1
Hamiltonian Formalism

A major role in the development of the theory of classical dynamical systems is played by the Hamiltonian formulation of the equations of dynamics. This chapter is intended to provide a basic knowledge of the Hamiltonian formalism, assuming that the Lagrangian formalism is known. A reader already familiar with the canonical formalism may want to skip the present chapter.

The canonical equations were first written by Giuseppe Luigi Lagrangia (best known by the French version of his name, Joseph Louis Lagrange) as the last improvement of his theory of secular motions of the planets [140]. The complete form, later developed in what we now call Hamiltonian formalism, is due to William Rowan Hamilton [106][107][108]. A short sketch concerning the anticipations of Hamilton's work can be found in the treatise of Edmund Taylor Whittaker [209], §109.

In view of the didactical purpose of the present notes, the exposition in this chapter follows the traditional lines. The chapter includes some basic tools: the algebra of Poisson brackets and the elementary integration methods. Many examples are also included in order to illustrate how to write the Hamiltonian function for some models, often investigated using Newton's or Lagrange's equations.

1.1 Phase Space and Hamilton's Equations

The dynamical state of a system with n degrees of freedom is identified with a point on a $2n$-dimensional differentiable manifold, denoted by $\mathscr{F}$, endowed with canonically conjugated coordinates $(q, p) \equiv (q_1, \ldots, q_n, p_1, \ldots, p_n)$. The object of investigation is the evolution of the state of the system. The manifold $\mathscr{F}$ was named a *phase space* by Josiah Willard Gibbs [75].

The evolution of the system is determined by a real-valued Hamiltonian function $H \colon \mathscr{F} \to \mathbb{R}$ through the vector field defined by *Hamilton's equations* (also called *canonical equations*),

$$(1.1) \qquad \dot{q}_j = \frac{\partial H}{\partial p_j} \ , \quad \dot{p}_j = -\frac{\partial H}{\partial q_j} \ , \quad j = 1, \ldots, n \ ,$$

where $H = H(q, p)$. The Hamiltonian will be assumed to be a smooth (differentiable) function. In most examples the cases of $C^\infty(\mathscr{F}, \mathbb{R})$ or even $C^\omega(\mathscr{F}, \mathbb{R})$ Hamiltonians will be considered.

An *orbit* of the system is a smooth curve $\big(q(t), p(t)\big)$, for t in some (possibly infinite) interval, which is a solution of the canonical equations (1.1); the initial condition is written $q(0) = q_0$, $p(0) = p_0$. As a general fact, the solutions of a system of differential equations may be interpreted as a flow in phase space which transports every point (q_0, p_0) to another point $\big(q(t), p(t)\big) = \phi^t(q_0, p_0)$ after a time interval t, thus representing in some sense the dynamics in phase space as the motion of a fluid; the symbol ϕ^t represents the action of the flow. The corresponding orbit is thus represented as $\Omega(q_0, p_0) = \bigcup_t \phi^t(q_0, p_0)$, the union being made over the (possibly infinite) time interval of existence of the solution.

1.1.1 Autonomous versus Non-autonomous Systems

In most treatises the general case of a time-dependent Hamiltonian $H(q, p, t)$ is considered. It is customary to call such a system *non-autonomous*, in contrast with the time-independent case that is called *autonomous*.[1]

As a matter of fact, the non-autonomous case can be reduced to the autonomous one by a standard technique, already used by Henri Poincaré ([190], vol. I, § 12). The suggestion is to introduce the *extended phase space* by adding one more pair of canonically conjugated coordinates q_+, p_+, thus increasing the dimension of the phase space by 2; the special role played by the new variables will be emphasized by the special notation (q, q_+, p, p_+), although all canonical pairs should be considered on the same footing. Having given the Hamiltonian $H(q, p, t)$, let us introduce a new Hamiltonian

$$(1.2) \qquad \tilde{H}\big(q, q_+, p, p_+\big) = H(q, p, q_+) + p_+ \ .$$

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[1] The non-autonomous case typically arises when one considers a small system subjected to the time-dependent action of some external device, the state of which is not affected by the small part one is interested in. Examples are the restricted problem of three bodies, where the motion of a small body (e.g., an asteroid or a spacecraft) under the action of two big bodies moving on Keplerian orbits (e.g., Sun–Jupiter or Earth–Moon) is investigated; an electric charge acted on by an electromagnetic wave; a particle in an accelerator; &c. The autonomous case arises when an isolated system is considered.

Thus, one has to consider the extended set of canonical equations

$$
(1.3) \qquad
\begin{aligned}
\dot{q}_j &= \frac{\partial H}{\partial p_j} \;, \quad \dot{p}_j = -\frac{\partial H}{\partial q_j} \;, \quad j = 1, \ldots, n \;, \\
\dot{q}_+ &= 1 \quad , \quad \dot{p}_+ = -\frac{\partial H}{\partial q_+} \;.
\end{aligned}
$$

The third equation has the trivial solution $q_+(t) = t - t_0$, where t_0 is the initial time. In view of this, and having fixed the initial time t_0, the first two equations actually coincide with (1.1). Suppose now that we know a solution $q(t), p(t)$ of (1.1); if we replace it in the equation for $\dot{p}_+$, the r.h.s. turns out to be a known function of t only, so that the equation can be solved by a quadrature.[2] Therefore, the autonomous system (1.1) and the non-autonomous one (1.3) are fully equivalent.

Extending the phase space in order to transform a non-autonomous system into an autonomous one may appear to be an unnecessary complication: we have one more variable to take care of, although in a straightforward manner. However, this will offer us the opportunity of developing most of the theory in the framework of the autonomous systems, which makes the exposition definitely simpler.

1.1.2   *Connection with the Lagrangian Formalism*

It is traditional to introduce the Hamiltonian formalism starting from the equations of Lagrange.[3] The Hamiltonian function is introduced as the Legendre transform of the Lagrangian. One considers an n-dimensional differentiable manifold (the *configuration space*) endowed with (local) co-ordinates $q_1, \ldots, q_n$, and its tangent space described by the generalized components $\dot{q}_1, \ldots, \dot{q}_n$ of the velocities. The dynamical state of the system

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[2] Actually no quadrature is needed, for an autonomous Hamiltonian $H(q, p)$ is a first integral, i.e., along the orbit $q(t), p(t)$ one has $H\big(q(t), p(t)\big) = E$, where $E = H\big(q(0), p(0)\big)$ is the initial value – in most mechanical models it is the energy of the system. Since the Hamiltonian $\tilde{H}(q, q_+, p, p_+)$ in (1.2) is seen as an autonomous one we get the first integral

$$
\tilde{H}(q(t), t - t_0, p(t), p_+(t)) = H(q(t), p(t), t) + p_+(t) = C \;.
$$

The dynamics does not depend on the value of the constant C; hence we may well restrict the study to the manifold $\tilde{H} = 0$; this gives $p_+(t) = -H(q(t), p(t), t)$. Thus, the canonical momentum p_+ conjugated to the time t is identified with the energy E, which depends on time.

[3] For a deduction of the Lagrangian form of the equations of a mechanical system from Newton's equations see, for instance, [209], § 26. The original formulation can be found in Lagrange's treatise [139] and [143]. For a deduction of Hamilton's equations see [209], § 109 or [144], ch. VI.

at a given time t is completely determined by the knowledge of $q(t), \dot{q}(t)$. The dynamics is determined by the Lagrangian function $L(q, \dot{q}, t)$ through the n differential equations of second order

(1.4) $$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0 , \quad 1 \le j \le n .$$

In order to give the equations the Hamiltonian form, one introduces the momenta $p_1, \ldots, p_n$ conjugated to $q_1, \ldots, q_n$ defined as

(1.5) $$p_j = \frac{\partial L}{\partial \dot{q}_j} , \quad 1 \le j \le n ;$$

thus, the momenta are given as functions of q, $\dot{q}$ and t. The latter equations together with

(1.6) $$\dot{p}_j = \frac{\partial L}{\partial q_j} , \quad 1 \le j \le n$$

are equivalent to the equations of Lagrange (1.4). If the condition

$$\det\left(\frac{\partial^2 L}{\partial \dot{q}_j \partial \dot{q}_k}\right) \ne 0$$

is fulfilled, then (1.5) can be solved with respect to $\dot{q}_1, \ldots, \dot{q}_n$, thus giving $\dot{q}_j = \dot{q}_j(q, p, t)$, and the momenta can be used in place of the velocities $\dot{q}$ in order to determine the dynamical state.

The Hamiltonian function is defined as

(1.7) $$H(q, p, t) = \sum_{j=1}^{n} p_j \dot{q}_j - L(q, \dot{q}, t)\Big|_{\dot{q}=\dot{q}(q,p,t)} ,$$

where $\dot{q}$ must be replaced everywhere with its expression as a function of q, p, t. With the latter function the dynamics is determined by the canonical equations (1.1). The latter claim is proved as follows. Differentiate the function H as depending on the $3n+1$ variables q, $\dot{q}$, p and t, and then substitute the functions $\dot{q}(q, p, t)$, thus getting

$$dH = \sum_{j=1}^{n}\left(\dot{q}_j \, dp_j + p_j \, d\dot{q}_j - \frac{\partial L}{\partial \dot{q}_j} \, d\dot{q}_j - \frac{\partial L}{\partial q_j} \, dq_j\right) - \frac{\partial L}{\partial t} \, dt$$

$$= \sum_{j=1}^{n}\left(\dot{q}_j \, dp_j - \frac{\partial L}{\partial q_j} \, dq_j\right) - \frac{\partial L}{\partial t} \, dt .$$

Here the definition, (1.5), of the momenta p has been used. The latter expression must be compared with the differential of H as a function of q, p, t, namely

$$dH = \sum_{j=1}^{n} \left(\frac{\partial H}{\partial p_j} \, dp_j + \frac{\partial H}{\partial q_j} \, dq_j \right) + \frac{\partial H}{\partial t} \, dt \ .$$

By comparison of coefficients, also recalling (1.6), we get

$$\frac{\partial H}{\partial p_j} = \dot{q}_j \ , \quad \frac{\partial H}{\partial q_j} = -\frac{\partial L}{\partial q_j} = -\dot{p}_j \ , \quad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \ ,$$

namely the equations of Hamilton plus a relation between the time derivatives of H and L.

