example, the configuration of a door can be represented by a single number, the angle θ about its hinge. The configuration of a point on a plane can be described by two coordinates, (x, y). The configuration of a coin lying heads up on a flat table can be described by three coordinates: two coordinates (x, y) that specify the location of a particular point on the coin, and one coordinate (θ) that specifies the coin's orientation. (See Figure 2.1).

The above coordinates all take values over a continuous range of real numbers. The number of **degrees of freedom** (**dof**) of a robot is the smallest number of real-valued coordinates needed to represent its configuration. In the example above, the door has one degree of freedom. The coin lying heads up on a table has three degrees of freedom. Even if the coin could lie either heads up or tails up, its configuration space still would have only three degrees of freedom; a fourth variable, representing which side of the coin faces up, takes values in the discrete set {heads, tails}, and not over a continuous range of real values like the other three coordinates.

Definition 2.1 The **configuration** of a robot is a complete specification of the position of every point of the robot. The minimum number n of real-valued coordinates needed to represent the configuration is the number of **degrees of freedom** (**dof**) of the robot. The n-dimensional space containing all possible configurations of the robot is called the **configuration space** (**C-space**). The configuration of a robot is represented by a point in its C-space.

In this chapter we study the C-space and degrees of freedom of general robots.

[1] Compare with trying to represent the configuration of a soft object like a pillow.
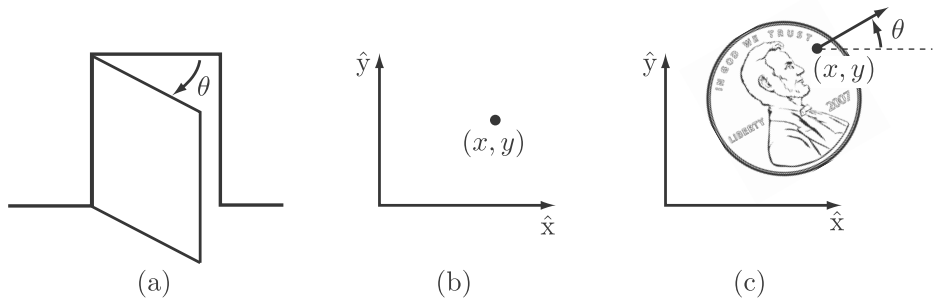
Figure 2.1 (a) The configuration of a door is described by the angle θ. (b) The configuration of a point in a plane is described by coordinates (x, y). (c) The configuration of a coin on a table is described by (x, y, θ), where θ defines the direction in which Abraham Lincoln is looking.

Since our robots are constructed from rigid links, we examine first the degrees of freedom of a single rigid body, and then the degrees of freedom of general multi-link robots. Next we study the shape (or topology) and geometry of C-spaces and their mathematical representation. The chapter concludes with a discussion of the C-space of a robot's end-effector, its **task space**. In the following chapter we study in more detail the mathematical representation of the C-space of a single rigid body.

2.1   Degrees of Freedom of a Rigid Body

Continuing with the example of the coin lying on the table, choose three points A, B, and C on the coin (Figure 2.2(a)). Once a coordinate frame $\hat{x}$–$\hat{y}$ is attached to the plane,[2] the positions of these points in the plane are written (x_A, y_A), (x_B, y_B), and (x_C, y_C). If the points could be placed independently anywhere in the plane, the coin would have six degrees of freedom – two for each of the three points. But, according to the definition of a rigid body, the distance between point A and point B, denoted $d(A, B)$, is always constant regardless of where the coin is. Similarly, the distances $d(B, C)$ and $d(A, C)$ must be constant. The following equality constraints on the coordinates (x_A, y_A), (x_B, y_B), and (x_C, y_C) must therefore always be satisfied:

$$d(A, B) = \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2} = d_{AB},$$
$$d(B, C) = \sqrt{(x_B - x_C)^2 + (y_B - y_C)^2} = d_{BC},$$
$$d(A, C) = \sqrt{(x_A - x_C)^2 + (y_A - y_C)^2} = d_{AC}.$$

To determine the number of degrees of freedom of the coin on the table, first

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[2] The unit axes of coordinate frames are written with a hat, indicating they are unit vectors, and in a non-italic font, e.g., $\hat{x}$, $\hat{y}$, and $\hat{z}$.
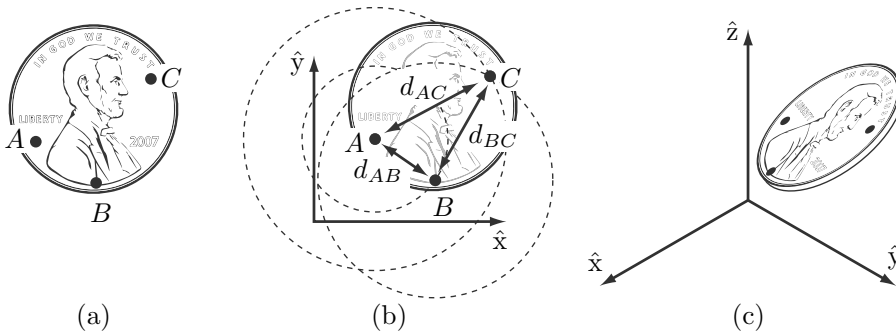
Figure 2.2 (a) Choosing three points fixed to the coin. (b) Once the location of A is chosen, B must lie on a circle of radius d_{AB} centered at A. Once the location of B is chosen, C must lie at the intersection of circles centered at A and B. Only one of these two intersections corresponds to the "heads up" configuration. (c) The configuration of a coin in three-dimensional space is given by the three coordinates of A, two angles to the point B on the sphere of radius d_{AB} centered at A, and one angle to the point C on the circle defined by the intersection of the a sphere centered at A and a sphere centered at B.

choose the position of point A in the plane (Figure 2.2(b)). We may choose it to be anything we want, so we have two degrees of freedom to specify, namely (x_A, y_A). Once (x_A, y_A) is specified, the constraint $d(A, B) = d_{AB}$ restricts the choice of (x_B, y_B) to those points on the circle of radius d_{AB} centered at A. A point on this circle can be specified by a single parameter, e.g., the angle specifying the location of B on the circle centered at A. Let's call this angle ϕ_{AB} and define it to be the angle that the vector $\overrightarrow{AB}$ makes with the $\hat{\mathrm{x}}$-axis.

Once we have chosen the location of point B, there are only two possible locations for C: at the intersections of the circle of radius d_{AC} centered at A and the circle of radius d_{BC} centered at B (Figure 2.2(b)). These two solutions correspond to heads or tails. In other words, once we have placed A and B and chosen heads or tails, the two constraints $d(A, C) = d_{AC}$ and $d(B, C) = d_{BC}$ eliminate the two apparent freedoms provided by (x_C, y_C), and the location of C is fixed. The coin has exactly three degrees of freedom in the plane, which can be specified by (x_A, y_A, ϕ_{AB}).

Suppose that we choose to specify the position of an additional point D on the coin. This introduces three additional constraints: $d(A, D) = d_{AD}$, $d(B, D) = d_{BD}$, and $d(C, D) = d_{CD}$. One of these constraints is **redundant**, i.e., it provides no new information; only two of the three constraints are independent. The two freedoms apparently introduced by the coordinates (x_D, y_D) are then immediately eliminated by these two independent constraints. The same would hold for any other newly chosen point on the coin, so that there is no need to consider additional points.

We have been applying the following general rule for determining the number

of degrees of freedom of a system:

$$\text{degrees of freedom} = (\text{sum of freedoms of the points}) -$$
$$(\text{number of independent constraints}). \qquad (2.1)$$

This rule can also be expressed in terms of the number of variables and independent equations that describe the system:

$$\text{degrees of freedom} = (\text{number of variables}) -$$
$$(\text{number of independent equations}). \qquad (2.2)$$

This general rule can also be used to determine the number of freedoms of a rigid body in three dimensions. For example, assume our coin is no longer confined to the table (Figure 2.2(c)). The coordinates for the three points A, B, and C are now given by (x_A, y_A, z_A), (x_B, y_B, z_B), and (x_C, y_C, z_C), respectively. Point A can be placed freely (three degrees of freedom). The location of point B is subject to the constraint $d(A, B) = d_{AB}$, meaning it must lie on the sphere of radius d_{AB} centered at A. Thus we have $3 - 1 = 2$ freedoms to specify, which can be expressed as the latitude and longitude for the point on the sphere. Finally, the location of point C must lie at the intersection of spheres centered at A and B of radius d_{AC} and d_{BC}, respectively. In the general case the intersection of two spheres is a circle, and the location of point C can be described by an angle that parametrizes this circle. Point C therefore adds $3 - 2 = 1$ freedom. Once the position of point C is chosen, the coin is fixed in space.

In summary, a rigid body in three-dimensional space has six freedoms, which can be described by the three coordinates parametrizing point A, the two angles parametrizing point B, and one angle parametrizing point C, provided A, B, and C are noncollinear. Other representations for the configuration of a rigid body are discussed in Chapter 3.

We have just established that a rigid body moving in three-dimensional space, which we call a **spatial rigid body**, has six degrees of freedom. Similarly, a rigid body moving in a two-dimensional plane, which we henceforth call a **planar rigid body**, has three degrees of freedom. This latter result can also be obtained by considering the planar rigid body to be a spatial rigid body with six degrees of freedom but with the three independent constraints $z_A = z_B = z_C = 0$.

Since our robots consist of rigid bodies, Equation (2.1) can be expressed as follows:

$$\text{degrees of freedom} = (\text{sum of freedoms of the bodies}) -$$
$$(\text{number of independent constraints}). \qquad (2.3)$$

Equation (2.3) forms the basis for determining the degrees of freedom of general robots, which is the topic of the next section.

2.2   Degrees of Freedom of a Robot

Consider once again the door example of Figure 2.1(a), consisting of a single rigid body connected to a wall by a hinge joint. From the previous section we know that the door has only one degree of freedom, conveniently represented by the hinge joint angle θ. Without the hinge joint, the door would be free to move in three-dimensional space and would have six degrees of freedom. By connecting the door to the wall via the hinge joint, five independent constraints are imposed on the motion of the door, leaving only one independent coordinate (θ). Alternatively, the door can be viewed from above and regarded as a planar body, which has three degrees of freedom. The hinge joint then imposes two independent constraints, again leaving only one independent coordinate (θ). The door's C-space is represented by some range in the interval $[0, 2\pi)$ over which θ is allowed to vary.

In both cases the joints constrain the motion of the rigid body, thus reducing the overall degrees of freedom. This observation suggests a formula for determining the number of degrees of freedom of a robot, simply by counting the number of rigid bodies and joints. In this section we derive precisely such a formula, called Grübler's formula, for determining the number of degrees of freedom of planar and spatial robots.

2.2.1   Robot Joints

Figure 2.3 illustrates the basic joints found in typical robots. Every joint connects exactly two links; joints that simultaneously connect three or more links are not allowed. The **revolute joint** (R), also called a hinge joint, allows rotational motion about the joint axis. The **prismatic joint** (P), also called a sliding or linear joint, allows translational (or rectilinear) motion along the direction of the joint axis. The **helical joint** (H), also called a screw joint, allows simultaneous rotation and translation about a screw axis. Revolute, prismatic, and helical joints all have one degree of freedom.

Joints can also have multiple degrees of freedom. The **cylindrical joint** (C) has two degrees of freedom and allows independent translations and rotations about a single fixed joint axis. The **universal joint** (U) is another two-degree-of-freedom joint that consists of a pair of revolute joints arranged so that their joint axes are orthogonal. The **spherical joint** (S), also called a ball-and-socket joint, has three degrees of freedom and functions much like our shoulder joint.

A joint can be viewed as providing freedoms to allow one rigid body to move relative to another. It can also be viewed as providing constraints on the possible motions of the two rigid bodies it connects. For example, a revolute joint can be viewed as allowing one freedom of motion between two rigid bodies in space, or it can be viewed as providing five constraints on the motion of one rigid body relative to the other. Generalizing, the number of degrees of freedom of a rigid body (three for planar bodies and six for spatial bodies) minus the number of
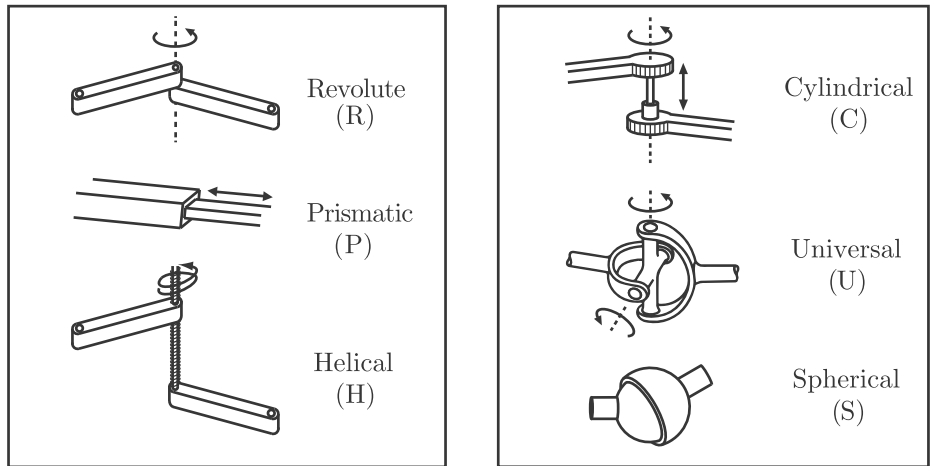
Figure 2.3 Typical robot joints.

		Constraints c between two planar rigid bodies	Constraints c between two spatial rigid bodies
Joint type	dof f		
Revolute (R)	1	2	5
Prismatic (P)	1	2	5
Helical (H)	1	N/A	5
Cylindrical (C)	2	N/A	4
Universal (U)	2	N/A	4
Spherical (S)	3	N/A	3

Table 2.1 The number of degrees of freedom f and constraints c provided by common joints.

constraints provided by a joint must equal the number of freedoms provided by that joint.

The freedoms and constraints provided by the various joint types are summarized in Table 2.1.

2.2.2   Grübler's Formula

The number of degrees of freedom of a mechanism with links and joints can be calculated using **Grübler's formula**, which is an expression of Equation (2.3).

Proposition 2.2   *Consider a mechanism consisting of N links, where ground is also regarded as a link. Let J be the number of joints, m be the number of degrees of freedom of a rigid body ($m = 3$ for planar mechanisms and $m = 6$ for spatial mechanisms), f_i be the number of freedoms provided by joint i, and c_i be the number of constraints provided by joint i, where $f_i + c_i = m$ for all i. Then*
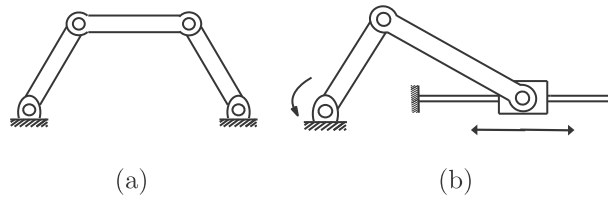
Figure 2.4 (a) Four-bar linkage. (b) Slider–crank mechanism.

Grübler's formula for the number of degrees of freedom of the robot is

$$\text{dof} = \underbrace{m(N-1)}_{\text{rigid body freedoms}} - \underbrace{\sum_{i=1}^{J} c_i}_{\text{joint constraints}}$$

$$= m(N-1) - \sum_{i=1}^{J}(m - f_i)$$

$$= m(N-1-J) + \sum_{i=1}^{J} f_i. \tag{2.4}$$

This formula holds in "generic" cases, but fails under certain configurations of the links and joints, such as when the joint constraints are not independent.

Below we apply Grübler's formula to several planar and spatial mechanisms. We distinguish between two types of mechanism: **open-chain mechanisms** (also known as **serial mechanisms**) and **closed-chain mechanisms**. A closed-chain mechanism is any mechanism that has a closed loop. A person standing with both feet on the ground is an example of a closed-chain mechanism, since a closed loop can be traced from the ground, through the right leg, through the waist, through the left leg, and back to ground (recall that the ground itself is a link). An open-chain mechanism is any mechanism without a closed loop; an example is your arm when your hand is allowed to move freely in space.

Example 2.3 (Four-bar linkage and slider–crank mechanism)   The planar four-bar linkage shown in Figure 2.4(a) consists of four links (one of them ground) arranged in a single closed loop and connected by four revolute joints. Since all the links are confined to move in the same plane, we have $m = 3$. Substituting $N = 4$, $J = 4$, and $f_i = 1$, $i = 1, \ldots, 4$, into Grübler's formula, we see that the four-bar linkage has one degree of freedom.

The slider–crank closed-chain mechanism of Figure 2.4(b) can be analyzed in two ways: (i) the mechanism consists of three revolute joints and one prismatic joint ($J = 4$ and each $f_i = 1$) and four links ($N = 4$, including the ground link), or (ii) the mechanism consists of two revolute joints ($f_i = 1$) and one RP joint (the RP joint is a concatenation of a revolute and prismatic joint, so that $f_i = 2$)
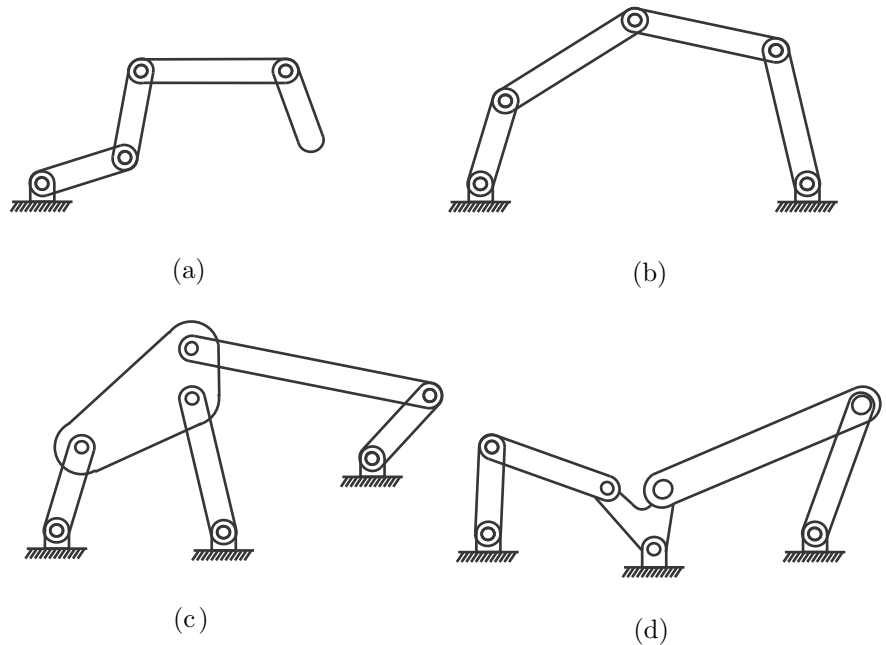
Figure 2.5 (a) A k-link planar serial chain. (b) Five-bar planar linkage. (c) Stephenson six-bar linkage. (d) Watt six-bar linkage.

and three links ($N = 3$; remember that each joint connects precisely two bodies). In both cases the mechanism has one degree of freedom.

Example 2.4 (Some classical planar mechanisms)   Let us now apply Grübler's formula to several classical planar mechanisms. The k-link planar serial chain of revolute joints in Figure 2.5(a) (called a kR robot for its k revolute joints) has $N = k + 1$ links (k links plus ground), and $J = k$ joints, and, since all the joints are revolute, $f_i = 1$ for all i. Therefore,

$$\text{dof} = 3((k + 1) - 1 - k) + k = k$$

as expected. For the planar five-bar linkage of Figure 2.5(b), $N = 5$ (four links plus ground), $J = 5$, and since all joints are revolute, each $f_i = 1$. Therefore,

$$\text{dof} = 3(5 - 1 - 5) + 5 = 2.$$

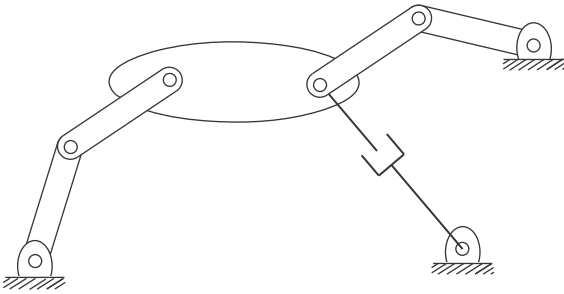For the Stephenson six-bar linkage of Figure 2.5(c), we have $N = 6$, $J = 7$, and $f_i = 1$ for all i, so that

$$\text{dof} = 3(6 - 1 - 7) + 7 = 1.$$

Finally, for the Watt six-bar linkage of Figure 2.5(d), we have $N = 6$, $J = 7$,

and $f_i = 1$ for all i, so that, like the Stephenson six-bar linkage,

$$\text{dof} = 3(6 - 1 - 7) + 7 = 1.$$

Figure 2.6 A planar mechanism with two overlapping joints.

Example 2.5 (A planar mechanism with overlapping joints)   The planar mechanism illustrated in Figure 2.6 has three links that meet at a single point on the right of the large link. Recalling that a joint by definition connects exactly two links, the joint at this point of intersection should not be regarded as a single revolute joint. Rather, it is correctly interpreted as two revolute joints overlapping each other. Again, there is more than one way to derive the number of degrees of freedom of this mechanism using Grübler's formula: (i) The mechanism consists of eight links ($N = 8$), eight revolute joints, and one prismatic joint. Substituting into Grübler's formula yields

$$\text{dof} = 3(8 - 1 - 9) + 9(1) = 3.$$

(ii) Alternatively, the lower-right revolute–prismatic joint pair can be regarded as a single two-dof joint. In this case the number of links is $N = 7$, with seven revolute joints, and a single two-dof revolute–prismatic pair. Substituting into Grübler's formula yields

$$\text{dof} = 3(7 - 1 - 8) + 7(1) + 1(2) = 3.$$

Example 2.6 (Redundant constraints and singularities)   For the parallelogram linkage of Figure 2.7(a), $N = 5$, $J = 6$, and $f_i = 1$ for each joint. From Grübler's

(a)   (b)
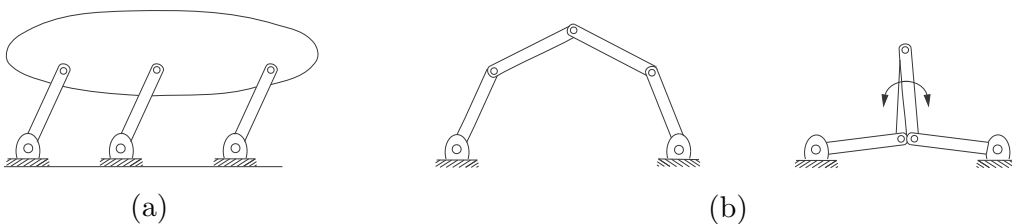
Figure 2.7 (a) A parallelogram linkage. (b) The five-bar linkage in a regular and singular configuration.

formula, the number of degrees of freedom is $3(5 - 1 - 6) + 6 = 0$. A mechanism with zero degrees of freedom is by definition a rigid structure. It is clear from examining the figure, though, that the mechanism can in fact move with one degree of freedom. Indeed, any one of the three parallel links, with its two joints, has no effect on the motion of the mechanism, so we should have calculated $\text{dof} = 3(4 - 1 - 4) + 4 = 1$. In other words, the constraints provided by the joints are not independent, as required by Grübler's formula.

A similar situation arises for the two-dof planar five-bar linkage of Figure 2.7(b). If the two joints connected to ground are locked at some fixed angle, the five-bar linkage should then become a rigid structure. However, if the two middle links are the same length and overlap each other, as illustrated in Figure 2.7(b), these overlapping links can rotate freely about the two overlapping joints. Of course, the link lengths of the five-bar linkage must meet certain specifications in order for such a configuration to even be possible. Also note that if a different pair of joints is locked in place, the mechanism does become a rigid structure as expected.

Example 2.7 (Delta robot)   The Delta robot of Figure 2.8 consists of two platforms – the lower one mobile, the upper one stationary – connected by three legs. Each leg contains a parallelogram closed chain and consists of three revolute joints, four spherical joints, and five links. Adding the two platforms, there are $N = 17$ links and $J = 21$ joints (nine revolute and 12 spherical). By Grübler's formula,

$$\text{dof} = 6(17 - 1 - 21) + 9(1) + 12(3) = 15.$$
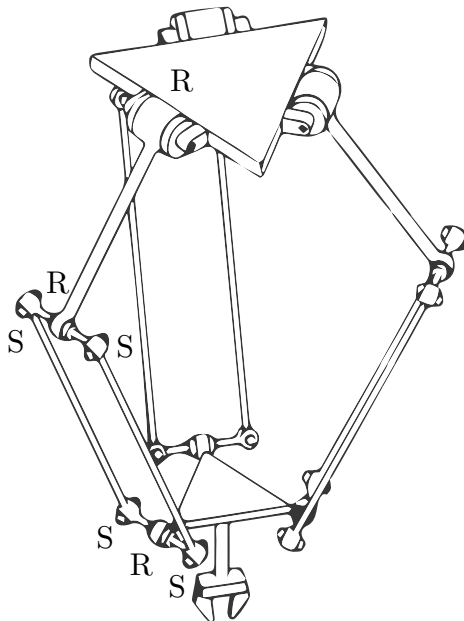
Figure 2.8 The Delta robot.

Of these 15 degrees of freedom, however, only three are visible at the end-effector on the moving platform. In fact, the parallelogram leg design ensures that the moving platform always remains parallel to the fixed platform, so that the Delta robot acts as an x–y–z Cartesian positioning device. The other 12 internal degrees of freedom are accounted for by torsion of the 12 links in the parallelograms (each of the three legs has four links in its parallelogram) about their long axes.