Example 1.1: *Free particle.* Let a point of mass m move in space under no forces. Using rectangular coordinates, the configuration space is identified with $\mathbb{R}^3$. The Lagrangian function coincides with the kinetic energy and reads

$$L = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \ .$$

The momenta conjugated to the coordinates are $p_x = m\dot{x}, p_y = m\dot{y}, p_z = m\dot{z}$; they coincide with the components of the momentum of the particle. The phase space in this case is naturally identified with $\mathbb{R}^6$, and the Hamiltonian coincides with the kinetic energy, being

$$(1.8) \qquad H = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) \ .$$

If the particle is acted on by a force depending on a potential $V(x, y, z)$, then the Lagrangian and the Hamiltonian functions are, respectively,

$$(1.9) \qquad \begin{aligned} L &= \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(x, y, z) \ , \\ H &= \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + V(x, y, z) \ . \end{aligned}$$

In particular, the Hamiltonian represents the energy of the particle, the kinetic energy having been expressed in terms of the momenta. *E.D.*

Example 1.2: *One-dimensional harmonic oscillator.* Let now a point of mass m move on a straight line under the action of a perfectly elastic spring. The configuration space is $\mathbb{R}$, and the Lagrangian function is

$$L = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2 \ .$$

Introducing the *angular frequency* $\omega = \sqrt{k/m}$, the Lagrangian changes to the following:[4]

$$L = \frac{1}{2} \dot{x}^2 - \frac{1}{2} \omega^2 x^2 \ .$$

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[4] Recall that if the Lagrangian function is multiplied by an arbitrary non-zero factor, the equations of Lagrange do not change.

The momentum conjugated to x is $p = \dot{x}$, so that the phase space is naturally identified with $\mathbb{R}^2$. The Hamiltonian is again the energy and is written as

$$\text{(1.10)} \qquad\qquad H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 x^2 \ . \qquad\qquad \text{E.D.}$$

Example 1.3: *The pendulum.* A mass point is moving without friction on a vertical circle, so that it is subjected to gravity. The configuration space can be chosen to be a one-dimensional torus $\mathbb{T} = \mathbb{R}/(2\pi\,\mathbb{Z})$, namely the real line where the points x and $x + 2k\pi$ ($k \in \mathbb{Z}$) are identified. The resulting angular coordinate will be denoted by ϑ. The Lagrangian function may be written as

$$L = \frac{1}{2}\dot{\vartheta}^2 + \frac{g}{l}\cos\vartheta \ ,$$

where l is the length of the pendulum and g the constant acceleration due to local gravity. The momentum conjugated to ϑ is $p = \dot{\vartheta} \in \mathbb{R}$, and the phase space is naturally identified with $\mathbb{T} \times \mathbb{R}$. The Hamiltonian is

$$\text{(1.11)} \qquad\qquad H = \frac{1}{2}p^2 - \frac{g}{l}\cos\vartheta \ . \qquad\qquad \text{E.D.}$$

Example 1.4: *Motion under central forces.* The problem is to investigate the motion of a particle P of mass m acted on by a force in the direction PO, where O is a fixed point. Let us assume that the force field is spherically symmetric, so that there exists a potential $V(r)$, where r is the distance of P from the fixed point O. In view of the symmetry of the problem it is convenient to use spherical coordinates r, ϑ, φ, related to the rectangular coordinates x, y, z by

$$x = r\sin\vartheta\cos\varphi \ , \quad y = r\sin\vartheta\sin\varphi \ , \quad z = r\cos\vartheta \ .$$

Thus, the configuration space is $\mathbb{R}^+ \times (0, \pi) \times \mathbb{T}$ and the Lagrangian function reads

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\vartheta}^2 + r^2\dot{\varphi}^2\sin^2\vartheta\right) - V(r) \ .$$

The momenta conjugated to the coordinates are $p_r = m\dot{r}$, $p_\vartheta = mr^2\dot{\vartheta}$ and $p_\varphi = mr^2\dot{\varphi}\sin^2\vartheta$. The Hamiltonian is

$$\text{(1.12)} \qquad\qquad H = \frac{1}{2m}\left(p_r^2 + \frac{p_\vartheta^2}{r^2} + \frac{p_\varphi^2}{r^2\sin^2\vartheta}\right) + V(r) \ ,$$

on the phase space $\mathbb{R}^+ \times (0, \pi) \times \mathbb{T} \times \mathbb{R}^3$. $\qquad\qquad$ E.D.

Exercise 1.5: In galactic dynamics one often assumes the galaxy to be composed of a central nucleus with a large number of stars, and many other stars distributed in some way close to a plane orthogonal to the axis of rotation of the galaxy. The simplest approach consists in assuming that the distribution of the stars possesses an axial symmetry. Thus, one is led to

study the motion of a single star close to the galactic plane, far from the nucleus, as subjected to an average potential $V(r, z)$, independent of ϑ in view of the axial symmetry. Show that in cylindrical coordinates r, ϑ, z with

$$x = r \cos \vartheta \ , \quad y = r \sin \vartheta$$

the Hamiltonian reads

$$H = \frac{1}{2} \left(p_r^2 + \frac{p_\vartheta^2}{r^2} + p_z^2 \right) + V(r, z) \ . \qquad \text{A.E.L.}$$

Example 1.6: *The problem of n bodies.* The problem is to investigate the motion of a system of n particles in space, with a two-body interaction due to some potential. Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ be the positions of the n bodies in a rectangular frame, so that the configuration space can be identified with $\mathbb{R}^{3n}$, and let $m_1, \ldots, m_n$ be the masses of the bodies. Then the Lagrangian function reads

$$L = \frac{1}{2} \sum_{j=1}^{n} m_j \dot{\mathbf{x}}_j^2 - V(\mathbf{x}_1, \ldots, \mathbf{x}_n) \ ,$$

where $V(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ is the potential energy. The momenta conjugated to the coordinates are $\mathbf{p}_j = m_j \dot{\mathbf{x}}_j$, $1 \le j \le n$. Therefore, the phase space can be identified with $\mathbb{R}^{6n}$, and the Hamiltonian reads

$$(1.13) \qquad H = \sum_{j=1}^{n} \frac{\mathbf{p}_j^2}{2m_j} + V(\mathbf{x}_1, \ldots, \mathbf{x}_n) \ .$$

The cases $n = 2$ and $n = 3$ are particularly interesting. The former one, the *problem of two bodies*, is the most natural approximation for the motion of a planet revolving around the Sun when the law of gravitation is applied in its correct form (i.e., the gravitational action of the planet on the Sun is taken into account). The latter one is the celebrated *problem of three bodies*, which was at the origin of the discovery of chaos by Poincaré. *E.D.*

Example 1.7: *Particle in a rotating frame.* We consider again the problem of a particle of mass m moving in space, acted on by some potential, and investigate its motion with respect to a frame uniformly rotating around a fixed axis. Let ξ, η, ζ be the coordinates of the particle with respect to a fixed rectangular frame, with the ζ axis coinciding with the fixed axis of the rotating frame. Let now x, y, z be rectangular coordinates in the rotating frame, chosen so that the two frames have a common origin, and the axes ζ and z coincide. Therefore, the two sets of coordinates are related via

$$\xi = x \cos \omega t - y \sin \omega t \ , \quad \eta = x \sin \omega t + y \cos \omega t \ , \quad \zeta = z \ ,$$

ω being the angular velocity of the rotating frame with respect to the fixed one. By substitution of the latter transformation in the Lagrangian function for a point in a fixed frame (see Example 1.1) one has

$$L = \frac{1}{2}m\left[(\dot{x} - \omega y)^2 + (\dot{y} + \omega x)^2 + \dot{z}^2\right] - V(x, y, z, t) ,$$

$V(x, y, z, t)$ being the potential energy, which may depend on time due to the change of coordinates. The momenta conjugated to the coordinates are

$$p_x = m(\dot{x} - \omega y) , \quad p_y = m(\dot{y} + \omega x) , \quad p_z = m\dot{z} .$$

The Hamiltonian reads

(1.14) $$H = \frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right) + \omega\left(yp_x - xp_y\right) + V(x, y, z, t) .$$ *E.D.*

Example 1.8: *The circular restricted problem of three bodies.* A remarkable example, which plays a fundamental role in Celestial Mechanics, is the so-called *restricted problem of three bodies*: the particle (named the *planetoid*) is attracted by two much bigger masses (the *primaries*, e.g., Sun–Jupiter or Earth–Moon) moving on Keplerian orbits without being affected by the planetoid. This model is a fundamental one when dealing with the dynamics of small bodies (e.g., inner planets, asteroids or spacecraft). The simplest approach is to assume that the primaries revolve uniformly on a circular orbit around their common centre of mass. It is also common to choose the units as follows: (i) the unit of length is the distance between the primaries; (ii) the period of the primaries is $T = 2\pi$; (iii) the masses of the primaries are set to be μ and $1 - \mu$ with $0 < \mu < 1$, so that the total mass coincides with the mass unit. By the way, in these units the gravitational constant G takes the value $G = 1$. The rotating system is chosen so that the origin coincides with the barycentre and the primaries are in fixed positions $1 - \mu$ and μ on the x axis. Therefore the potential turns out to be independent of t, since

$$V(x, y, z) = -\frac{m(1 - \mu)}{\sqrt{(x - \mu)^2 + y^2 + z^2}} - \frac{m\mu}{\sqrt{(x + 1 - \mu)^2 + y^2 + z^2}} .$$

We are also allowed to remove the factor m from the Lagrangian, which is tantamount to substituting $m = 1$. Therefore, the Hamiltonian reads

$$H(x, y, z, p_x, p_y, p_z) = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) - \omega x p_y + \omega y p_x$$
$$- \frac{1 - \mu}{\sqrt{(x - \mu)^2 + y^2 + z^2}} - \frac{\mu}{\sqrt{(x + 1 - \mu)^2 + y^2 + z^2}} .$$
E.D.