Example 2.8 (Stewart–Gough platform)   The Stewart–Gough platform of Figure 1.1(b) consists of two platforms – the lower one stationary and regarded as ground, the upper one mobile – connected by six universal–prismatic–spherical (UPS) legs. The total number of links is 14 ($N = 14$). There are six universal joints (each with two degrees of freedom, $f_i = 2$), six prismatic joints (each with a single degree of freedom, $f_i = 1$), and six spherical joints (each with three degrees of freedom, $f_i = 3$). The total number of joints is 18. Substituting these values into Grübler's formula with $m = 6$ yields

$$\text{dof} = 6(14 - 1 - 18) + 6(1) + 6(2) + 6(3) = 6.$$

In some versions of the Stewart–Gough platform the six universal joints are replaced by spherical joints. By Grübler's formula this mechanism has 12 degrees of freedom; replacing each universal joint by a spherical joint introduces an extra degree of freedom in each leg, allowing torsional rotations about the leg axis. Note, however, that this torsional rotation has no effect on the motion of the mobile platform.

The Stewart–Gough platform is a popular choice for car and airplane cockpit simulators, as the platform moves with the full six degrees of freedom of motion of a rigid body. On the one hand, the parallel structure means that each leg needs to support only a fraction of the weight of the payload. On the other hand, this structure also limits the range of translational and rotational motion of the platform relative to the range of motion of the end-effector of a six-dof open chain.

2.3   Configuration Space: Topology and Representation

2.3.1   Configuration Space Topology

Until now we have been focusing on one important aspect of a robot's C-space – its dimension, or the number of degrees of freedom. However, the *shape* of the space is also important.

Consider a point moving on the surface of a sphere. The point's C-space is two dimensional, as the configuration can be described by two coordinates, latitude and longitude. As another example, a point moving on a plane also has a two-dimensional C-space, with coordinates (x, y). While both a plane and the surface of a sphere are two dimensional, clearly they do not have the same shape – the plane extends infinitely while the sphere wraps around.

Figure 2.9 An open interval of the real line, denoted (a, b), can be deformed to an open semicircle. This open semicircle can then be deformed to the real line by the mapping illustrated: beginning from a point at the center of the semicircle, draw a ray that intersects the semicircle and then a line above the semicircle. These rays show that every point of the semicircle can be stretched to exactly one point on the line, and vice versa. Thus an open interval can be continuously deformed to a line, so an open interval and a line are topologically equivalent.

Unlike the plane, a larger sphere has the same shape as the original sphere, in that it wraps around in the same way. Only its size is different. For that matter, an oval-shaped American football also wraps around similarly to a sphere. The only difference between a football and a sphere is that the football has been stretched in one direction.

The idea that the two-dimensional surfaces of a small sphere, a large sphere, and a football all have the same kind of shape, which is different from the shape of a plane, is expressed by the **topology** of the surfaces. We do not attempt a rigorous treatment in this book,[3] but we say that two spaces are **topologically equivalent** if one can be continuously deformed into the other without cutting or gluing. A sphere can be deformed into a football simply by stretching, without cutting or gluing, so those two spaces are topologically equivalent. You cannot turn a sphere into a plane without cutting it, however, so a sphere and a plane are not topologically equivalent.

Topologically distinct one-dimensional spaces include the circle, the line, and a closed interval of the line. The circle is written mathematically as S or S^1, a one-dimensional "sphere." The line can be written as $\mathbb{E}$ or $\mathbb{E}^1$, indicating a one-dimensional Euclidean (or "flat") space. Since a point in $\mathbb{E}^1$ is usually represented by a real number (after choosing an origin and a length scale), it is often written as $\mathbb{R}$ or $\mathbb{R}^1$ instead. A closed interval of the line, which contains its endpoints, can be written $[a, b] \subset \mathbb{R}^1$. (An open interval (a, b) does not include the endpoints a and b and is topologically equivalent to a line, since the open interval can be stretched to a line, as shown in Figure 2.9. A closed interval is not topologically equivalent to a line, since a line does not contain endpoints.)

In higher dimensions, $\mathbb{R}^n$ is the n-dimensional Euclidean space and S^n is the n-dimensional surface of a sphere in $(n + 1)$-dimensional space. For example, S^2 is the two-dimensional surface of a sphere in three-dimensional space.

Note that the topology of a space is a fundamental property of the space itself and *is independent of how we choose coordinates to represent points in the space.*

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[3] For those familiar with concepts in topology, all the spaces we consider can be viewed as embedded in a higher-dimensional Euclidean space, inheriting the Euclidean topology of that space.

For example, to represent a point on a circle, we could refer to the point by the angle θ from the center of the circle to the point, relative to a chosen zero angle. Or, we could choose a reference frame with its origin at the center of the circle and represent the point by the two coordinates (x, y) subject to the constraint $x^2 + y^2 = 1$. No matter what our choice of coordinates is, the space itself does not change.
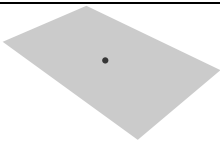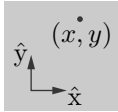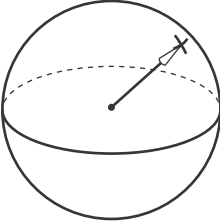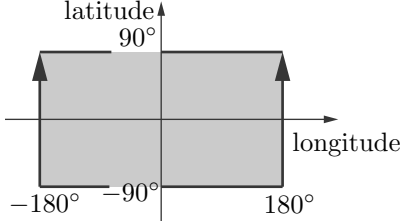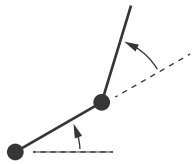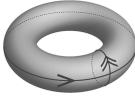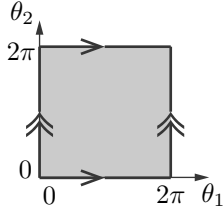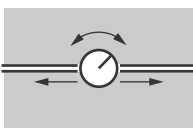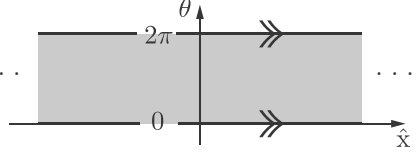
Some C-spaces can be expressed as the **Cartesian product** of two or more spaces of lower dimension; that is, points in such a C-space can be represented as the union of the representations of points in the lower-dimensional spaces. For example:

- The C-space of a rigid body in the plane can be written as $\mathbb{R}^2 \times S^1$, since the configuration can be represented as the concatenation of the coordinates (x, y) representing $\mathbb{R}^2$ and an angle θ representing S^1.
- The C-space of a PR robot arm can be written $\mathbb{R}^1 \times S^1$. (We will occasionally ignore joint limits, i.e., bounds on the travel of the joints, when expressing the topology of the C-space; with joint limits, the C-space is the Cartesian product of two closed intervals of the line.)
- The C-space of a 2R robot arm can be written $S^1 \times S^1 = T^2$, where T^n is the n-dimensional surface of a torus in an $(n+1)$-dimensional space. (See Table 2.2.) Note that $S^1 \times S^1 \times \cdots \times S^1$ (n copies of S^1) is equal to T^n, not S^n; for example, a sphere S^2 is not topologically equivalent to a torus T^2.
- The C-space of a planar rigid body (e.g., the chassis of a mobile robot) with a 2R robot arm can be written as $\mathbb{R}^2 \times S^1 \times T^2 = \mathbb{R}^2 \times T^3$.
- As we saw in Section 2.1 when we counted the degrees of freedom of a rigid body in three dimensions, the configuration of a rigid body can be described by a point in $\mathbb{R}^3$, plus a point on a two-dimensional sphere S^2, plus a point on a one-dimensional circle S^1, giving a total C-space of $\mathbb{R}^3 \times S^2 \times S^1$.

2.3.2 Configuration Space Representation

To perform computations, we must have a numerical **representation** of the space, consisting of a set of real numbers. We are familiar with this idea from linear algebra – a vector is a natural way to represent a point in a Euclidean space. It is important to keep in mind that the representation of a space involves a choice, and therefore it is not as fundamental as the topology of the space, which is independent of the representation. For example, the same point in a three-dimensional space can have different coordinate representations depending on the choice of reference frame (the origin and the direction of the coordinate axes) and the choice of length scale, but the topology of the underlying space is the same regardless of theses choices.

While it is natural to choose a reference frame and length scale and to use a vector to represent points in a Euclidean space, representing a point on a

system	topology	sample representation
point on a plane	$\mathbb{E}^2$	$\mathbb{R}^2$
spherical pendulum	S^2	$[-180°, 180°) \times [-90°, 90°]$
2R robot arm	$T^2 = S^1 \times S^1$	$[0, 2\pi) \times [0, 2\pi)$
rotating sliding knob	$\mathbb{E}^1 \times S^1$	$\mathbb{R}^1 \times [0, 2\pi)$

Table 2.2 Four topologically different two-dimensional C-spaces and example coordinate representations. In the latitude–longitude representation of the sphere, the latitudes $-90°$ and $90°$ each correspond to a single point (the south pole and the north pole, respectively), and the longitude parameter wraps around at $180°$ and $-180°$; the edges with arrows are glued together. Similarly, the coordinate representations of the torus and cylinder wrap around at the edges marked with corresponding arrows.

curved space, such as a sphere, is less obvious. One solution for a sphere is to use latitude and longitude coordinates. A choice of n coordinates, or parameters, to represent an n-dimensional space is called an **explicit parametrization** of the space. Such an explicit parametrization is valid for a particular range of the parameters (e.g., $[-90°, 90°]$ for latitude and $[-180°, 180°)$ for longitude for a sphere, where, on Earth, negative values correspond to "south" and "west," respectively).

The latitude–longitude representation of a sphere is unsatisfactory if you are walking near the North Pole (where the latitude equals $90°$) or South Pole (where

the latitude equals $-90°$), where taking a very small step can result in a large change in the coordinates. The North and South Poles are **singularities** of the representation, and the existence of singularities is a result of the fact that a sphere does not have the same topology as a plane, i.e., the space of the two real numbers that we have chosen to represent the sphere (latitude and longitude). The location of these singularities has nothing to do with the sphere itself, which looks the same everywhere, and everything to do with the chosen representation of it. Singularities of the parametrization are particularly problematic when representing velocities as the time rate of change of coordinates, since these representations may tend to infinity near singularities even if the point on the sphere is moving at a constant speed $\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$ (which is what the speed would be had you represented the point as (x, y, z) instead).

If you can assume that the configuration never approaches a singularity of the representation, you can ignore this issue. If you cannot make this assumption, there are two ways to overcome the problem.

- Use more than one **coordinate chart** on the space, where each coordinate chart is an explicit parametrization covering only a portion of the space such that, within each chart, there is no singularity. As the configuration representation approaches a singularity in one chart, e.g., the North or South Pole, you simply switch to another chart where the North and South Poles are far from singularities.

  If we define a set of singularity-free coordinate charts that overlap each other and cover the entire space, like the two charts above, the charts are said to form an **atlas** of the space, much as an atlas of the Earth consists of several maps that together cover the Earth. An advantage of using an atlas of coordinate charts is that the representation always uses the minimum number of numbers. A disadvantage is the extra bookkeeping required to switch representations between coordinate charts to avoid singularities. (Note that Euclidean spaces can be covered by a single coordinate chart without singularities.)

- Use an **implicit representation** of the space instead of an explicit parametrization. An implicit representation views the n-dimensional space as embedded in a Euclidean space of more than n dimensions, just as a two-dimensional unit sphere can be viewed as a surface embedded in a three-dimensional Euclidean space. An implicit representation uses the coordinates of the higher-dimensional space (e.g., (x, y, z) in the three-dimensional space), but subjects these coordinates to constraints that reduce the number of degrees of freedom (e.g., $x^2 + y^2 + z^2 = 1$ for the unit sphere).

  A disadvantage of this approach is that the representation has more numbers than the number of degrees of freedom. An advantage is that there are no singularities in the representation – a point moving smoothly around the sphere is represented by a smoothly changing (x, y, z), even at

the North and South poles. A single representation is used for the whole sphere; multiple coordinate charts are not needed.

Another advantage is that while it may be very difficult to construct an explicit parametrization, or atlas, for a closed-chain mechanism, it is easy to find an implicit representation: the set of all joint coordinates subject to the **loop-closure equations** that define the closed loops (Section 2.4).

We will use implicit representations throughout the book, beginning in the next chapter. In particular, we use nine numbers, subject to six constraints, to represent the three orientation freedoms of a rigid body in space. This is called a **rotation matrix**. In addition to being singularity-free (unlike three-parameter representations such as roll–pitch–yaw angles[4]), the rotation matrix representation allows us to use linear algebra to perform computations such as rotating a rigid body or changing the reference frame in which the orientation of a rigid body is expressed.[5]

In summary, the non-Euclidean shape of many C-spaces motivates our use of implicit representations of C-space throughout this book. We return to this topic in the next chapter.

2.4   Configuration and Velocity Constraints

For robots containing one or more closed loops, usually an implicit representation is more easily obtained than an explicit parametrization. For example, consider the planar four-bar linkage of Figure 2.10, which has one degree of freedom. The fact that the four links always form a closed loop can be expressed by the following three equations:

$$L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2) + \cdots + L_4 \cos(\theta_1 + \cdots + \theta_4) = 0,$$
$$L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2) + \cdots + L_4 \sin(\theta_1 + \cdots + \theta_4) = 0,$$
$$\theta_1 + \theta_2 + \theta_3 + \theta_4 - 2\pi = 0.$$

These equations are obtained by viewing the four-bar linkage as a serial chain with four revolute joints in which (i) the tip of link L_4 always coincides with the origin and (ii) the orientation of link L_4 is always horizontal.