Example 1.9: *Forced oscillations.* A simple but remarkable example in Physics is a harmonic oscillator acted on by a (small) periodic forcing. It is usual to write immediately the equation as, for example,

(1.15) $$\ddot{x} + \omega^2 x = \varepsilon \cos \nu t ,$$

where ω is the proper frequency of the oscillator and ν is the frequency of the forcing term, while ε is a real parameter. The equation may be derived from the Lagrangian function

$$L = \frac{1}{2}\dot{x}^2 - \frac{\omega^2}{2}x^2 + \varepsilon x \cos \nu t .$$

The Hamiltonian is easily found to be

$$H(x, p, t) = \frac{1}{2}(p^2 + \omega^2 x^2) - \varepsilon x \cos \nu t ,$$

where $p = \dot{x}$ is the momentum. The reader will notice that the Hamiltonian has the generic form $H = T + V(x, t)$, where $T = \dot{x}^2/2$ is the kinetic energy and $V(x, t)$ is the potential energy. An immediate generalization is to replace the forcing term with a more general function $xf(t)$, where $f(t)$ is time-periodic (i.e., $f(t + T) = f(t)$ for all t and some fixed T, e.g., $T = 2\pi/\nu$ in the preceding case). One may also add multiple periods or even include a non-periodic dependence on time, when useful. *E.D.*

Example 1.10: *The forced pendulum.* A definitely more difficult problem arises when one considers a forced *nonlinear* oscillator. A widely studied model is the forced pendulum, as described by the equation (denoting again by ϑ the coordinate)

$$(1.16) \qquad \ddot{\vartheta} + \sin \vartheta = \varepsilon \cos \nu t ,$$

where the forcing term may be replaced by a generic periodic function. The equation may be derived from the Lagrangian function

$$L = \frac{1}{2}\dot{\vartheta}^2 + \cos \vartheta + \varepsilon \vartheta \cos \nu t .$$

The Hamiltonian is easily found to be

$$H(\vartheta, p, t) = \frac{1}{2}p^2 - \cos \vartheta - \varepsilon \vartheta \cos \nu t .$$

A slightly different form of the Hamiltonian that is often used is

$$H_\varepsilon(\vartheta, p) = \frac{p^2}{2} - \cos \vartheta - \varepsilon \cos(\vartheta - t) , \quad \vartheta \in \mathbb{T} , \ p \in \mathbb{R} . \qquad \text{*E.D.*}$$

Example 1.11: *Autonomous mechanical system.* The general form of the Lagrangian function for an autonomous mechanical system may be written as

$$L(q, \dot{q}) = T(q, \dot{q}) - V(q) , \quad T(q, \dot{q}) = \frac{1}{2}\sum_{j,k=1}^{n} g_{j,k}(q)\dot{q}_j\dot{q}_k ,$$

where $T(q, \dot{q})$ is the kinetic energy in a general coordinate system, defined through a symmetric non-degenerate matrix $g(q)$ with elements $g_{j,k}(q)$, and

$V(q)$ is the potential energy, independent of the velocities. The relation be-
tween generalized velocities and momenta is

$$p_j = \sum_{k=1}^{n} g_{jk}(q)\,\dot{q}_k \ , \quad \dot{q}_k = \sum_{j=1}^{n} [g^{-1}]_{kj}(q) p_j \ ,$$

where we have denoted by $[g^{-1}]_{j,k}$ the elements of the inverse matrix $g^{-1}(q)$.
Thus we get

$$(1.17) \qquad\qquad H = \frac{1}{2} \sum_{j,k=1}^{n} [g^{-1}]_{j,k} p_j p_k + V(q) \ .$$

<div align="right">*E.D.*</div>

1.1.3   *Compact Notation for the Canonical Equations*

The canonical equations may be given a more compact form by introducing
the $2n$ column vector $\mathbf{z}$ and the $2n \times 2n$ skew symmetric matrix J as

$$(1.18) \qquad \mathbf{z} = (q_1, \ldots, q_n,\, p_1, \ldots, p_n)^\top \ , \quad \mathsf{J} = \begin{pmatrix} 0 & \mathsf{I}_n \\ -\mathsf{I}_n & 0 \end{pmatrix} \ ,$$

where I_n is the $n \times n$ identity matrix. Remark that $\mathsf{J}^2 = -\mathsf{I}_{2n}$. We shall also
introduce the operator

$$(1.19) \qquad\qquad \partial_z = \left(\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_{2n}} \right)^\top \ .$$

With these notations the canonical equations take the compact form

$$(1.20) \qquad\qquad \dot{\mathbf{z}} = \mathsf{J} \partial_z H \ .$$

The compact notation used here turns out to be useful in a few cases, since
it allows us to write shorter expressions. Although most of the present notes
are written in the traditional language, in some cases the compact notation
will be adopted in order to simplify some calculations.

1.2   **Dynamical Variables and First Integrals**

A *dynamical variable* is a differentiable real function $f : \mathscr{F} \to \mathbb{R}$ with do-
main on the phase space (e.g., the kinetic energy, the potential energy and
the Hamiltonian itself are dynamical variables, and so are the coordinates
themselves). If the canonical coordinates evolve in time as $\big(q(t), p(t)\big)$, so does
the dynamical variable $f\big(q(t), p(t)\big)$. In particular, if the flow $\big(q(t), p(t)\big)$ is
determined by the canonical equations, (1.1), then the time evolution of
$f(p, q)$ obeys

$$\dot{f} = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial q_j} \frac{\partial H}{\partial p_j} - \frac{\partial f}{\partial p_j} \frac{\partial H}{\partial q_j} \right) .$$

This is the time derivative of f along the Hamiltonian flow induced by H, called the *Lie derivative* of f and denoted by $L_H f$. In canonical coordinates the operator L_H representing the Lie derivative takes the form

$$(1.21) \qquad L_H := \sum_{j=1}^{n} \left(\frac{\partial H}{\partial p_j} \frac{\partial}{\partial q_j} - \frac{\partial H}{\partial q_j} \frac{\partial}{\partial p_j} \right) .$$

1.2.1   The Algebra of Poisson Brackets

Let $f(q,p)$ and $g(q,p)$ be differentiable dynamical variables; then the *Poisson bracket* is defined as

$$(1.22) \qquad \{g, f\} = \sum_{j=1}^{n} \left(\frac{\partial g}{\partial q_j} \frac{\partial f}{\partial p_j} - \frac{\partial g}{\partial p_j} \frac{\partial f}{\partial q_j} \right) .$$

It is immediately seen that the latter expression is the Lie derivative of the function g along the Hamiltonian field generated by f. Thus, it is natural to associate to any function f the *Lie derivative* $L_f \cdot = \{\cdot, f\}$, defined in coordinates as in (1.21).

The operation of the Poisson brackets satisfies some relevant properties which may be stated either in traditional notations or in terms of the Lie derivative. It is also useful to introduce the notation $[\cdot, \cdot]$ for the commutator between two operators; for example, $[L_f, L_g] = L_f L_g - L_g L_f$.

Proposition 1.12:   *If f, g and h are differentiable dynamical variables and α is a real constant, we have:*
   (i)   *linearity, $\{f, g+h\} = \{f, g\} + \{f, h\}$, $\{f, \alpha g\} = \alpha \{f, g\}$;*
   (ii)   *anticommutativity, $\{f, g\} = - \{g, f\}$;*
  (iii)   *Jacobi's identity,*

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0 .$$

In terms of Lie derivatives the same properties are written as

$$(1.23) \qquad L_f(g+h) = L_f g + L_f h , \quad L_f(\alpha g) = \alpha L_f g ,$$
$$(1.24) \qquad L_f g = -L_g f ,$$
$$(1.25) \qquad [L_f, L_g] = L_{L_f g} = L_{\{g, f\}} .$$

Jacobi's identity (1.25) deserves particular attention. The left member is a composition of two differential operators; hence it is natural to imagine that it is a second-order operator. This is untrue, in fact. The second member

is a first-order operator, with a remarkable interpretation: the commutation between the Hamiltonian vector fields generated by the functions f and g is the Hamiltonian vector field generated by $\{g, f\}$.

Corollary 1.13:   *The following additional properties apply*

(iv) distributivity over the product: $\{f, gh\} = g\{f, h\} + \{f, g\} h$;

(v) Leibniz's rule: for $1 \leq j \leq n$

$$\frac{\partial}{\partial q_j}\{f, g\} = \left\{\frac{\partial f}{\partial q_j}, g\right\} + \left\{f, \frac{\partial g}{\partial q_j}\right\} , \quad \frac{\partial}{\partial p_j}\{f, g\} = \left\{\frac{\partial f}{\partial p_j}, g\right\} + \left\{f, \frac{\partial g}{\partial p_j}\right\} .$$

In terms of Lie derivatives the same properties are written as

$$(1.26) \qquad L_f(gh) = g L_f h + h L_f g ,$$

$$(1.27) \qquad \left[\frac{\partial}{\partial q_j}, L_f\right] = L_{\frac{\partial f}{\partial q_j}} , \quad \left[\frac{\partial}{\partial p_j}, L_f\right] = L_{\frac{\partial f}{\partial p_j}} , \quad 1 \leq j \leq n .$$

Proof of Proposition 1.12 and of Corollary 1.13.   We leave to the reader the easy check that the properties (1.23) to (1.27) are just a rewriting of the corresponding properties (i)–(v). Also, with the exception of Jacobi's identity (iii) or equivalently (1.25), the proof of the other identities is an easy matter and is left to the reader. So let us prove (1.25). Begin by calculating

$$\begin{aligned}
L_f L_g &= \sum_{j,k} \left(\frac{\partial f}{\partial p_j}\frac{\partial}{\partial q_j} - \frac{\partial f}{\partial q_j}\frac{\partial}{\partial p_j}\right)\left(\frac{\partial g}{\partial p_k}\frac{\partial}{\partial q_k} - \frac{\partial g}{\partial q_k}\frac{\partial}{\partial p_k}\right) \\
&= \sum_{j,k}\left[\frac{\partial f}{\partial p_j}\left(\frac{\partial^2 g}{\partial q_j \partial p_k}\frac{\partial}{\partial q_k} - \frac{\partial^2 g}{\partial q_j \partial q_k}\frac{\partial}{\partial p_k}\right)\right] \\
&\quad - \sum_{j,k}\left[\frac{\partial f}{\partial q_j}\left(\frac{\partial^2 g}{\partial p_j \partial p_k}\frac{\partial}{\partial q_k} - \frac{\partial^2 g}{\partial p_j \partial q_k}\frac{\partial}{\partial p_k}\right)\right] \\
&\quad + \sum_{j,k}\left[\frac{\partial f}{\partial p_j}\left(\frac{\partial g}{\partial p_k}\frac{\partial^2}{\partial q_j \partial q_k} - \frac{\partial g}{\partial q_k}\frac{\partial^2}{\partial q_j \partial p_k}\right)\right] \\
&\quad - \sum_{j,k}\left[\frac{\partial f}{\partial q_j}\left(\frac{\partial g}{\partial p_k}\frac{\partial^2}{\partial p_j \partial q_k} - \frac{\partial g}{\partial p_k}\frac{\partial^2}{\partial p_j \partial p_k}\right)\right] .
\end{aligned}$$

The similar calculation for $L_g L_f$ consists merely in exchanging the symbols f and g in the previous expression. Hence when subtracting $L_f L_g - L_g L_f$, all terms that generate a second-order differentiation (namely the two last lines in the formula, with an exchange of the summation indices j, k where useful) compensate each other, and we are left with

$$L_f L_g - L_g L_f = \sum_{j,k} \left[\frac{\partial f}{\partial p_j} \left(\frac{\partial^2 g}{\partial q_j \partial p_k} \frac{\partial}{\partial q_k} - \frac{\partial^2 g}{\partial q_j \partial q_k} \frac{\partial}{\partial p_k} \right) \right]$$

$$- \sum_{j,k} \left[\frac{\partial f}{\partial q_j} \left(\frac{\partial^2 g}{\partial p_j \partial p_k} \frac{\partial}{\partial q_k} - \frac{\partial^2 g}{\partial p_j \partial q_k} \frac{\partial}{\partial p_k} \right) \right]$$

$$- \sum_{j,k} \left[\frac{\partial g}{\partial p_j} \left(\frac{\partial^2 f}{\partial q_j \partial p_k} \frac{\partial}{\partial q_k} + \frac{\partial^2 f}{\partial q_j \partial q_k} \frac{\partial}{\partial p_k} \right) \right]$$

$$+ \sum_{j,k} \left[\frac{\partial g}{\partial q_j} \left(\frac{\partial^2 f}{\partial p_j \partial p_k} \frac{\partial}{\partial q_k} - \frac{\partial^2 f}{\partial p_j \partial q_k} \frac{\partial}{\partial p_k} \right) \right] .$$

Letting the index k be fixed and collecting the coefficients of the operator $\frac{\partial}{\partial q_k}$, we get

$$\sum_j \left(\frac{\partial f}{\partial p_j} \frac{\partial}{\partial q_j} - \frac{\partial f}{\partial q_j} \frac{\partial}{\partial p_j} \right) \frac{\partial g}{\partial p_k} - \left(\frac{\partial g}{\partial p_j} \frac{\partial}{\partial q_j} - \frac{\partial g}{\partial q_j} \frac{\partial}{\partial p_j} \right) \frac{\partial f}{\partial p_k}$$

$$= \left\{ \frac{\partial g}{\partial p_k}, f \right\} + \left\{ g, \frac{\partial f}{\partial p_k} \right\} = \frac{\partial}{\partial p_k} \{g, f\} .$$

Here Leibniz's rule (iii) has been used. Similarly, the coefficients of the operator $\frac{\partial}{\partial p_k}$ give $-\frac{\partial}{\partial q_k} \{g, f\}$. Thus, restoring the sum over k, we get

$$L_f L_g - L_g L_f = \frac{\partial}{\partial p_k} \{g, f\} \frac{\partial}{\partial q_k} - \frac{\partial}{\partial q_k} \{g, f\} \frac{\partial}{\partial p_k} = L_{\{g,f\}} ,$$

namely Jacobi's identity in the form (1.25). *Q.E.D.*

As we have seen, in terms of Poisson brackets (or Lie derivatives) the time evolution of a differentiable dynamical variable $f(q, p)$ satisfies the partial differential equation $\dot f = \{f, H\}$, or equivalently $\dot f = L_H f$. As a consequence, the Hamiltonian dynamics can be expressed in terms of Poisson brackets (see [129]). In particular, the canonical equations may be written in one of the following more symmetric forms:

$$(1.28) \qquad \dot q_j = \{q_j, H\} , \quad \dot p_j = \{p_j, H\} , \quad 1 \le j \le n ,$$

or

$$(1.29) \qquad \dot q_j = L_H q_j , \quad \dot p_j = L_H p_j , \quad 1 \le j \le n .$$

This exploits the fact that the coordinates themselves can be considered as dynamical variables.

1.2.2   First Integrals

A dynamical variable $\Phi(p,q)$ is said to be a *first integral* if it keeps its value constant under the canonical flow generated by $H(p,q)$, namely by Eq. (1.1). The name *constant of motion* is often used. If $\Phi(p,q)$ is differentiable, then one has $\dot\Phi = 0$ under the flow. Thus, a differentiable first integral $\Phi(q,p)$ must satisfy the partial differential equation

$$(1.30) \qquad\qquad\qquad L_H \Phi = 0 \ .$$

From the properties of the Lie derivative the following results are immediately obtained:

(i) The Hamiltonian of an autonomous system is a first integral;[5] for a typical autonomous mechanical system this is actually the total energy, so that it is usually named the energy integral.

(ii) If $\Phi(q,p)$ and $\Psi(q,p)$ are differentiable first integrals, then $\{\Phi,\Psi\}$ is a first integral.

The first statement follows from anticommutativity of the Poisson bracket; the second one is a straightforward consequence of Jacobi's identity and is often referred to as *Poisson's theorem*.

Everybody knows that the existence of first integrals for a system of differential equations may greatly help in investigating the behaviour of the flow, possibly leading to a complete integration of the equations when a sufficient number of independent first integrals is known.[6] In Chapter 3 the

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[5] A time-dependent dynamical variable $f(q,p,t)$ satisfies $\dot f = \{f,H\} + \frac{\partial f}{\partial t}$. Thus, for a non-autonomous Hamiltonian $H(q,p,t)$ one has $\frac{dH}{dt} = \frac{\partial H}{\partial t}$. This is nothing but the equation $\dot p_+ = -\frac{\partial H}{\partial t}$ in (1.3).

[6] For an autonomous system $\dot x = f(x)$ of ordinary differential equations on an n-dimensional manifold, a differentiable first integral defines a $(n-1)$-dimensional manifold through the implicit function theorem, which is invariant for the flow. Similarly, if $k < n$ independent first integrals are known, $\varphi_1(x) = c_1, \ldots, \varphi_k(x) = c_k$ say, then the implicit function theorem states the existence of an invariant $(n-k)$-dimensional manifold. If $k = n-1$, then the resulting one-dimensional invariant manifold is a set of orbits. In the latter case a complete integration can be worked out by suitably choosing one of the coordinates, x_j say, and determining the remaining $n-1$ as functions of x_j and of the $n-1$ constants $c_1, \ldots, c_{n-1}$. Thus, one is left with a single one-dimensional differential equation of the form $\dot x_j = g(x_j; c_1, \ldots, c_{n-1})$, depending on the constants as parameters, which may be solved by a quadrature. This makes it evident that no more than $n-1$ independent first integrals may be found. By the way, the name 'first integrals' comes from the remark that finding the complete solution of a system of differential equations of order n requires n integrations, thus introducing n integration constants. The knowledge of a function which is constant under the flow corresponds to having performed one of the

particular case of systems possessing a sufficient number of first integrals will be investigated – luckily, an exceptional case.

The argument goes as follows. Consider the subset of $\mathscr{F}$,

$$M_\Phi = \{(q, p) \in \mathscr{F} : \Phi(q, p) = \Phi(q_0, p_0)\}\,,$$

where $(q_0, p_0) = (q_{0,1}, \ldots, q_{0,n}, p_{0,1}, \ldots, p_{0,n}) \in \mathscr{F}$ is the initial point. Then the orbit through (q_0, p_0) is entirely contained in M_Φ. In particular, if the canonical vector field $\mathsf{J}\partial_z \Phi$ (with the compact notation of Section 1.1.3) does not vanish on M_Φ, then M_Φ is a differentiable submanifold of $\mathscr{F}$ which is invariant for the flow induced by H.