These equations are sometimes referred to as **loop-closure equations**. For the four-bar linkage they are given by a set of three equations in four unknowns. The set of all solutions forms a one-dimensional curve in the four-dimensional joint space and constitutes the C-space.

[4] Roll–pitch–yaw angles and *Euler* angles use three parameters for the space of rotations $S^2 \times S^1$ (two for S^2 and one for S^1), and therefore are subject to singularities as discussed above.

[5] Another singularity-free implicit representation of orientations, the unit quaternion, uses only four numbers subject to the constraint that the 4-vector be of unit length. In fact, this representation is a double covering of the set of orientations: for every orientation, there are two unit quaternions.
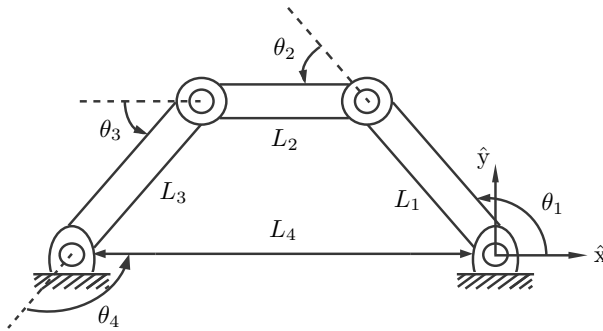
Figure 2.10 The four-bar linkage.

In this book, when vectors are used in a linear algebra computation, they are generally treated as column vectors, e.g., $p = [1\ 2\ 3]^{\mathrm{T}}$. When a computation is not imminent, however, we often think of a vector simply as an ordered list of variables, e.g., $p = (1, 2, 3)$.

Thus, for general robots containing one or more closed loops, the configuration space can be implicitly represented by the column vector $\theta = [\theta_1\ \cdots\ \theta_n]^{\mathrm{T}} \in \mathbb{R}^n$ and loop-closure equations of the form

$$g(\theta) = \begin{bmatrix} g_1(\theta_1, \ldots, \theta_n) \\ \vdots \\ g_k(\theta_1, \ldots, \theta_n) \end{bmatrix} = 0, \tag{2.5}$$

a set of k independent equations, with $k \leq n$. Such constraints are known as **holonomic constraints**, ones that reduce the dimension of the C-space.[6] The C-space can be viewed as a surface of dimension $n-k$ (assuming that all constraints are independent) embedded in $\mathbb{R}^n$.

Suppose that a closed-chain robot with loop-closure equations $g(\theta) = 0$, $g : \mathbb{R}^n \to \mathbb{R}^k$, is in motion, following the time trajectory $\theta(t)$. Differentiating both sides of $g(\theta(t)) = 0$ with respect to t, we obtain

$$\frac{d}{dt}g(\theta(t)) = 0; \tag{2.6}$$

thus

$$\begin{bmatrix} \dfrac{\partial g_1}{\partial \theta_1}(\theta)\dot{\theta}_1 + \cdots + \dfrac{\partial g_1}{\partial \theta_n}(\theta)\dot{\theta}_n \\ \vdots \\ \dfrac{\partial g_k}{\partial \theta_1}(\theta)\dot{\theta}_1 + \cdots + \dfrac{\partial g_k}{\partial \theta_n}(\theta)\dot{\theta}_n \end{bmatrix} = 0.$$

---

[6] Viewing a rigid body as a collection of points, the distance constraints between the points, as we saw earlier, can be viewed as holonomic constraints.

This can be expressed as a matrix multiplying a column vector $[\dot{\theta}_1 \; \cdots \; \dot{\theta}_n]^{\mathrm{T}}$:

$$
\begin{bmatrix}
\dfrac{\partial g_1}{\partial \theta_1}(\theta) & \cdots & \dfrac{\partial g_1}{\partial \theta_n}(\theta) \\
\vdots & \ddots & \vdots \\
\dfrac{\partial g_k}{\partial \theta_1}(\theta) & \cdots & \dfrac{\partial g_k}{\partial \theta_n}(\theta)
\end{bmatrix}
\begin{bmatrix}
\dot{\theta}_1 \\
\vdots \\
\dot{\theta}_n
\end{bmatrix} = 0,
$$

which we can write as

$$
\frac{\partial g}{\partial \theta}(\theta)\dot{\theta} = 0. \tag{2.7}
$$

Here, the joint-velocity vector $\dot{\theta}_i$ denotes the derivative of θ_i with respect to time t, $\partial g(\theta)/\partial \theta \in \mathbb{R}^{k \times n}$, and $\theta, \dot{\theta} \in \mathbb{R}^n$. The constraints (2.7) can be written

$$
A(\theta)\dot{\theta} = 0, \tag{2.8}
$$

where $A(\theta) \in \mathbb{R}^{k \times n}$. Velocity constraints of this form are called **Pfaffian constraints**. For the case of $A(\theta) = \partial g(\theta)/\partial \theta$, one could regard $g(\theta)$ as being the "integral" of $A(\theta)$; for this reason, holonomic constraints of the form $g(\theta) = 0$ are also called **integrable constraints** – the velocity constraints that they imply can be integrated to give equivalent configuration constraints.

We now consider another class of Pfaffian constraints that are fundamentally different from the holonomic type. To illustrate this with a concrete example, consider an upright coin of radius r rolling on a plane as shown in Figure 2.11. The configuration of the coin is given by the contact point (x, y) on the plane, the steering angle ϕ, and the angle of rotation θ. The C-space of the coin is therefore $\mathbb{R}^2 \times T^2$, where T^2 is the two-dimensional torus parametrized by the angles ϕ and θ. This C-space is four dimensional.
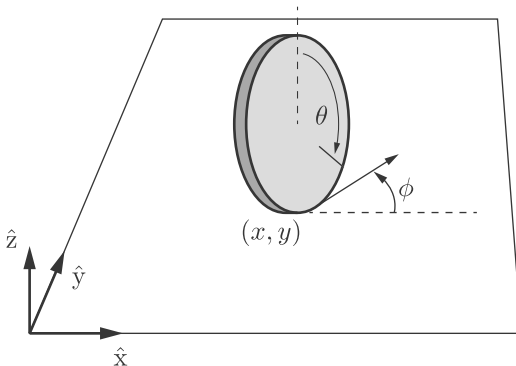
Figure 2.11 A coin rolling on a plane without slipping.

We now express, in mathematical form, the fact that the coin rolls without slipping. The coin must always roll in the direction indicated by $(\cos\phi, \sin\phi)$, with forward speed $r\dot{\theta}$:

$$
\begin{bmatrix}
\dot{x} \\
\dot{y}
\end{bmatrix} = r\dot{\theta}
\begin{bmatrix}
\cos\phi \\
\sin\phi
\end{bmatrix}. \tag{2.9}
$$

Collecting the four C-space coordinates into a single vector $q = [q_1 \; q_2 \; q_3 \; q_4]^{\mathrm{T}} = [x \; y \; \phi \; \theta]^{\mathrm{T}} \in \mathbb{R}^2 \times T^2$, the above no-slip rolling constraint can then be expressed in the form

$$\begin{bmatrix} 1 & 0 & 0 & -r\cos q_3 \\ 0 & 1 & 0 & -r\sin q_3 \end{bmatrix} \dot{q} = 0. \tag{2.10}$$

These are Pfaffian constraints of the form $A(q)\dot{q} = 0$, $A(q) \in \mathbb{R}^{2 \times 4}$.

These constraints are not integrable; that is, for the $A(q)$ given in (2.10), there does not exist a differentiable function $g : \mathbb{R}^4 \to \mathbb{R}^2$ such that $\partial g(q)/\partial q = A(q)$. If this were not the case then there would have to exist a differentiable $g_1(q)$ that satisfied the following four equalities:

$$
\begin{array}{llll}
\partial g_1(q)/\partial q_1 & = & 1 & \longrightarrow \quad g_1(q) = q_1 + h_1(q_2, q_3, q_4) \\
\partial g_1(q)/\partial q_2 & = & 0 & \longrightarrow \quad g_1(q) = h_2(q_1, q_3, q_4) \\
\partial g_1(q)/\partial q_3 & = & 0 & \longrightarrow \quad g_1(q) = h_3(q_1, q_2, q_4) \\
\partial g_1(q)/\partial q_4 & = & -r\cos q_3 & \longrightarrow \quad g_1(q) = -rq_4 \cos q_3 + h_4(q_1, q_2, q_3),
\end{array}
$$

for some h_i, $i = 1, \ldots, 4$, differentiable in each of its variables. By inspection it should be clear that no such $g_1(q)$ exists. Similarly, it can be shown that $g_2(q)$ does not exist, so that the constraint (2.10) is nonintegrable. A Pfaffian constraint that is nonintegrable is called a **nonholonomic constraint.** Such constraints reduce the dimension of the feasible velocities of the system but do not reduce the dimension of the reachable C-space. The rolling coin can reach any point in its four-dimensional C-space despite the two constraints on its velocity.[7] See Exercise 2.30.

In a number of robotics contexts nonholonomic constraints arise that involve the conservation of momentum and rolling without slipping, e.g., wheeled vehicle kinematics and grasp contact kinematics. We examine nonholonomic constraints in greater detail in our study of wheeled mobile robots in Chapter 13.

2.5 Task Space and Workspace

We now introduce two more concepts relating to the configuration of a robot: the task space and the workspace. Both relate to the configuration of the end-effector of a robot, not to the configuration of the entire robot.

The **task space** is a space in which the robot's task can be naturally expressed. For example, if the robot's task is to plot with a pen on a piece of paper, the task space would be $\mathbb{R}^2$. If the task is to manipulate a rigid body, a natural representation of the task space is the C-space of a rigid body, representing the position and orientation of a frame attached to the robot's end-effector. This is the default representation of task space. The decision of how to define the task space is driven by the task, independently of the robot.

---

[7] Some texts define the number of degrees of freedom of a system to be the dimension of the feasible velocities, e.g., two for the rolling coin. We always refer to the dimension of the C-space as the number of degrees of freedom.
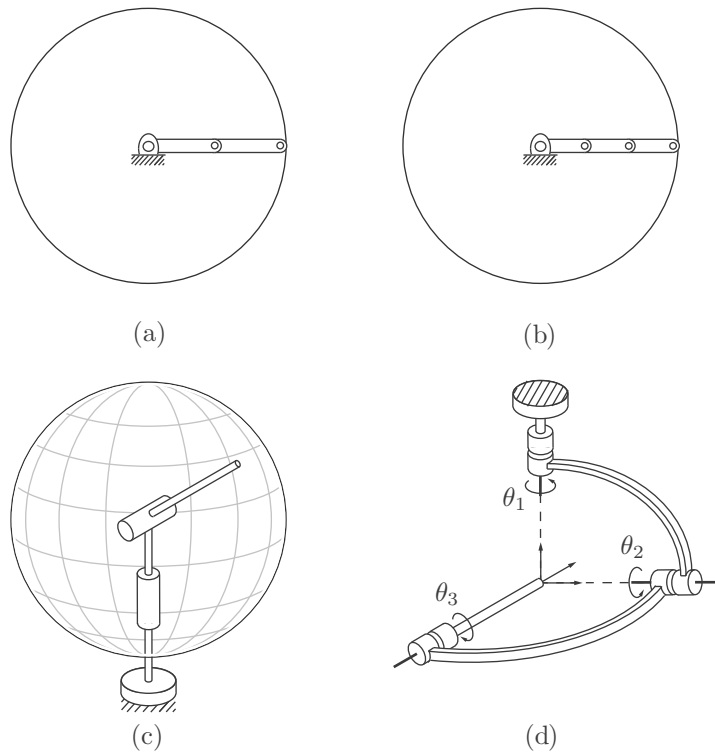
Figure 2.12 Examples of workspaces for various robots: (a) a planar 2R open chain; (b) a planar 3R open chain; (c) a spherical 2R open chain; (d) a 3R orienting mechanism.

The **workspace** is a specification of the configurations that the end-effector of the robot can reach. The definition of the workspace is primarily driven by the robot's structure, independently of the task.

Both the task space and the workspace involve a choice by the user; in particular, the user may decide that some freedoms of the end-effector (e.g., its orientation) do not need to be represented.

The task space and the workspace are distinct from the robot's C-space. A point in the task space or the workspace may be achievable by more than one robot configuration, meaning that the point is not a full specification of the robot's configuration. For example, for an open-chain robot with seven joints, the six-dof position and orientation of its end-effector does not fully specify the robot's configuration.

Some points in the task space may not be reachable at all by the robot, such as some points on a chalkboard. By definition, however, all points in the workspace are reachable by at least one configuration of the robot.

Two mechanisms with different C-spaces may have the same workspace. For

example, considering the end-effector to be the Cartesian tip of the robot (e.g., the location of a plotting pen) and ignoring orientations, the planar 2R open chain with links of equal length three (Figure 2.12(a)) and the planar 3R open chain with links of equal length two (Figure 2.12(b)) have the same workspace despite having different C-spaces.

Two mechanisms with the same C-space may also have different workspaces. For example, taking the end-effector to be the Cartesian tip of the robot and ignoring orientations, the 2R open chain of Figure 2.12(a) has a planar disk as its workspace, while the 2R open chain of Figure 2.12(c) has the surface of a sphere as its workspace.

Attaching a coordinate frame to the tip of the tool of the 3R open chain "wrist" mechanism of Figure 2.12(d), we see that the frame can achieve any orientation by rotating the joints but the Cartesian position of the tip is always fixed. This can be seen by noting that the three joint axes always intersect at the tip. For this mechanism, we would probably define the workspace to be the three-dof space of orientations of the frame, $S^2 \times S^1$, which is different from the C-space T^3. The task space depends on the task; if the job is to point a laser pointer, then rotations about the axis of the laser beam are immaterial and the task space would be S^2, the set of directions in which the laser can point.

Example 2.9   The SCARA robot of Figure 2.13 is an RRRP open chain that is widely used for tabletop pick-and-place tasks. The end-effector configuration is completely described by the four parameters (x, y, z, ϕ), where (x, y, z) denotes the Cartesian position of the end-effector center point and ϕ denotes the orientation of the end-effector in the x–y-plane. Its task space would typically be defined as $\mathbb{R}^3 \times S^1$, and its workspace would typically be defined as the reachable points in (x, y, z) Cartesian space, since all orientations $\phi \in S^1$ can be achieved at all reachable points.

Example 2.10   A standard 6R industrial manipulator can be adapted to spray-painting applications as shown in Figure 2.14. The paint spray nozzle attached to the tip can be regarded as the end-effector. What is important to the task is the Cartesian position of the spray nozzle, together with the direction in which the spray nozzle is pointing; rotations about the nozzle axis (which points in the direction in which paint is being sprayed) do not matter. The nozzle configuration can therefore be described by five coordinates: (x, y, z) for the Cartesian position of the nozzle and spherical coordinates (θ, ϕ) to describe the direction in which the nozzle is pointing. The task space can be written as $\mathbb{R}^3 \times S^2$. The workspace could be the reachable points in $\mathbb{R}^3 \times S^2$, or, to simplify visualization, the user could define the workspace to be the subset of $\mathbb{R}^3$ corresponding to the reachable Cartesian positions of the nozzle.
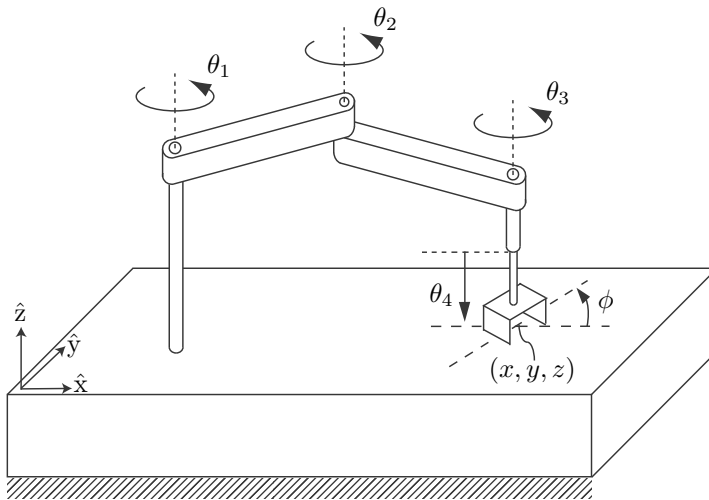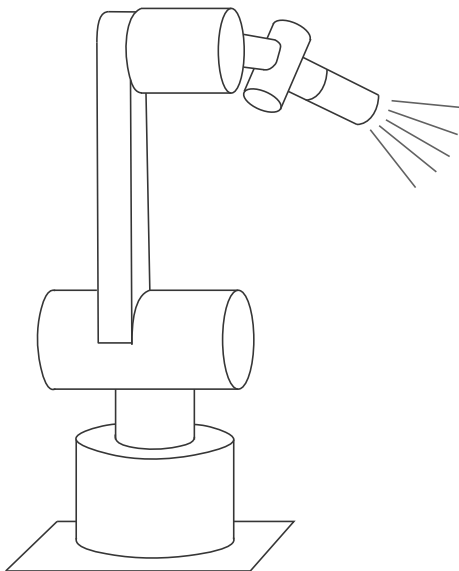
Figure 2.13 SCARA robot.

Figure 2.14 A spray-painting robot.

2.6 Summary

- A robot is mechanically constructed from links that are connected by various types of joint. The links are usually modeled as rigid bodies. An end-effector

such as a gripper may be attached to some link of the robot. Actuators deliver forces and torques to the joints, thereby causing motion of the robot.

- The most widely used one-dof joints are the revolute joint, which allows rotation about the joint axis, and the prismatic joint, which allows translation in the direction of the joint axis. Some common two-dof joints include the cylindrical joint, which is constructed by serially connecting a revolute and prismatic joint, and the universal joint, which is constructed by orthogonally connecting two revolute joints. The spherical joint, also known as the ball-and-socket joint, is a three-dof joint whose function is similar to the human shoulder joint.

- The configuration of a rigid body is a specification of the location of all its points. For a rigid body moving in the plane, three independent parameters are needed to specify the configuration. For a rigid body moving in three-dimensional space, six independent parameters are needed to specify the configuration.

- The configuration of a robot is a specification of the configuration of all its links. The robot's configuration space is the set of all possible robot configurations. The dimension of the C-space is the number of degrees of freedom of a robot.

- The number of degrees of freedom of a robot can be calculated using Grübler's formula,

$$\text{dof} = m(N - 1 - J) + \sum_{i=1}^{J} f_i,$$

where $m = 3$ for planar mechanisms and $m = 6$ for spatial mechanisms, N is the number of links (including the ground link), J is the number of joints, and f_i is the number of degrees of freedom of joint i.

- A robot's C-space can be parametrized explicitly or represented implicitly. For a robot with n degrees of freedom, an explicit parametrization uses n coordinates, the minimum necessary. An implicit representation involves m coordinates with $m \geq n$, with the m coordinates subject to $m - n$ constraint equations. With an implicit parametrization, a robot's C-space can be viewed as a surface of dimension n embedded in a space of higher dimension m.

- The C-space of an n-dof robot whose structure contains one or more closed loops can be implicitly represented using k loop-closure equations of the form $g(\theta) = 0$, where $\theta \in \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^k$. Such constraint equations are called holonomic constraints. Assuming that θ varies with time t, the holonomic constraints $g(\theta(t)) = 0$ can be differentiated with respect to t to yield

$$\frac{\partial g}{\partial \theta}(\theta)\dot{\theta} = 0,$$

where $\partial g(\theta)/\partial \theta$ is a $k \times m$ matrix.

- A robot's motion can also be subject to velocity constraints of the form

$$A(\theta)\dot{\theta} = 0,$$

  where $A(\theta)$ is a $k \times m$ matrix that cannot be expressed as the differential of some function $g(\theta)$. In other words, there does not exist any $g(\theta), g : \mathbb{R}^m \to \mathbb{R}^k$, such that

$$A(\theta) = \frac{\partial g}{\partial \theta}(\theta).$$

  Such constraints are said to be nonholonomic constraints, or nonintegrable constraints. These constraints reduce the dimension of feasible velocities of the system but do not reduce the dimension of the reachable C-space. Nonholonomic constraints arise in robot systems subject to conservation of momentum or rolling without slipping.
- A robot's task space is a space in which the robot's task can be naturally expressed. A robot's workspace is a specification of the configurations that the end-effector of the robot can reach.

2.7   Notes and References

In the kinematics literature, structures that consist of links connected by joints are also called mechanisms or linkages. The number of degrees of freedom of a mechanism, also referred to as its mobility, is treated in most texts on mechanism analysis and design, e.g., Erdman and Sandor (1996) or McCarthy and Soh (2011). The notion of a robot's configuration space was first formulated in the context of motion planning by Lozano-Perez (1980); more recent and advanced treatments can be found in Latombe (1991), LaValle (2006), and Choset et al. (2005). As apparent from examples in this chapter, a robot's configuration space can be nonlinear and curved, as can its task space. Such spaces often have the mathematical structure of a differentiable manifold, which are the central objects of study in differential geometry. Some accessible introductions to differential geometry are Millman and Parker (1977), do Carmo (1976) and Boothby (2002).

2.8   Exercises

In the exercises below, if you are asked to "describe" a C-space, you should indicate its dimension and whatever you know about its topology (e.g., using $\mathbb{R}$, S, and T, as with the examples in Sections 2.3.1 and 2.3.2).

Exercise 2.1   Using the methods of Section 2.1 derive a formula, in terms of n, for the number of degrees of freedom of a rigid body in n-dimensional space. Indicate how many of these dof are translational and how many are rotational. Describe the topology of the C-space (e.g., for $n = 2$, the topology is $\mathbb{R}^2 \times S^1$).
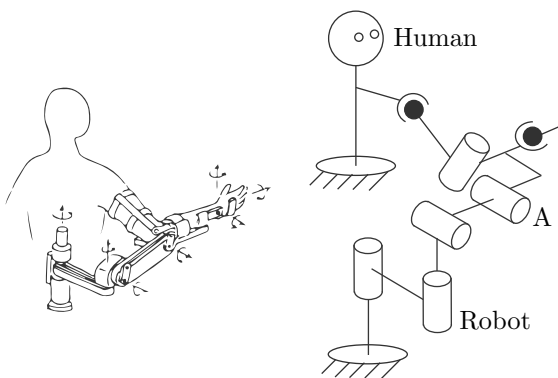
Exercise 2.2   Find the number of degrees of freedom of your arm, from your torso to your palm (just past the wrist, not including finger degrees of freedom). Keep the center of the ball-and-socket joint of your shoulder stationary (do not "hunch" your shoulders). Find the number of degrees of freedom in two ways:

(a) add up the degrees of freedom at the shoulder, elbow, and wrist joints;
(b) fix your palm flat on a table with your elbow bent and, without moving the center of your shoulder joint, investigate with how many degrees of freedom you can still move your arm.

Do your answers agree? How many constraints were placed on your arm when you placed your palm at a fixed configuration on the table?

Exercise 2.3   In the previous exercise, we assumed that your arm is a serial chain. In fact, between your upper arm bone (the humerus) and the bone complex just past your wrist (the carpal bones), your forearm has two bones, the radius and the ulna, which are part of a closed chain. Model your arm, from your shoulder to your palm, as a mechanism with joints and calculate the number of degrees of freedom using Grübler's formula. Be clear on the number of freedoms of each joint you use in your model. Your joints may or may not be of the standard types studied in this chapter (R, P, H, C, U, and S).

Exercise 2.4   Assume each of your arms has n degrees of freedom. You are driving a car, your torso is stationary relative to the car (owing to a tight seatbelt!), and both hands are firmly grasping the wheel, so that your hands do not move relative to the wheel. How many degrees of freedom does your arms-plus-steering wheel system have? Explain your answer.
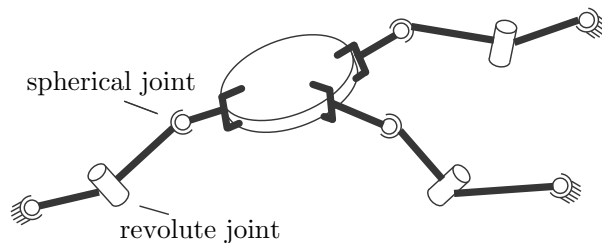

Figure 2.15 Robot used for human arm rehabilitation.

Exercise 2.5   Figure 2.15 shows a robot used for human arm rehabilitation. Determine the number of degrees of freedom of the chain formed by the human arm and the robot.
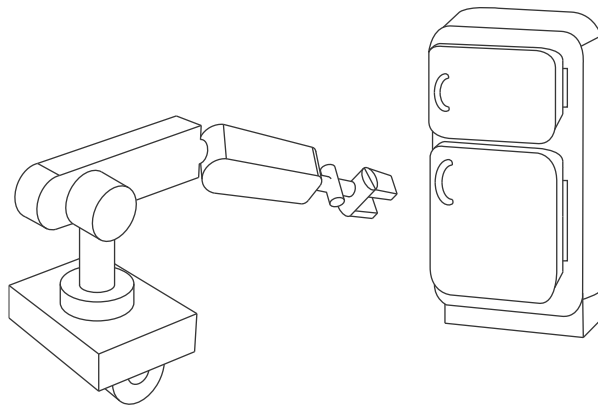
Figure 2.16 Mobile manipulator.

spherical joint

revolute joint

Figure 2.17 Three cooperating SRS arms grasping a common object.

Exercise 2.6 The mobile manipulator of Figure 2.16 consists of a 6R arm and multi-fingered hand mounted on a mobile base with a single wheel. You can think of the wheeled base as the same as the rolling coin in Figure 2.11 – the wheel (and base) can spin together about an axis perpendicular to the ground, and the wheel rolls without slipping. The base always remains horizontal. (Left unstated are the means to keep the base horizontal and to spin the wheel and base about an axis perpendicular to the ground.)