Let now $\Phi_1(q, p), \ldots, \Phi_r(q, p)$ be independent first integrals, that is, assume that the Jacobian $r \times 2n$ matrix satisfies

$$\mathrm{rank}
\begin{pmatrix}
\frac{\partial \Phi_1}{\partial q_1} & \cdots & \frac{\partial \Phi_1}{\partial q_n} & \frac{\partial \Phi_1}{\partial p_1} & \cdots & \frac{\partial \Phi_1}{\partial p_n} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\frac{\partial \Phi_r}{\partial q_1} & \cdots & \frac{\partial \Phi_r}{\partial q_n} & \frac{\partial \Phi_r}{\partial p_1} & \cdots & \frac{\partial \Phi_r}{\partial p_n}
\end{pmatrix} = r\,.$$

Then the equations

$$\Phi_1(q, p) = \Phi_1(q_0, p_0), \ldots, \Phi_r(q, p) = \Phi_r(q_0, p_0)$$

define a $(2n - r)$-dimensional submanifold of $\mathscr{F}$ which is still invariant for the flow induced by H.

As already remarked, an autonomous Hamiltonian system always possesses a first integral – the Hamiltonian itself, also named energy integral. Therefore the orbits of an autonomous Hamiltonian system with one degree of freedom lie on one-dimensional manifolds determined as the level curves of $H(q, p) = E$. This is enough in order to perform a complete integration of the system, as will be discussed in Section 1.3.

For systems with more than one degree of freedom the existence of the energy integral is not enough to perform a complete integration. According to the argument in note 6 one could naively expect that $2n - 1$ independent first integrals must be found to integrate the system by quadratures (where n is the number of degrees of freedom). It is a remarkable fact that the Hamiltonian structure requires the existence of only n independent first integrals, provided they satisfy the further condition of being in involution, namely that $\{\Phi_j, \Phi_k\} = 0$ for every pair j, k. This is the contents of Liouville's theorem; it will be discussed in Chapter 3. This interesting fact has a reverse side: as a general fact, no more than n nontrivial first integrals in involution

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necessary integrations, the constant being the value of the function at the initial point. The discussion in this section is essentially an adaptation of the general argument to the particular case of Hamiltonian systems.

exist in a global sense,[7] due to the peculiar behaviour of the orbits. On the other hand, as discovered by Poincaré, integrability itself turns out to be an exceptional property.

Example 1.14: *Autonomous Hamiltonian system with one degree of freedom.* As already remarked, the Hamiltonian $H(q, p)$ is a first integral and the orbits are subsets of the level sets defined by the equation $H(q, p) = E$, where $E = H\big(p(0), q(0)\big)$ is a constant determined by the initial point. Let us begin with two examples of linear systems (i.e., with a quadratic Hamiltonian), which describe the typical behaviour in a neighbourhood of an equilibrium point. In the case of an elastic attractive force (the harmonic oscillator) the Hamiltonian is $H(x, p) = p^2/2 + \omega^2 x^2/2$ (see panel (a) of Fig. 1.1). For $E = 0$ the orbit is a point, namely the equilibrium state $x = p = 0$. For $E > 0$ the level curves are ellipses with centre at the origin, representing oscillations around the equilibrium. Thus the equilibrium is stable. The case of an elastic repulsive force (see panel (b) of Fig. 1.1) is described by the Hamiltonian $H(p, x) = p^2/2 - \lambda^2 x^2/2$. For $E = 0$ the level curves are two straight lines intersecting at the origin, named *separatrices*. They actually represent five different orbits. Two of them represent orbits asymptotic to the origin for $t \to -\infty$ (unstable manifolds); two represent orbits asymptotic to the origin for $t \to +\infty$ (stable manifolds); the fifth orbit is the origin, which is an unstable equilibrium. For $E \neq 0$ the level curves are hyperbolæ representing orbits reflected back by the repulsive potential for $E < 0$, and orbits overtaking the equilibrium position for $E > 0$. A typical non-linear example is the pendulum. In this case the phase space is a cylinder, and the Hamiltonian is $H(\vartheta, p) = p^2/2 - \cos\vartheta$ (see panel (c) of Fig. 1.1). For $E = -1$ the orbit is the equilibrium point $\vartheta = p = 0$, similar to the equilibrium of the harmonic oscillator. For $E = 1$ the orbits are the upper unstable equilibrium $\vartheta = \pi$, with stable and unstable manifolds emanating from it (recall that ϑ is an angle, so the points $\vartheta = \pi$ and $\vartheta = -\pi$ are just different representations of the same point). In the neighbourhood of the unstable equilibrium the figure looks similar to that of the repulsive force, but far from the equilibrium one sees that the stable and unstable manifolds coincide: the corresponding orbits are asymptotic to the unstable equilibrium both for $t \to -\infty$ and for $t \to +\infty$. The stable and unstable manifolds separate orbits representing oscillations from orbits representing rotations. This should be enough to explain how to represent the phase portrait for the Hamiltonian of any natural mechanical system with one degree of freedom. *E.D.*

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[7] It is relevant here to distinguish between *local* and *global* first integrals. Local first integrals can always be found for a system of differential equations provided that the vector field satisfies suitable regularity constraints (e.g., Lipschitz condition). Global first integrals are a more delicate matter. The examples in the rest of the section should help to clarify this matter.

Figure 1.1 Phase portrait for three paradigmatic systems:

(a) elastic attractive force,
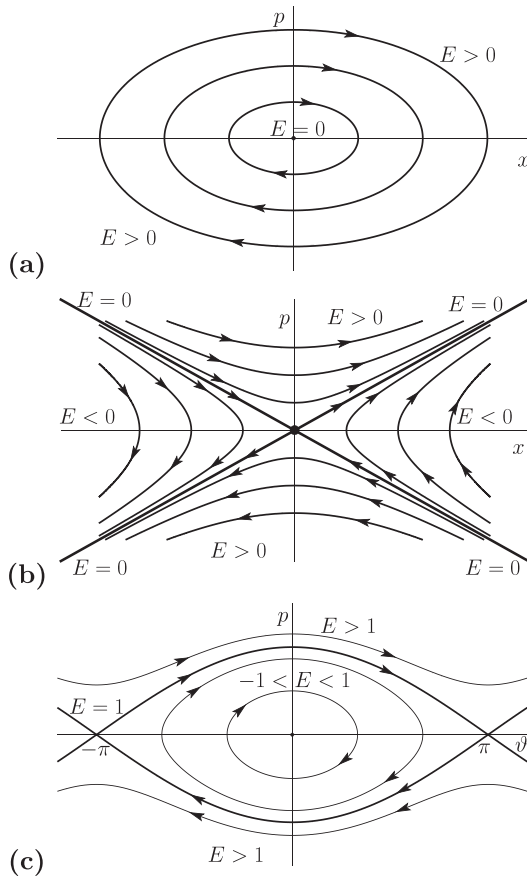
$$H = p^2/2 + \omega^2 x^2/2 \; ;$$

(b) elastic repulsive force,

$$H = p^2/2 - \lambda^2 x^2/2 \; ;$$

(c) the pendulum,

$$H = p^2/2 - \cos \vartheta \; .$$

The curves are level lines of energy, $H(x,p) = E$. The arrows indicate the direction of the flow ϕ^t.

(a)

(b)

(c)

Exercise 1.15:   Draw the phase portrait for the Hamiltonian

$$H(x,p) = \frac{p^2}{2} + V(x)$$

representing the motion of a particle with unit mass moving on a straight line under the potential $V(x)$. Consider the following cases:

(i)	$V(x) = \frac{1}{2}x^2 + \frac{1}{4}x^4$;		(v)	$V(x) = \frac{1}{3}x^3$;
(ii)	$V(x) = \frac{1}{2}x^2 + \frac{1}{3}x^3$;		(vi)	$V(x) = -\frac{1}{4}x^4$;
(iii)	$V(x) = \frac{1}{2}x^2 - \frac{1}{4}x^4$;		(vii)	$V(x) = -\frac{1}{4}x^4$;
(iv)	$V(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4$;		(viii)	$V(x) = \frac{\cos x}{1+x^2}$. A.E.L.

Exercise 1.16:   Sometimes it is useful to consider potentials $V(x)$ which are not differentiable or even not continuous at some point. This may be useful

for describing, for example, collisions. With a little attention, the reader will realize that a phase portrait can still be traced. Here are some examples.

(ix) $V(x) = \begin{cases} 0 & \text{for } x < 0 \text{ ,} \\ x & \text{for } x \geq 0 \text{ .} \end{cases}$ (xii) $V(x) = \begin{cases} -1 & \text{for } |x| < 1 \text{ ,} \\ 1 & \text{for } |x| \geq 1 \text{ .} \end{cases}$

(x) $V(x) = |x|$.

(xi) $V(x) = \begin{cases} -1 & \text{for } x < 0 \text{ ,} \\ 1 & \text{for } x \geq 0 \text{ .} \end{cases}$ (xiii) $V(x) = \begin{cases} 0 & \text{for } x < 0 \text{ ,} \\ \infty & \text{for } x \geq 0 \text{ .} \end{cases}$

A.E.L.

Exercise 1.17:   Draw the phase portrait for the Hamiltonians

$$ H = p(1 - p)\sin\vartheta \quad \text{and} \quad H = \cos p \sin\vartheta \text{ ,} \quad \vartheta \in \mathbb{T} \text{ , } p \in \mathbb{R} \text{ .} $$
A.E.L.