(a) Ignoring the multi-fingered hand, describe the configuration space of the mobile manipulator.
(b) Now suppose that the robot hand rigidly grasps a refrigerator door handle and, with its wheel and base completely stationary, opens the door using only its arm. With the door open, how many degrees of freedom does the mechanism formed by the arm and open door have?
(c) A second identical mobile manipulator comes along, and after parking its mobile base, also rigidly grasps the refrigerator door handle. How many degrees of freedom does the mechanism formed by the two arms and the open refrigerator door have?

Exercise 2.7 Three identical SRS open-chain arms are grasping a common object, as shown in Figure 2.17.

(a) Find the number of degrees of freedom of this system.
(b) Suppose there are now a total of n such arms grasping the object. How many degrees of freedom does this system have?
(c) Suppose the spherical wrist joint in each of the n arms is now replaced by a universal joint. How many degrees of freedom does this system have?

Exercise 2.8   Consider a spatial parallel mechanism consisting of a moving plate connected to a fixed plate by n identical legs. For the moving plate to have six degrees of freedom, how many degrees of freedom should each leg have, as a function of n? For example, if $n = 3$ then the moving plate and fixed plate are connected by three legs; how many degrees of freedom should each leg have for the moving plate to move with six degrees of freedom? Solve for arbitrary n.

Exercise 2.9   Use the planar version of Grübler's formula to determine the number of degrees of freedom of the mechanisms shown in Figure 2.18. Comment on whether your results agree with your intuition about the possible motions of these mechanisms.

Exercise 2.10   Use the planar version of Grübler's formula to determine the number of degrees of freedom of the mechanisms shown in Figure 2.19. Comment on whether your results agree with your intuition about the possible motions of these mechanisms.

Exercise 2.11   Use the spatial version of Grübler's formula to determine the number of degrees of freedom of the mechanisms shown in Figure 2.20. Comment on whether your results agree with your intuition about the possible motions of these mechanisms.

Exercise 2.12   Use the spatial version of Grübler's formula to determine the number of degrees of freedom of the mechanisms shown in Figure 2.21. Comment on whether your results agree with your intuition about the possible motions of these mechanisms.

Exercise 2.13   In the parallel mechanism shown in Figure 2.22, six legs of identical length are connected to a fixed and moving platform via spherical joints. Determine the number of degrees of freedom of this mechanism using Grübler's formula. Illustrate all possible motions of the upper platform.

Exercise 2.14   The $3 \times$ UPU platform of Figure 2.23 consists of two platforms – the lower one stationary, the upper one mobile – connected by three UPU legs.

(a) Using the spatial version of Grübler's formula, verify that it has three degrees of freedom.
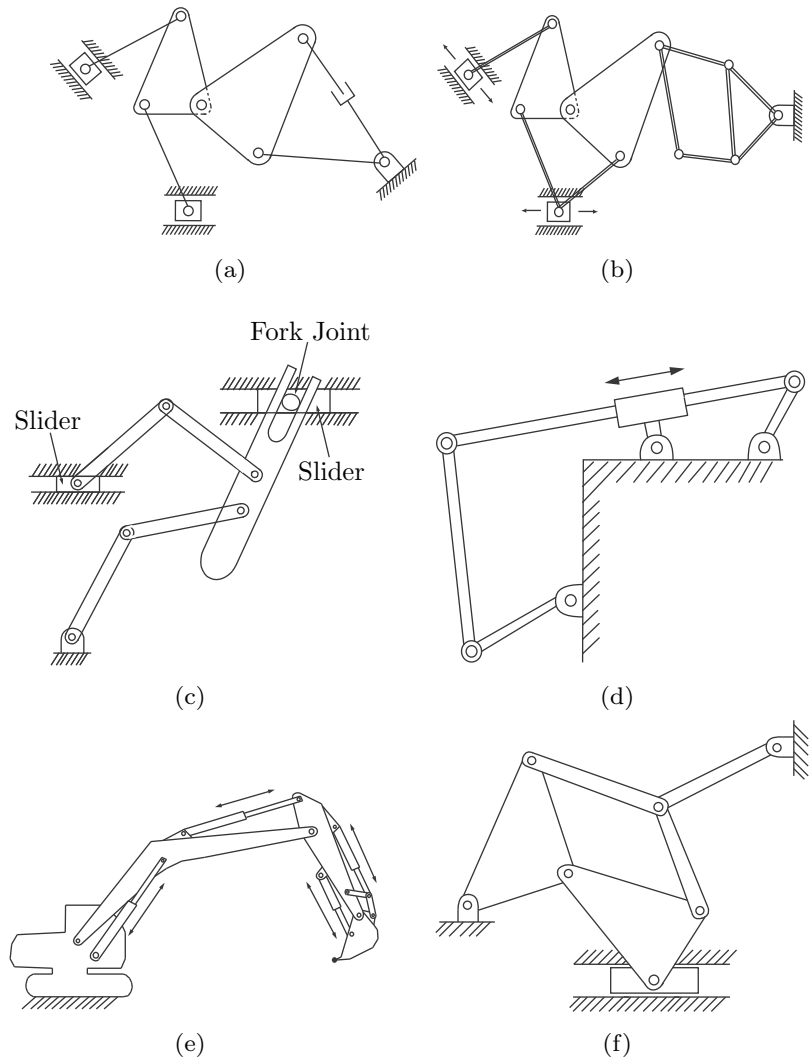
Figure 2.18 A first collection of planar mechanisms.

(b) Construct a physical model of the $3 \times$ UPU platform to see if it does indeed have three degrees of freedom. In particular, lock the three P joints in place; does the robot become a structure as predicted by Grübler's formula, or does it move?

Exercise 2.15   Consider the mechanisms of Figures 2.24(a) and 2.24(b).

(a) The mechanism of Figure 2.24(a) consists of six identical squares arranged in a single closed loop, connected by revolute joints. The bottom square is
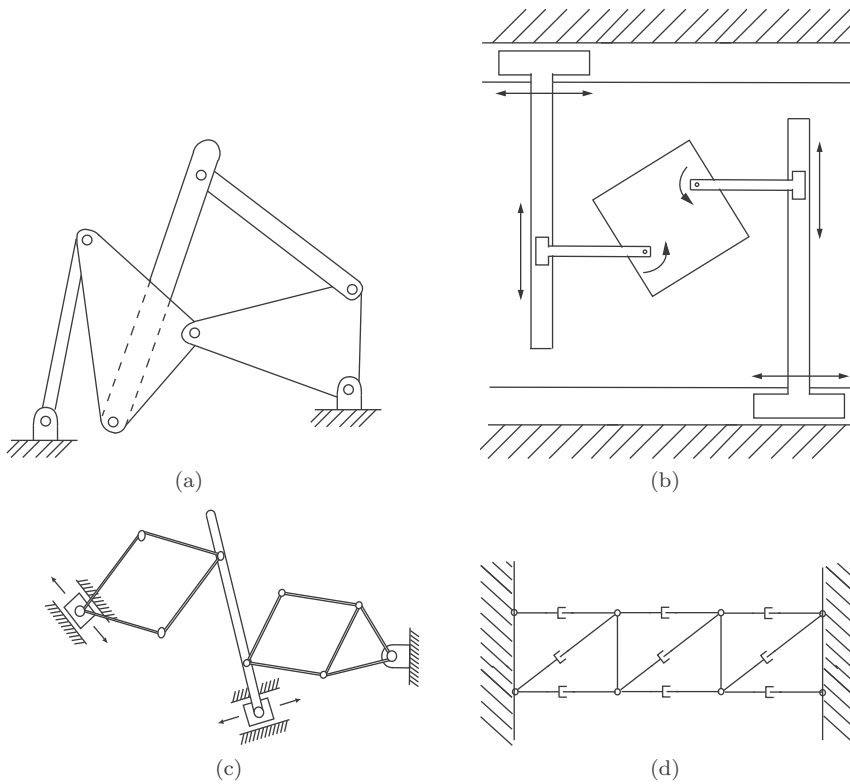
Figure 2.19 A second collection of planar mechanisms.

fixed to ground. Determine the number of degrees of freedom using Grübler's
formula.
(b) The mechanism of Figure 2.24(b) also consists of six identical squares con-
nected by revolute joints, but arranged differently (as shown). Determine
the number of degrees of freedom using Grübler's formula. Does your result
agree with your intuition about the possible motions of this mechanism?

Exercise 2.16   Figure 2.25 shows a spherical four-bar linkage, in which four
links (one of the links is the ground link) are connected by four revolute joints
to form a single-loop closed chain. The four revolute joints are arranged so that
they lie on a sphere such that their joint axes intersect at a common point.

(a) Use Grübler's formula to find the number of degrees of freedom. Justify your
choice of formula.
(b) Describe the configuration space.
(c) Assuming that a reference frame is attached to the center link, describe its
workspace.
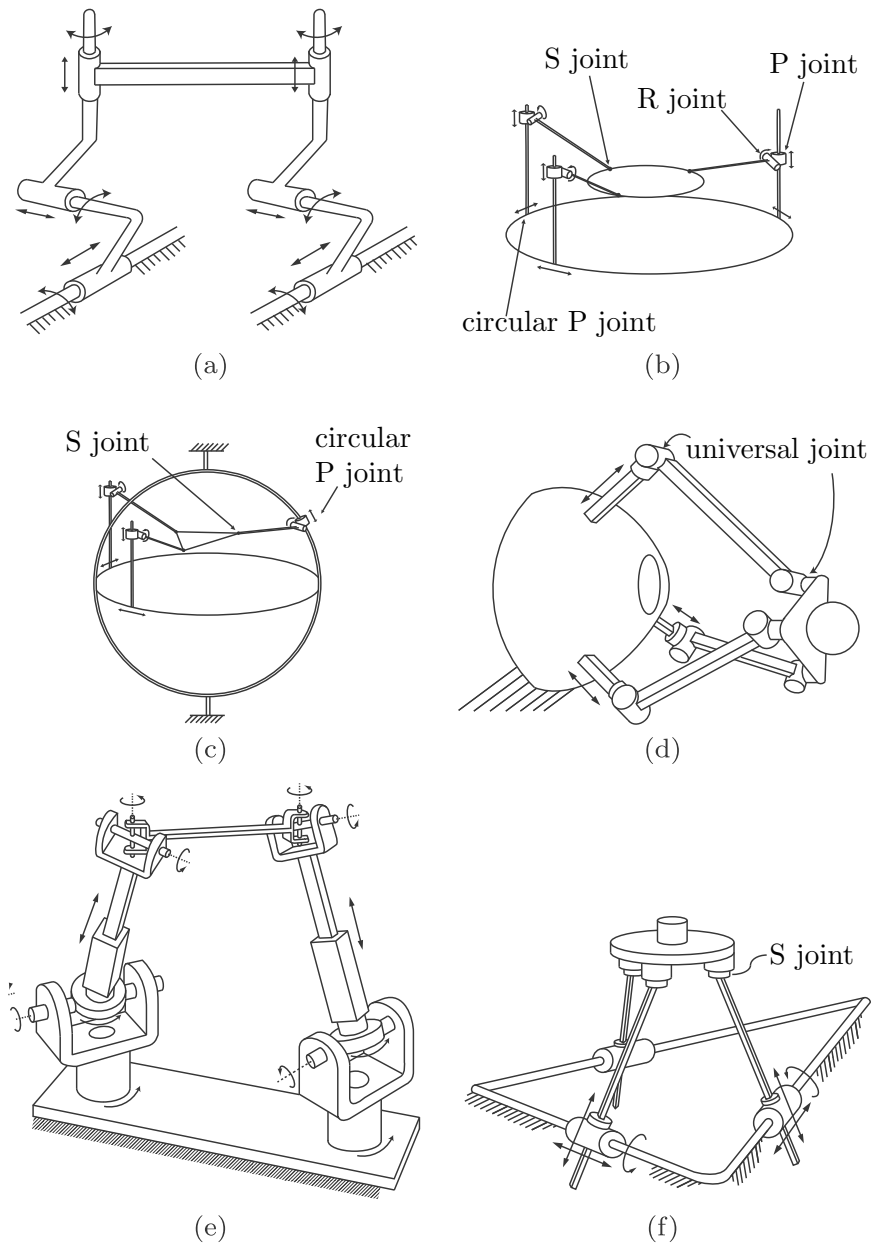
Figure 2.20 A first collection of spatial parallel mechanisms.