Example 1.18: *Free particle.*   The Hamiltonian is $H = \frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right)$ (see Example 1.1). Besides the Hamiltonian, the following quantities are immediately checked to be first integrals:

$$ p_x \, , \ p_y \, , \ p_z \, , \ M_x = yp_z - zp_y \, , \ M_y = zp_x - xp_z \, , \ M_z = xp_y - yp_x $$

(the linear and the angular momentum). Not all of them are independent, of course,[8] but we can select five independent first integrals among them, for instance: $p_x \, , \ p_y \, , \ p_z \, , \ M_x$ and M_y. As a curiosity, one could ask what happens if one tries to construct new first integrals via Poisson brackets (recall that the Poisson bracket between two first integrals is a first integral). The answer is found in Table 1.1. Using the first integral, we can easily construct the orbit. For the intersection in $\mathbb{R}^6$ of the planes $p_x = c_1$, $p_y = c_2$, $p_z = c_3$, $M_x = c_4$, $M_y = c_5$, where $c_1, \ldots, c_5$ are constants determined by the initial point, is a straight line representing the orbit.   *E.D.*

Example 1.19: *The problem of n bodies.*   The Hamiltonian reads (see Example 1.6)

$$ H = \sum_{j=1}^{n} \frac{\mathbf{p}_j^2}{2m_j} + V(\mathbf{x}_1, \ldots, \mathbf{x}_n) \text{ .} $$

The system possesses seven independent first integrals, namely: the total energy H, the three components of the linear momentum

$$ P_x = \sum_j p_{j,x} \, , \quad P_y = \sum_j p_{j,y} \, , \quad P_z = \sum_j p_{j,z} $$

(where $p_{j,x}$, $p_{j,y}$ and $p_{j,z}$ denote the x , y and z components, respectively, of the momentum $\mathbf{p}_j$ of the jth particle) and the three components of the angular momentum

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[8]   Recall that in general a system of differential equations on an n-dimensional manifold cannot possess more than $n-1$ independent first integrals (see note 6).

Table 1.1 Poisson brackets between the components of the momentum and the angular momentum for one particle.

$\{\cdot,\cdot\}$	p_x	p_y	p_z	M_x	M_y	M_z
p_x	0	0	0	0	p_z	$-p_y$
p_y	0	0	0	$-p_z$	0	p_x
p_z	0	0	0	p_y	$-p_x$	0
M_x	0	p_z	$-p_y$	0	M_z	$-M_y$
M_y	$-p_z$	0	p_x	$-M_z$	0	M_x
M_z	p_y	$-p_x$	0	M_y	$-M_x$	0

$$M_x = \sum_j y_j p_{j,z} - z_j p_{j,y}\,,\ M_y = \sum_j z_j p_{j,x} - x_j p_{j,z}\,,\ M_z = \sum_j x_j p_{j,y} - y_j p_{j,x}\,.$$

No new first integrals can be constructed by Poisson brackets between the known ones: the results of Table 1.1 still apply. For $n = 2$ (the problem of two bodies) the existence of such integrals allows us to conclude that the orbit lies on five-dimensional manifolds in the phase space $\mathbb{R}^{12}$. At first sight, this appears insufficient in order to perform a complete integration, seemingly contradicting the well-known fact that the system is actually integrable. However, we have already anticipated that the existence of n (in this case 6) independent first integrals which are in involution is enough. The common choice for the six first integrals is H, P_x, P_y, P_z, M_z and $\Gamma^2 = M_x^2 + M_y^2 + M_z^2$. For $n > 2$ the seven known first integrals are insufficient, and we should look for more. However, the theorems of Heinrich Bruns and Henry Poincaré state that in general there are no more first integrals. *E.D.*

Example 1.20: *Motion under central forces.* In Cartesian coordinates the Hamiltonian reads

$$H = \frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right) + V(r)\,,$$

where $r = \sqrt{x^2 + y^2 + z^2}$. The phase space is $\mathbb{R}^6$, and the system possesses four independent first integrals, namely the Hamiltonian H and the three components M_x, M_y and M_z of the angular momentum (check it). We conclude that the orbits lie on a two-dimensional manifold in the phase space. *E.D.*

Example 1.21: *The Keplerian case.*   According to Kepler's laws, the orbit of a planet in physical space is an ellipse. This does not follow from the general discussion of the previous example, because we could only conclude that the orbit in phase space lies on a two-dimensional manifold.[9] The fact that the orbit in a Keplerian potential is closed is due to the existence of a further first integral known as the *Runge–Lenz vector* (but already known to Laplace). In Cartesian coordinates the vector is

$$(1.31) \qquad\qquad \mathbf{A} = \mathbf{p} \wedge \mathbf{M} - \frac{km\mathbf{x}}{r} \; ,$$

where k is the constant in the Keplerian potential and $\mathbf{M}$ is the angular momentum. The vector $\mathbf{A}$ points to the pericentre of the orbit. Not all the components of the vector are independent of the four first integrals of the general case. However, one can extract a fifth integral which is independent of them, so that the invariant manifold has dimension 1 and the orbit is closed.[10]   *E.D.*

Exercise 1.22:   Let the phase space be $\mathscr{F} = \mathbb{T}^2 \times \mathbb{R}^2$, with $q \in \mathbb{T}^2$ and $p \in \mathbb{R}^2$, and consider the Hamiltonian

$$H(q,p) = \omega_1 p_1 + \omega_2 p_2$$

with real ω_1, ω_2. The momenta p_1 and p_2 are obviously first integrals, so that every orbit lies on a torus $\mathbb{T}^2$. Check that the function $\omega_1 q_2 - \omega_2 q_1$ is a third first integral, which, however, is only *local*, in general, in the sense that it does not provide a global one-value function on the torus (it is not periodic in q_1, q_2). Prove that if ω_1/ω_2 is an irrational number, then there are no more global first integrals. If, however, $\omega_1/\omega_2 = r/s$, where r, s are integers, then the function $f(q) = \sin(rq_2 - sq_1)$ is a global first integral.[11] Try to extend the example to a higher-dimensional case $\mathscr{F} = \mathbb{T}^n \times \mathbb{R}^n$ with Hamiltonian $H(q,p) = \omega_1 p_1 + \cdots + \omega_n p_n$. For a detailed discussion see Section 4.1.1.   *A.E.L.*

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[9]   A theorem by Newton, later proved in extended form by Joseph Bertrand [25], states that there are only two potentials for which all bounded orbits are closed: (a) the Keplerian potential $V(r) = -k/r$ for negative energy, the orbit being an ellipse with the centre of force being placed in one of the foci, and (b) the harmonic potential $V(r) = kr^2$, the orbit being an ellipse with centre coinciding with the centre of force. For a discussion of Newton's theorem see [101]. For a proof of Bertrand's theorem see also [206].

[10]   For a short historical note see [102]. See also [44].

[11]   In some classical books, e.g., Wintner's treatise [212], it is usual to distinguish between *isolating* first integrals, which define a global invariant surface, and *non-isolating* ones, which are only local in the language adopted here.

1.3   Use of First Integrals

In the previous section the first integrals have been used in order to get qualitative information on the orbits, that is, without proceeding to integrate the dynamical equations and finding the time dependence of the evolution. Actually we can do better. The classical approach consists in exploiting the first integrals in order to perform the so-called *integration by quadratures:* one looks for a solution in terms of algebraic operations, including inversion of functions, and calculation of integrals of known functions. The general theory in this sense will be developed in Chapter 3, with the theorems of Liouville and of Arnold–Jost. In this section a first simplified approach is made, with the aim of describing in detail the process of integration by quadratures.

1.3.1   Motion on a One-Dimensional Manifold

The simplest case is the motion of a particle with mass m on a straight line (or possibly a circle, or another smooth curve) under the action of a given autonomous potential, which is assumed to be smooth. This is the case considered in Examples 1.7 and Exercises 1.15 and 1.16, where the energy integral has been used in order to draw the phase portrait; we have seen that the energy integral provides a complete qualitative description of the dynamics. Here we perform the next step: to write the solution of the canonical equations in a somehow explicit form. This is indeed the process of integration by quadrature. The discussion here provides the basis for the further extension to the case of higher-dimensional systems which possess a sufficient number of first integrals.

The Hamiltonian has the general form

$$(1.32) \qquad H(x,p) = \frac{p^2}{2m} + V(x) \ ,$$

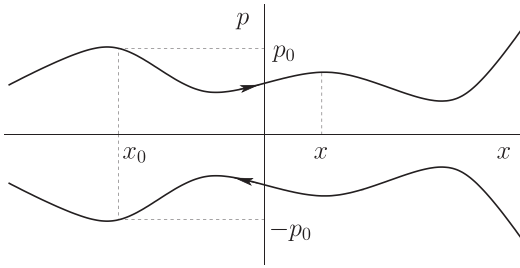where the potential $V(x)$ is assumed to be a smooth function, and the corresponding Hamilton's equations are

$$(1.33) \qquad \dot{x} = \frac{p}{m} \ , \quad \dot{p} = -\frac{dV}{dx} \ .$$

The general integration scheme goes as follows. Solve Eq. (1.32) for p, thus getting

$$(1.34) \qquad p = \pm\sqrt{2m\big(E - V(x)\big)} \ ,$$

and substitute it into Eq. (1.33), thus getting

$$(1.35) \qquad \dot{x} = \pm\sqrt{\frac{2}{m}\big(E - V(x)\big)} \ .$$

Figure 1.2 Illustrating the case of an orbit in the phase plane without reflection points.

This is a separable differential equation, and the solution satisfying the initial condition $q(t_0) = q_0$, $p(t_0) = p_0$ is written as

$$(1.36) \qquad t - t_0 = \pm \sqrt{\frac{m}{2}} \int_{x_0}^{x} \frac{d\xi}{\sqrt{E - V(\xi)}} \ .$$

This is the general solution. However, understanding this formula may be not completely immediate; so let us add a short discussion that an expert reader may skip.

Let us refer to the phase portrait, which is obtained by drawing the level curves implicitly defined by the equation $H(x, p) = E$, the value of the energy. Remark that for systems with Hamiltonian of the form (1.32) all equilibria are located on the axis $p = 0$ and correspond to stationary points of the potential. On the other hand, all curves that intersect the axis $p = 0$ on a point x which is not an equilibrium have a vertical tangent at that point.

We may focus our attention on four examples of curves in the phase plane:

(i) isolated equilibrium points;

(ii) open curves not containing equilibrium points;

(iii) closed curves topologically equivalent to a circle;

(iv) curves which intersect on an equilibrium.