Exercise 2.17 Figure 2.26 shows a parallel robot used for surgical applications. As shown in Figure 2.26(a), leg A is an RRRP chain, while legs B and C are RRRUR chains. A surgical tool is rigidly attached to the end-effector.
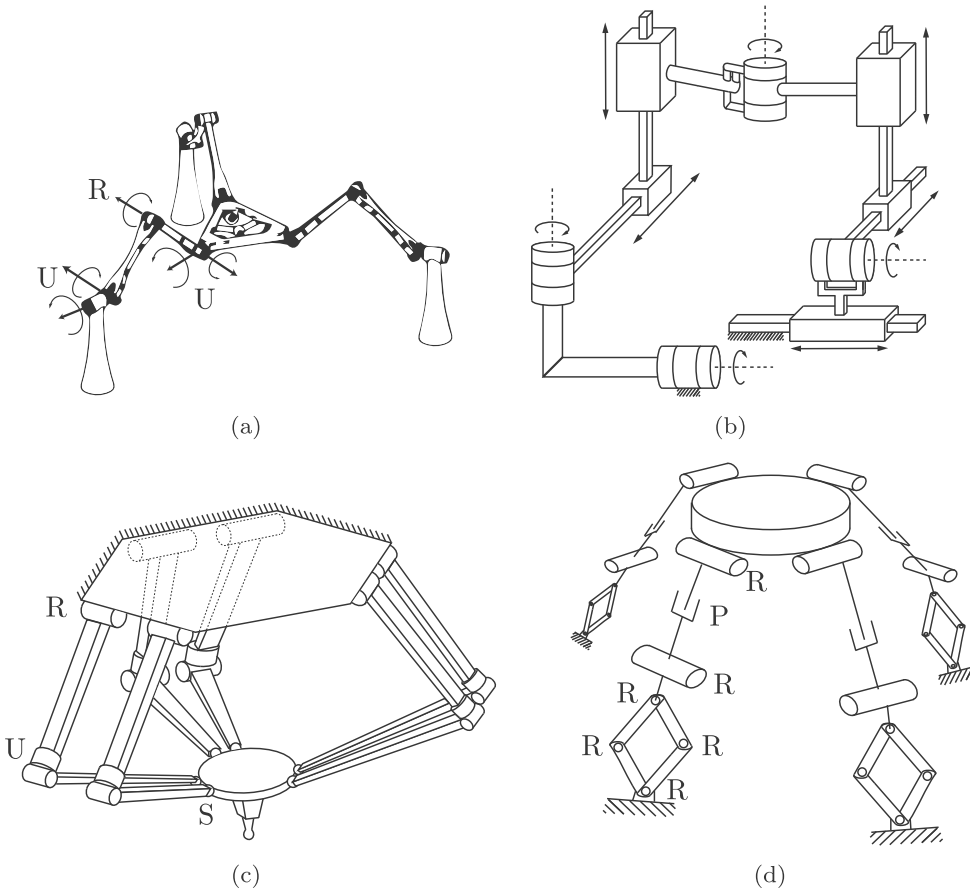
Figure 2.21 A second collection of spatial parallel mechanisms.

(a) Use Grübler's formula to find the number of degrees of freedom of the mechanism in Figure 2.26(a).

(b) Now suppose that the surgical tool must always pass through point A in Figure 2.26(a). How many degrees of freedom does the manipulator have?

(c) Legs A, B, and C are now replaced by three identical RRRR legs as shown in Figure 2.26(b). Furthermore, the axes of all R joints pass through point A. Use Grübler's formula to derive the number of degrees of freedom of this mechanism.

Exercise 2.18 Figure 2.27 shows a $3 \times$ PUP platform, in which three identical PUP legs connect a fixed base to a moving platform. The P joints on both the fixed base and moving platform are arranged symmetrically. Use Grübler's formula to find the number of degrees of freedom. Does your answer agree with your intuition about this mechanism? If not, try to explain any discrepancies without resorting to a detailed kinematic analysis.
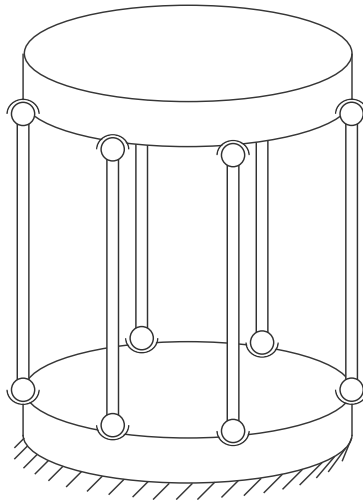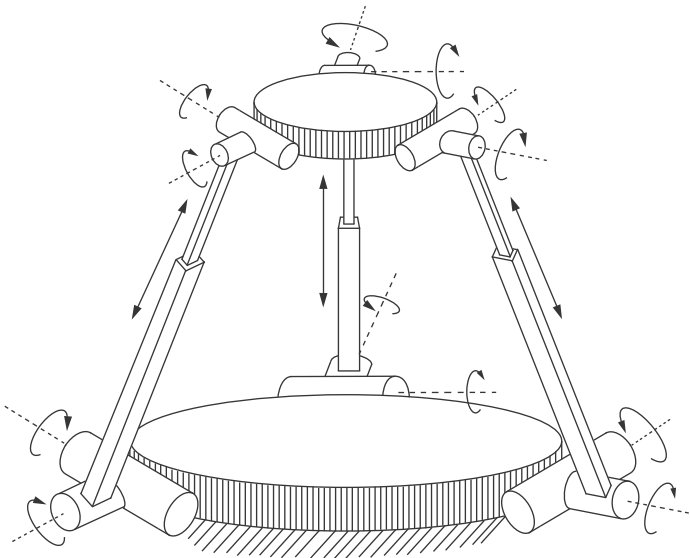
Figure 2.22 A 6×SS platform.

Figure 2.23 The 3×UPU platform.

Exercise 2.19   The dual-arm robot of Figure 2.28 is rigidly grasping a box. The box can only slide on the table; the bottom face of the box must always be in contact with the table. How many degrees of freedom does this system have?
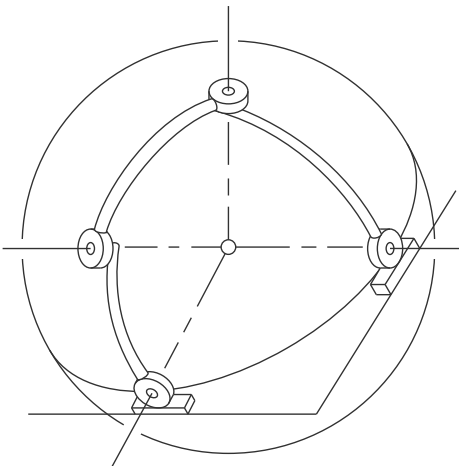
Exercise 2.20   The dragonfly robot of Figure 2.29 has a body, four legs, and four wings as shown. Each leg is connected to each adjacent leg by a USP linkage. Use Grübler's formula to answer the following questions.

(a) Suppose the body is fixed and only the legs and wings can move. How many degrees of freedom does the robot have?
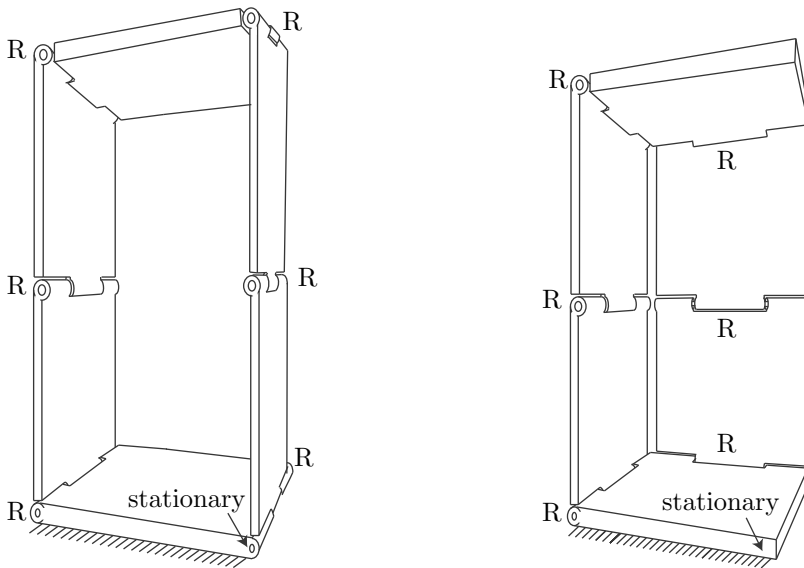
Figure 2.24 Two mechanisms.


Figure 2.25 The spherical four-bar linkage.

(b) Now suppose the robot is flying in the air. How many degrees of freedom does the robot have?

(c) Now suppose the robot is standing with all four feet in contact with the ground. Assume that the ground is uneven and that each foot–ground contact can be modeled as a point contact with no slip. How many degrees of freedom does the robot have? Explain your answer.

Exercise 2.21 A caterpillar robot.

(a) A caterpillar robot is hanging by its tail end as shown in Figure 2.30(a).
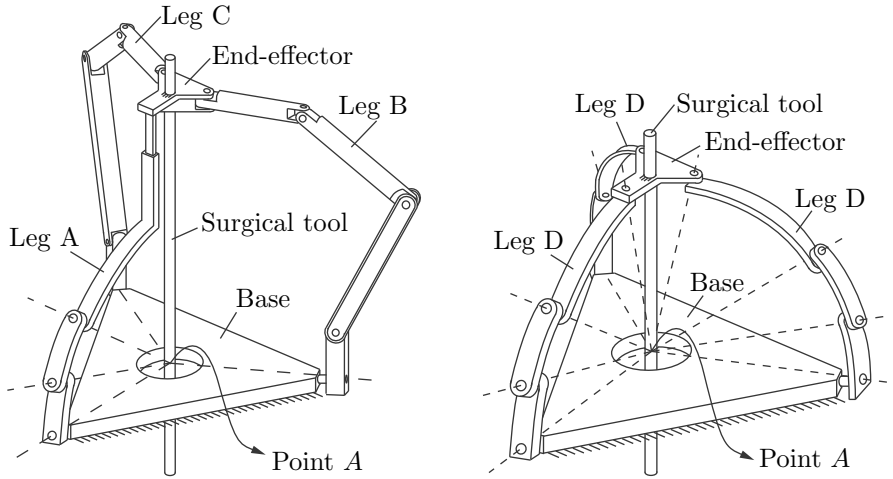
Figure 2.26 Surgical manipulator.
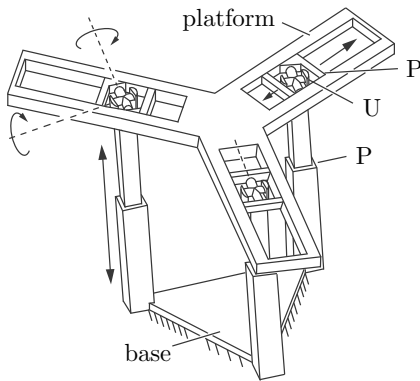
Figure 2.27 The 3×PUP platform.

The robot consists of eight serially connected rigid links (one head, one tail, and six body links). The six body links are connected by revolute–prismatic–revolute joints, while the head and tail are connected to the body by revolute joints. Find the number of degrees of freedom of this robot.

(b) The caterpillar robot is now crawling on a leaf as shown in Figure 2.30(b). Suppose that all six body links must make contact with the leaf at all times but the links can slide and rotate on the leaf. Find the number of degrees of freedom of this robot during crawling.

(c) Now suppose the caterpillar robot crawls on the leaf as shown in Figure 2.30(c), with only the first and last body links in contact with the leaf. Find the number of degrees of freedom of this robot during crawling.

Exercise 2.22 The four-fingered hand of Figure 2.31(a) consists of a palm and four URR fingers (the U joints connect the fingers to the palm).
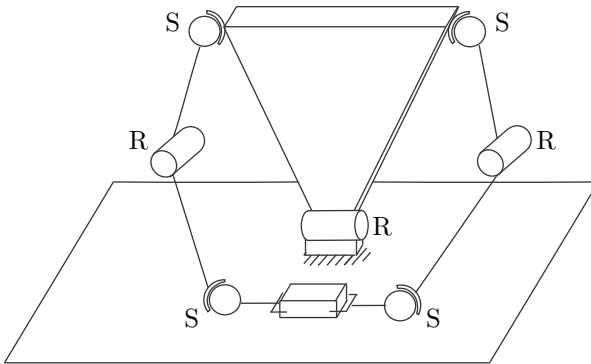
Figure 2.28 Dual arm robot.
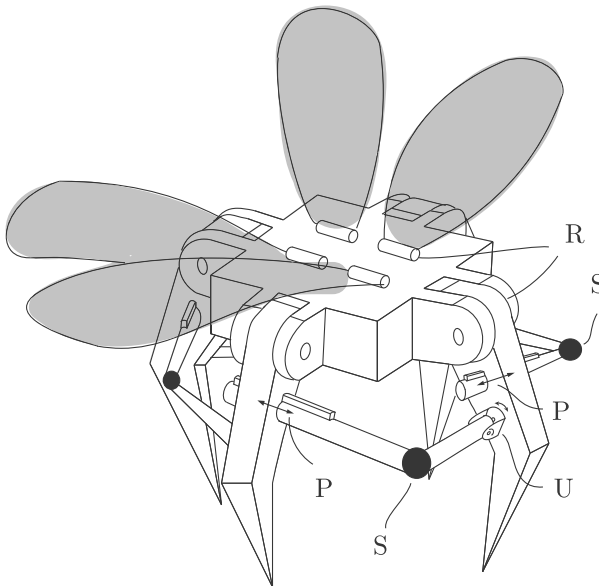
Figure 2.29 Dragonfly robot.

(a) Assume that the fingertips are points and that one fingertip is in contact with the table surface (sliding of the fingertip point-contact is allowed). How many degrees of freedom does the hand have? What if two fingertips are in sliding point contact with the table? Three? All four?

(b) Repeat part (a) but with each URR finger replaced by an SRR finger (each universal joint is replaced by a spherical joint).

(c) The hand (with URR fingers) now grasps an ellipsoidal object, as shown in Figure 2.31(b). Assume that the palm is fixed in space and that no slip occurs between the fingertips and object. How many degrees of freedom does the system have?

(d) Now assume that the fingertips are hemispheres as shown in Figure 2.31(c). Each fingertip can roll on the object but cannot slip or break contact with the object. How many degrees of freedom does the system have? For a single
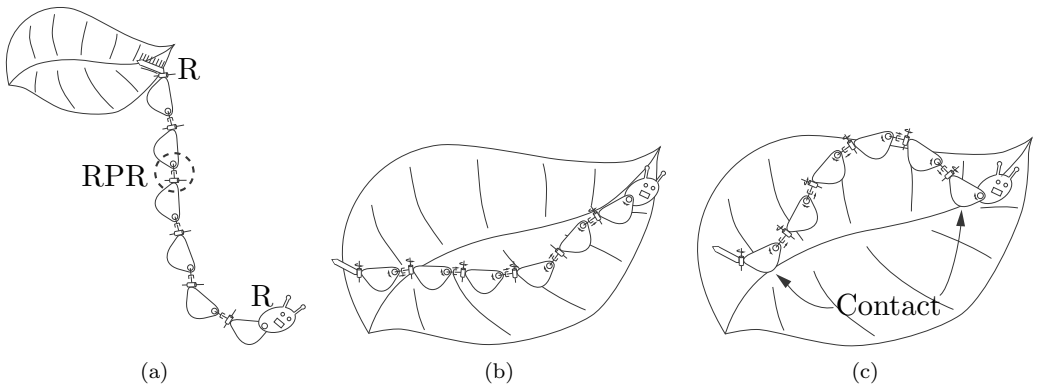
Figure 2.30 A caterpillar robot.
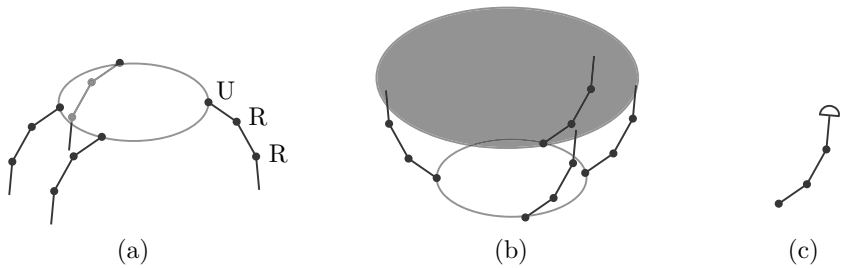
Figure 2.31 (a) A four-fingered hand with palm. (b) The hand grasping an ellipsoidal object. (c) A rounded fingertip that can roll on the object without sliding.

fingertip in rolling contact with the object, comment on the dimension of the space of feasible fingertip velocities relative to the object versus the number of parameters needed to represent the fingertip configuration relative to the object (the number of degrees of freedom). (Hint: You may want to experiment by rolling a ball around on a tabletop to get some intuition.)

Exercise 2.23 Consider the slider–crank mechanism of Figure 2.4(b). A rotational motion at the revolute joint fixed to ground (the "crank") causes a translational motion at the prismatic joint (the "slider"). Suppose that the two links connected to the crank and slider are of equal length. Determine the configuration space of this mechanism, and draw its projected version on the space defined by the crank and slider joint variables.

Exercise 2.24 The planar four-bar linkage.

(a) Use Grübler's formula to determine the number of degrees of freedom of a planar four-bar linkage floating in space.
(b) Derive an implicit parametrization of the four-bar's configuration space as follows. First, label the four links 1, 2, 3, and 4, and choose three points

A, B, C on link 1, D, E, F on link 2, G, H, I on link 3, and J, K, L on link 4. The four-bar linkage is constructed in such a way that the following four pairs of points are each connected by a revolute joint: C with D, F with G, I with J, and L with A. Write down explicit constraints on the coordinates for the eight points $A, \dots, H$ (assume that a fixed reference frame has been chosen, and denote the coordinates for point A by $p_A = (x_A, y_A, z_A)$, and similarly for the other points). Based on counting the number of variables and constraints, how many degrees of freedom does the configuration space have? If it differs from the result you obtained in (a), try to explain why.
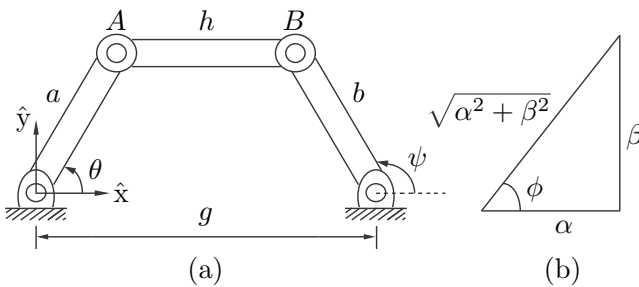

Figure 2.32 Planar four-bar linkage.

Exercise 2.25 In this exercise we examine in more detail the representation of the C-space for the planar four-bar linkage of Figure 2.32. Attach a fixed reference frame and label the joints and link lengths as shown in the figure. The (x, y) coordinates for joints A and B are given by

$$A(\theta) = (a \cos \theta, \ a \sin \theta),$$
$$B(\psi) = (g + b \cos \psi, \ b \sin \psi).$$

Using the fact that the link connecting A and B is of fixed length h, i.e., $\|A(\theta) - B(\psi)\|^2 = h^2$, we have the constraint

$$b^2 + g^2 + 2gb \cos \psi + a^2 - 2(a \cos \theta (g + b \cos \psi) + ab \sin \theta \sin \psi) = h^2.$$

Grouping the coefficients of $\cos \psi$ and $\sin \psi$, the above equation can be expressed in the form

$$\alpha(\theta) \cos \psi + \beta(\theta) \sin \psi = \gamma(\theta), \tag{2.11}$$

where

$$\alpha(\theta) = 2gb - 2ab \cos \theta,$$
$$\beta(\theta) = -2ab \sin \theta,$$
$$\gamma(\theta) = h^2 - g^2 - b^2 - a^2 + 2ag \cos \theta.$$

We now express ψ as a function of θ, by first dividing both sides of Equa-

tion (2.11) by $\sqrt{\alpha^2 + \beta^2}$ to obtain

$$\frac{\alpha}{\sqrt{\alpha^2 + \beta^2}}\cos\psi + \frac{\beta}{\sqrt{\alpha^2 + \beta^2}}\sin\psi = \frac{\gamma}{\sqrt{\alpha^2 + \beta^2}}. \tag{2.12}$$

Referring to Figure 2.32(b), the angle ϕ is given by $\phi = \tan^{-1}(\beta/\alpha)$, so that Equation (2.12) becomes

$$\cos(\psi - \phi) = \frac{\gamma}{\sqrt{\alpha^2 + \beta^2}}.$$

Therefore

$$\psi = \tan^{-1}\left(\frac{\beta}{\alpha}\right) \pm \cos^{-1}\left(\frac{\gamma}{\sqrt{\alpha^2 + \beta^2}}\right).$$

(a) Note that a solution exists only if $\gamma^2 \leq \alpha^2 + \beta^2$. What are the physical implications if this constraint is not satisfied?
(b) Note that, for each value of the input angle θ, there exist two possible values of the output angle ψ. What do these two solutions look like?
(c) Draw the configuration space of the mechanism in θ–ψ space for the following link length values: $a = b = g = h = 1$.
(d) Repeat (c) for the following link length values: $a = 1$, $b = 2$, $h = \sqrt{5}$, $g = 2$.
(e) Repeat (c) for the following link length values: $a = 1$, $b = 1$, $h = 1$, $g = \sqrt{3}$.

Exercise 2.26 The tip coordinates for the two-link planar 2R robot of Figure 2.33 are given by

$$x = 2\cos\theta_1 + \cos(\theta_1 + \theta_2)$$
$$y = 2\sin\theta_1 + \sin(\theta_1 + \theta_2).$$

(a) What is the robot's configuration space?
(b) What is the robot's workspace (i.e., the set of all points reachable by the tip)?
(c) Suppose infinitely long vertical barriers are placed at $x = 1$ and $x = -1$. What is the free C-space of the robot (i.e., the portion of the C-space that does not result in any collisions with the vertical barriers)?
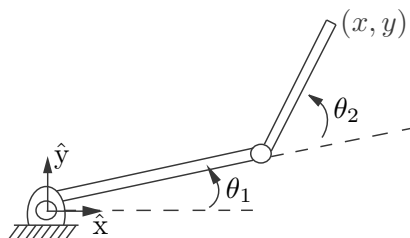
Figure 2.33 Two-link planar 2R open chain.

Exercise 2.27 The workspace of a planar 3R open chain.

(a) Consider a planar 3R open chain with link lengths (starting from the fixed base joint) 5, 2, and 1, respectively. Considering only the Cartesian point of the tip, draw its workspace.

(b) Now consider a planar 3R open chain with link lengths (starting from the fixed base joint) 1, 2, and 5, respectively. Considering only the Cartesian point of the tip, draw its workspace. Which of these two chains has a larger workspace?

(c) A not-so-clever designer claims that he can make the workspace of any planar open chain larger simply by increasing the length of the last link. Explain the fallacy behind this claim.

Exercise 2.28   Task space.

(a) Describe the task space for a robot arm writing on a blackboard.
(b) Describe the task space for a robot arm twirling a baton.

Exercise 2.29   Give a mathematical description of the topologies of the C-spaces of the following systems. Use cross products, as appropriate, of spaces such as a closed interval $[a, b]$ of a line and $\mathbb{R}^k$, S^m, and T^n, where k, m, and n are chosen appropriately.

(a) The chassis of a car-like mobile robot rolling on an infinite plane.
(b) The car-like mobile robot (chassis only) driving around on a spherical asteroid.
(c) The car-like mobile robot (chassis only) on an infinite plane with an RRPR robot arm mounted on it. The prismatic joint has joint limits, but the revolute joints do not.
(d) A free-flying spacecraft with a 6R arm mounted on it and no joint limits.

Exercise 2.30   Describe an algorithm that drives the rolling coin of Figure 2.11 from any arbitrary initial configuration in its four-dimensional C-space to any arbitrary goal configuration, despite the two nonholonomic constraints. The control inputs are the rolling speed $\dot{\theta}$ and the turning speed $\dot{\phi}$. You should explain clearly in words or pseudocode how the algorithm would work. It is not necessary to give actual code or formulas.

Exercise 2.31   A differential-drive mobile robot has two wheels that do not steer but whose speeds can be controlled independently. The robot goes forward and backward by spinning the wheels in the same direction at the same speed, and it turns by spinning the wheels at different speeds. The configuration of the robot is given by five variables: the (x, y) location of the point halfway between the wheels, the heading direction θ of the robot's chassis relative to the x-axis of the world frame, and the rotation angles ϕ_1 and ϕ_2 of the two wheels about the axis through the centers of the wheels (Figure 2.34). Assume that the radius of each wheel is r and the distance between the wheels is $2d$.
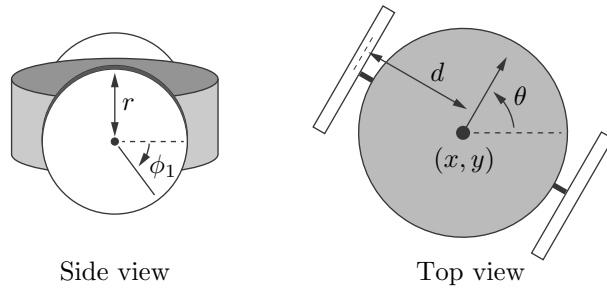
Figure 2.34 A side view and a top view of a differential-drive robot.

(a) Let $q = (x, y, \theta, \phi_1, \phi_2)$ be the configuration of the robot. If the two control inputs are the angular velocities of the wheels $\omega_1 = \dot{\phi}_1$ and $\omega_2 = \dot{\phi}_2$, write down the vector differential equation $\dot{q} = g_1(q)\omega_1 + g_2(q)\omega_2$. The vector fields $g_1(q)$ and $g_2(q)$ are called *control vector fields* (see Section 13.3) and express how the system moves when the respective unit control signal is applied.

(b) Write the corresponding Pfaffian constraints $A(q)\dot{q} = 0$ for this system. How many Pfaffian constraints are there?

(c) Are the constraints holonomic or nonholonomic? Or how many are holonomic and how many nonholonomic?

Exercise 2.32   Determine whether the following differential constraints are holonomic or nonholonomic:

(a) $(1 + \cos q_1)\dot{q}_1 + (1 + \cos q_2)\dot{q}_2 + (\cos q_1 + \cos q_2 + 4)\dot{q}_3 = 0.$

(b)
$$
\begin{aligned}
-\dot{q}_1 \cos q_2 + \dot{q}_3 \sin(q_1 + q_2) - \dot{q}_4 \cos(q_1 + q_2) &= 0 \\
\dot{q}_3 \sin q_1 - \dot{q}_4 \cos q_1 &= 0.
\end{aligned}
$$