If $\overline{x}$ is an equilibrium for (1.33), that is, if $\frac{dV}{dx}(\overline{x}) = 0$, then $x(t) = \overline{x}$, $p(t) = 0$ is a solution (check it). By the way, from the canonical equation (1.33) it is clear that equilibrium points (if any) are located on the x axis.

Let us consider the simplest case of a value E of energy such that $E > V(x)$ holds true for all x. Assuming also that $V(x)$ is bounded from below, we have two separate curves in the phase plane, symmetric with respect to the x axis, which lie on the upper and lower half plane, respectively, as illustrated in Fig. 1.2. Let us pick an initial point x_0 together with the fixed value $E > V(x_0)$ of the energy. The corresponding initial value p_0 is calculated by (1.34) by choosing the appropriate sign, thus selecting one between the upper or the lower curve (i.e., the direction of the motion). The integral formula (1.36) tells us how much time the point needs in order to move from the initial point x_0 to an arbitrary point x. This gives us a smooth function

Figure 1.3 Illustrating the case of an orbit in the phase plane with a reflection point.
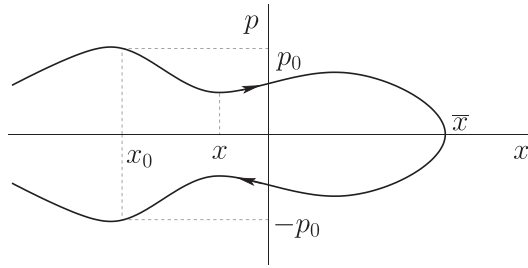
$t(x)$ which is clearly monotonic (the function under the integral does not change its sign), and so it may be inverted, thus giving the solution $x(t)$. The equation $p(x)$ of the curve is found by replacing $x(t)$ in (1.34), still keeping the chosen sign of the square root. Thus, the problem is completely solved.

The case illustrated in Fig. 1.3 requires some more attention. Here, we should assure that the argument of the square root is always non-negative. Thus, the upper limit x of the integral must satisfy $x \leq \overline{x}$, where $V(\overline{x}) = E$. Choose $p_0 > 0$, on the upper part of the curve. Then we are able to calculate the integral for any $x \leq \overline{x}$, as in the previous case. If we set the upper limit to $\overline{x}$, then we calculate a time

$$t_1 = t_0 + \sqrt{\frac{2}{m}} \int_{x_0}^{\overline{x}} \frac{d\xi}{\sqrt{E - V(\xi)}}$$

at which the point reaches $\overline{x}$. A moment's thought will allow us to realize that (for a smooth potential) such a time t_1 is finite.[12] Since $\overline{x}$ is not an equilibrium, the point on the phase plane must continue its motion along the curve: $\overline{x}$ is a reflection point for the motion. Thus, changing the sign of p, we may continue our integration by calculating

$$t - t_1 = -\sqrt{\frac{2}{m}} \int_{\overline{x}}^{x} \frac{d\xi}{\sqrt{E - V(\xi)}} = \int_{x}^{\overline{x}} \frac{d\xi}{\sqrt{E - V(\xi)}} \ ,$$

where $x < \overline{x}$ is still arbitrary. Remark that t is still increasing, so that we end up again with a monotonic function $t(x)$, to be inverted in order to obtain the motion $x(t)$. Remark also that the second integral actually coincides with the previous one (just replace the initial point x_0 with an arbitrary point x), so that we don't need to compute it again.[13]

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[12] Since $V(\overline{x}) = E$ and $V'(\overline{x}) \neq 0$, then we have $E - V(x) = E - V(\overline{x}) + O(x - \overline{x})$, so that the integral has a finite value.

[13] A reader who looks carefully at Eq. (1.35) may remark that we are in a paradoxical situation. If $E = V(x_0)$, then taking x_0 as initial condition and replacing
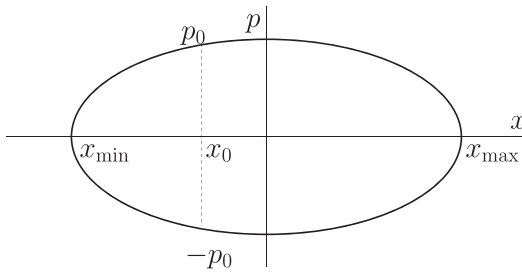
Figure 1.4 Illustrating the case of closed orbits in the phase plane.

A similar situation occurs if we consider a closed orbit, as in fig. 1.4. Here we have two reflection points $x_{\min}$ and $x_{\max}$, with $V(x_{\min}) = V(x_{\max}) = E$ and $V(x) > E$ in the interval $(x_{\min}, x_{\max})$. The point oscillates between the two extrema, and if we calculate

$$\tau = \sqrt{\frac{2}{m}} \int_{x_{\min}}^{x} \frac{d\xi}{\sqrt{E - V(\xi)}} \ , \quad x_{\min} \leq x \leq x_{\max},$$

we know the forward motion from $x_{\min}$ to $x_{\max}$. Then the backward motion from $x_{\max}$ to $x_{\min}$ is also known, by symmetry, so that we know the motion over one period. Since the motion is periodic, we know it for all times. The period T may be calculated as

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it in the right member of (1.33), we immediately realize that x_0 is an equilibrium, but at this point the Lipschitz condition is violated. In its simplest form the equation is $\dot{x} = \sqrt{x}$, a classical example of non-uniqueness of the solution: both functions $x(t) = 0$ and $x(t) = (t - t_0)^2/2$ are solutions satisfying the initial condition $x(t_0) = 0$, whatever t_0 is. E.g., if we cut the stalk of an apple it could happen that the apple stays on the tree for an arbitrary time, until it realizes that the stalk has been severed, and eventually it falls down: this is something that we can *imagine*, and we see sometimes in cartoons, but is hardly seen in real life. The reason for this apparent paradox lies precisely in the reduction that we have performed using conservation of energy. We must remember that the orbit in phase plane is defined *implicitly* by the conservation of energy, namely by (1.32). If $E = V(x_0)$, then we have $p = 0$ and so also $\frac{\partial H}{\partial p} = 0$, so that we cannot determine p as a function of x using the implicit function theorem. If we have also $\frac{\partial H}{\partial x}(x_0) = 0$, then x_0 is a true equilibrium, and the solution $x(t) = x_0$ is unique; else we should express x as a function of p and replace it in the second equation of (1.33), thus getting $\dot{p} = -\frac{dV}{dx}\big|_{x=x(p)}$. The latter equation includes the information that the acceleration is not zero at x_0, so that the solution is unique. Therefore, the paradox disappears if we calculate *local* solutions corresponding to different parts of the curve and then glue them together; the resulting solution is regular. In practice we avoid such a procedure: just solve Eq. (1.34) as explained in the text and throw away the spurious equilibrium solution.

Figure 1.5 Illustrating the case of curves that intersect in an equilibrium point.
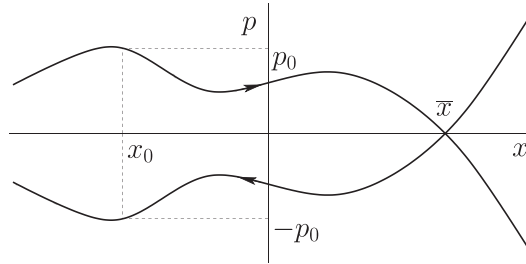
$$(1.37) \qquad T = 2\sqrt{\frac{2}{m}} \int_{x_{\min}}^{x_{\max}} \frac{d\xi}{\sqrt{E - V(\xi)}}.$$

The last case is concerned with curves that intersect at an equilibrium $\overline{x}$, as illustrated in Fig. 1.5. This happens when the potential has a local maximum in $\overline{x}$, and we set $E = V(\overline{x})$.

1.3.2 Systems with One Degree of Freedom

The discussion of Section 1.3.1 can be repeated, with a few minor changes, for the generic case of a smooth Hamiltonian $H(q, p)$ on a two-dimensional phase space $\mathscr{F}$ (e.g., the Hamiltonian of Exercise 1.17). Write Hamilton's equations

$$\dot{q} = \frac{\partial H}{\partial p} \ , \quad \dot{p} = -\frac{\partial H}{\partial q}$$

and recall that a point $(\overline{q}, \overline{p}) \in \mathscr{F}$ is an equilibrium if and only if $\frac{\partial H}{\partial q}(\overline{q}, \overline{p}) = \frac{\partial H}{\partial p}(\overline{q}, \overline{p}) = 0$. In this case $q(t) = \overline{q}$, $p(t) = \overline{p}$ is an orbit. If a point (q_0, p_0) is not an equilibrium, then at least one of $\frac{\partial H}{\partial q}(q_0, p_0)$ and $\frac{\partial H}{\partial p}(q_0, p_0)$ does not vanish. Considering the latter case and setting the value of the energy $E = H(q_0, p_0)$, we can solve the equation $H(q, p) = E$ with respect to p, thus getting $p = p(q, E)$ as a smooth function in a neighbourhood of q_0, p_0. Substituting the latter function in the equation for $\dot{q}$, we get

$$\dot{q} = \frac{\partial H}{\partial p}\big(q, p(q, E)\big) \ ,$$

which is easily integrated as

$$(1.38) \qquad \int_{q_0}^{q} \frac{d\xi}{\frac{\partial H}{\partial p}\big(\xi, p(\xi, E)\big)} = t - t_0 \ ,$$

t_0 being the initial time. This gives a monotonic function $t(q)$, which can be inverted, thus giving the solution $q(t)$. By substitution in $p(q, E)$, $p(t)$ also is found, so that the motion along the orbit is fully determined. The procedure

is, of course, local, but having reached a new point, say $q_1 = q(t_1), p_1 = p(t_1)$ with $t_1 \neq t_0$, the whole process can be repeated so that the solution is continued along the curve. The integrand in (1.38) becomes singular if at some point it happens that $\frac{\partial H}{\partial p}(\xi, p(\xi, E)) = 0$, but at that point one has $\frac{\partial H}{\partial q} \neq 0$, in view of the assumption that there are no equilibria on the phase curve. Thus, by exchanging the roles of q and p, the integration process can be continued again.

   If the phase portrait contains a closed curve with no equilibria, then the corresponding orbit is periodic, and the motion over one period can be integrated. The conclusion is: a system with one degree of freedom can be completely solved by quadratures (i.e., calculation of integrals of known functions).

1.3.3   The Period of Oscillation

Let us consider again a system with one degree of freedom, so that the phase space has dimension 2, and assume that $(\overline{q}, \overline{p})$ is a maximum or minimum for the Hamiltonian. Then $(\overline{q}, \overline{p})$ is an equilibrium, and in the phase portrait it is surrounded by closed curves representing oscillations of the system. It is often interesting to know the period of the oscillation, even forgetting the details about the orbit.

Example 1.23: *The period of oscillation of a pendulum.*   Let ℓ be the length of the pendulum, and g be the constant acceleration due to gravity. Then, using as coordinate the angle ϑ with respect to the vertical, the Hamiltonian writes

$$(1.39) \qquad\qquad H = \frac{p^2}{2} - \frac{g}{\ell} \cos \vartheta \ .$$

Let the initial energy E be fixed, with $-g/l < E < g/l$, so that the pendulum librates around the lower equilibrium point. According to (1.37) the period is given by

$$T = 2 \int_{-\vartheta_0}^{\vartheta_0} \frac{d\vartheta}{\sqrt{E + \frac{g}{\ell} \cos \vartheta}} \ ,$$

where ϑ_0 (the amplitude of the oscillation) is the solution of $E = \frac{g}{\ell} \cos \vartheta_0$. By replacing this value of E in the integral and exploiting the symmetry $V(-\vartheta) = V(\vartheta)$ of the potential, we get

$$T = \sqrt{\frac{8\ell}{g}} \int_0^{\vartheta_0} \frac{d\vartheta}{\sqrt{\cos \vartheta - \cos \vartheta_0}} \ .$$

Although apparently simple, this integral cannot be evaluated with elementary methods. However, it can be reduced to an elliptic integral as follows. Using the trigonometric formula $\cos \alpha = 1 - 2\sin^2 \frac{\alpha}{2}$, we get

$$T = 2\sqrt{\frac{\ell}{g}} \int_0^{\vartheta_0} \frac{d\vartheta}{\sqrt{\sin^2 \frac{\vartheta_0}{2} - \sin^2 \frac{\vartheta}{2}}} \ .$$

Next, we change the variable by setting

$$\sin \frac{\vartheta}{2} = \sin \frac{\vartheta_0}{2} \sin \varphi \ , \quad 0 \le \varphi \le \frac{\pi}{2} \ ,$$

which in turn gives

(1.40) $$T = 4\sqrt{\frac{l}{g}} K\left(\sin \frac{\vartheta_0}{2}\right) \ ,$$

where

(1.41) $$K(m) = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - m^2 \sin^2 \varphi}} \ , \quad 0 \le m < 1$$

is named the *complete elliptic integral of the first kind*. The numerical values of the integral depending on the parameter $m = \sin \frac{\vartheta_0}{2}$ (or sometimes of ϑ_0) can be found in numerical tables, for example, [1].

Let us see how the integral may by calculated using a series expansion. Recall the series expansion of the inverse of the square root:[14]

$$\frac{1}{\sqrt{1+x}} = 1 + \frac{1}{2}x + \frac{1 \cdot 3}{2 \cdot 4}x^2 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^3 + \cdots = \sum_{k \ge 0} \frac{(2k-1)!!}{(2k)!!}x^k \ .$$

This is an absolutely and uniformly convergent series for $|x| < 1$, and so it can be integrated term by term. Setting $x = m^2 \sin^2 \varphi$ and replacing the series expansion in (1.41), we get

$$K(m) = J_0 + \frac{1}{2}m^2 J_2 + \frac{1 \cdot 3}{2 \cdot 4}m^4 J_4 + \cdots = \sum_{k \ge 0} \frac{(2k-1)!!}{(2k)!!}m^{2k} J_{2k} \ ,$$

where

$$J_{2k} = \int_0^{\pi/2} \sin^{2k} \varphi \, d\varphi \ .$$

The latter integral may be computed by the recurrent formula

$$J_0 = \frac{\pi}{2} \ , \quad J_{2k} = \frac{2k-1}{2k}J_{2k-2} \ .$$

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[14] Here the symbol $n!!$ denotes the *semifactorial* (sometimes inappropriately called *double factorial*), namely $0!! = 1!! = 1$ and, in recurrent form, $n!! = n \cdot (n-2)!!$.

We conclude:

$$K(m) = \frac{\pi}{2}\left(1 + \frac{1}{4}m^2 + \frac{9}{64}m^4 + \cdots\right) = \sum_{k\geq 0}\left(\frac{(2k-1)!!}{(2k!!)}\right)^2 m^{2k}\ .$$

A rough indication of the change of the period due to the nonlinearity is found by writing the first terms of the expansion of the period T as a function of the amplitude ϑ_0. To this end use the expansion of the sine function

$$m = \sin\frac{\vartheta_0}{2} = \frac{\vartheta_0}{2} + \frac{\vartheta_0^3}{2^3 \cdot 3!} + \cdots$$

and replace it in the expression of the period, keeping just the first two terms. This gives the approximate value, for small values of ϑ_0,

$$T = 2\pi\sqrt{\frac{\ell}{g}}\left(1 + \frac{1}{16}\vartheta_0^2 + \cdots\right)\ ,$$

where the first forgotten term is of order ϑ_0^4. It is straightforward to check that if the amplitude ϑ_0 does not exceed 4×10^{-2}, which is about $2°$, then the relative change of the period is less than 10^{-4}. For higher values of the amplitude a longer calculation is needed.   *E.D.*

Exercise 1.24:   Find an approximate expression of the period for the potential $V(x) = x^2/2 - x^3/3$.   *A.E.L.*

Exercise 1.25:   Show how to calculate the period of rotation of the pendulum, that is, for $E > g/\ell$ in the Hamiltonian (1.39).   *A.E.L.*

Exercise 1.26:   Calculate the motion on the separatrix, that is, for $E = g/\ell$, for the Hamiltonian (1.39) of the pendulum.   *A.E.L.*

Exercise 1.27:   Calculate the period of oscillation for the potentials (x) and (xii) of Exercise 1.16.   *A.E.L.*

1.3.4   *Higher-Dimensional Models*

Some interesting Hamiltonian systems with more than one degree of freedom can be integrated by separating the variables, provided that enough first integrals are known. Here are a few examples.

Example 1.28: *Central forces.*   The most classical example is the motion under a central force (see Example 1.20).[15] Using the conservation of the

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[15] The procedure illustrated here is the common one which can be found in most books on Rational Mechanics or Classical Mechanics where the problem is discussed starting from Newton's or Lagrange equations. The reader may notice that the Hamiltonian approach is definitely more direct: the relevant facts, such as the conservation of angular momentum and the corresponding reduction to the radial problem with an effective potential, emerge rather spontaneously.

angular momentum one proves that the motion is a planar one. Introducing polar coordinates in the plane orthogonal to the direction of the angular momentum, we write the Lagrangian as

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\vartheta}^2) - V(r) \; .$$

Calculating the momenta as

$$p_r = m\dot{r} \; , \quad p_\vartheta = mr^2\dot{\vartheta},$$

the Hamiltonian is found to be

(1.42) $$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\vartheta^2}{r^2}\right) + V(r)$$

(this is just a straightforward simplification of Example 1.4). The Hamiltonian possesses two first integrals, namely the Hamiltonian itself and the angular momentum $\ell = p_\vartheta$. Hamilton's equations are

(1.43)
$$\dot{r} = \frac{p_r}{m} \; , \quad \dot{p}_r = \frac{p_\vartheta^2}{mr^3} - V'(r),$$
$$\dot{\vartheta} = \frac{p_\vartheta}{mr^2} \; , \quad \dot{p}_\vartheta = 0 \; .$$

Let $\ell = p_\vartheta(0)$ be the constant value of the angular momentum. It is an easy matter to check that a substitution of this value in the first two equations produces a one-degree-of-freedom system with Hamiltonian

(1.44) $$H = \frac{p_r^2}{2m} + \frac{\ell^2}{2mr^2} + V(r)$$

on the phase space $\mathscr{F} = (0, +\infty) \times \mathbb{R}$. Introducing the *effective potential*

$$V^*(r) = \frac{\ell^2}{2mr^2} + V(r),$$

the Hamiltonian is written as

$$H = \frac{p_r^2}{2m} + V^*(r) \; ,$$

belonging to the class of systems considered in Section 1.3.1. We may perform a complete integration, thus finding the solution $r(t)$. Replacing this function in the equation (1.43) for $\dot{\vartheta}$, the r.h.s. turns out to be a function of time only, so that the equation can be integrated. This shows that the problem can be completely solved by quadratures.   *E.D.*

Exercise 1.29:   The phase space for the Hamiltonian (1.42) is identified as $\mathscr{F} = (0, +\infty) \times \mathbb{T} \times \mathbb{R}^2$. Determine the form of the invariant surfaces defined by the two first integrals H and M_z.   *A.E.L.*

Exercise 1.30:   Consider the Hamiltonian of example 1.4, that is, the central motion in the spatial case. Show that the problem can be completely integrated.   *A.E.L.*

Exercise 1.31:   Consider the Hamiltonian of the *spherical pendulum*

$$H = \frac{1}{2}\left(p_\vartheta^2 + \frac{p_\varphi^2}{\sin^2\vartheta} \right) - \cos\vartheta,$$

with $0 < \vartheta < \pi$ and $\varphi \in \mathbb{T}$. Show that the problem can be completely integrated by quadratures.   *A.E.L.*