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TORSION ELEMENTS AND THE CLASSIFICATION OF VECTOR BUNDLES

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1. Introduction. There are many situations in algebraic topology when a geometric construction is possible if, and only if, a certain integral cohomology class, an obstruction is zero. When attempts are made to compute the obstruction, it often happens that it is relatively easy to show that m times the obstruction is zero, where m is an integer, and consequently the geometric construction is possible if the cohomology group in question has no elements of order m. The purpose of this paper is to give an example of this situation and to develop techniques for computing the obstruction when elements of order m are present.

We consider the problem of classifying vector bundles over an n-dimensional CW complex X. If ξ is a real vector bundle over X, the Stiefel-Whitney class of ξ in $H^i(X; \mathbf{Z}_2)$ is denoted by $w_i(\xi)$ and the Pontrjagin class in $H^{4i}(X; \mathbf{Z})$ by $P_i(\xi)$. If ξ is an n-plane bundle, the Euler class of ξ in $H^n(X; \mathbf{Z})$ is denoted by $\chi(\xi)$. If ω is a complex bundle, the Chern class in $H^{2i}(X; \mathbf{Z})$ is denoted by $c_i(\omega)$. Universal characteristic classes are denoted by w_i, $P(i)$, and $c(i)$. If ξ and η are two vector bundles over X and θ is a primary cohomology operation, $\theta_{\xi,\eta}$ denotes the functional cohomology operation associated with the action of θ on the cohomology sequence of the pair $(X \times I \cup B_F, B_F)$, where B_F is the mapping cylinder of the map $F : X \times \dot{I} \rightarrow B$ given by the classifying maps of ξ and η. Throughout this paper, we will take δSq^2 and δP^1 for θ, where δ is the Bockstein, Sq^2 the Steenrod square, and P^1 the Steenrod power mod 3. In the two theorems below, we assume that $n \leqq 8$ and that $H^n(X; \mathbf{Z})$ has no elements of order 2 if n is even in Theorem 1 and if $n = 8$ in Theorem 2. In Theorem 1, we assume that $n \neq 5$. Theorem 1 is true in the case $n = 5$, if the word *isomorphic* is replaced by the words *stably isomorphic*. In Theorem 2, we assume that n is even.

THEOREM 1. *If ξ and η are two orientable n-plane bundles over X, then ξ is isomorphic to η if, and only if, $w_2(\xi) = w_2(\eta)$; $P_i(\xi) = P_i(\eta)$, $i = 1, 2$; $0 \in \delta Sq^2_{\xi,\eta}(w_2)$; $0 \in \delta P^1_{\xi,\eta}(P(1))$; and $\chi(\xi) = \chi(\eta)$.*

THEOREM 2. *If ω and ζ are two complex $n/2$-bundles over X, then ω is isomorphic to ζ if, and only if, $c_i(\omega) = c_i(\zeta)$, $1 \leqq i \leqq 4$; $0 \in \delta Sq^2_{\omega,\zeta}(c(2))$; and $0 \in \delta P^1_{\omega,\zeta}(c(2))$.*

Theorem 1 contains the Dold-Whitney classification theorem for 7-complexes [**13**] which assumes that $H^4(X; \mathbf{Z})$ has no elements of order 2 and a

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classification theorem of Thomas for 8-complexes [**12**] which assumes that $H^4(X; \mathbf{Z})$ has no elements of order 2 and that $H^8(X; \mathbf{Z})$ has no elements of order 6. Theorems 1 and 2 give a classification of vector bundles over closed, orientable manifolds of appropriate dimensions because the top dimensional cohomology of such manifolds is torsion free.

2. The proofs of Theorems 1 and 2. We prove Theorem 1 and the comment on Theorem 2. The conditions in the theorem are clearly necessary. In view of Theorem 1.7 in [**3**] and Lemma 2 in [**13**], it is enough to show that the conditions in the theorem imply that ξ and η are stably isomorphic. It will then follow that they are isomorphic when n is odd, $n \neq 5$ [**3**], and the condition $\chi(\xi) = \chi(\eta)$ will be enough to imply isomorphism in the case n even [**13**].

We begin with a proposition which relates obstructions to a stable isomorphism and characteristic classes. In the proposition below, $\delta Sq^2_{\xi,\eta}$ and $\delta P^1_{\xi,\eta}$ denote the functional operations described in the introduction. If $w_2(\xi) = w_2(\eta)$, the operation $\delta Sq^2_{\xi,\eta}$ is defined on w_2 and the resulting subset of $H^4(X;\mathbf{Z})$ is denoted by $\delta Sq^2_{\xi,\eta}(w_2)$ and is a coset modulo $(P_1(\xi) - P_1(\eta))$, the subgroup generated by the difference $P_1(\xi) - P_1(\eta)$. If $P_1(\xi) = P_1(\eta)$, the operation $\delta P^1_{\xi,\eta}(P(1))$ is defined and is a subset of $H^8(X;\mathbf{Z})$ which is a coset of $(P_2(\xi) - P_2(\eta)) + \text{image } \delta P^1$. (See [**4**] or [**7**].) Let a_i be 1 for i even and 2 for i odd and if x is an integral class, $\bar{x}$ denotes its reduction mod 2.

PROPOSITION 2.1. *If ξ and η are two orientable stable bundles such that the integral obstruction to a stable isomorphism $O^{4i}(\xi, \eta)$ is nonvoid, $i = 1$ or 2, then*:

(2.2) $(2i-1)!a_i O^{4i}(\xi, \eta) = P_i(\xi) - P_i(\eta)$,

(2.3) $O^4(\xi, \eta) + (P_1(\xi) - P_1(\eta)) = \delta Sq^2_{\xi,\eta}(w_2)$,

(2.4) $\bar{O}^4(\xi, \eta) = w_4(\xi) - w_4(\eta)$,

(2.5) $2O^8(\xi, \eta) + (P_2(\xi) - P_2(\eta)) + \text{image } \delta P^1 = \delta P^1_{\xi,\eta}(P(1))$.

Proof. Formula (2.2) is just formula (b) in Theorem 6.15 of [**8**]. To prove (2.3), let $K(\mathbf{Z}_2, 2; \mathbf{Z}, 4, \delta Sq^2)$ be the total space of the fibration induced by δSq^2. If $f' : BSO \to K(\mathbf{Z}_2, 2; \mathbf{Z}, 4, \delta Sq^2)$ is a lifting of w_2, and $[g]$ in $\pi_4(BSO)$ is a generator, it follows from the Peterson-Stein definition of functional cohomology operation ([**6**, p. 159]) that the set $\{f_\#'[g] : f'^*\iota = w_2\}$ can be identified with the functional cohomology operation $\delta Sq^2_g(w_2)$. (See [**4**].) Direct computation shows that $\delta Sq^2_g(w_2)$ is the non-zero coset mod 2, and so the induced homomorphism $f_*' : H^4(X; \pi_4(BSO)) \to H^4(X; \mathbf{Z})$ may be taken to be the identity. If f and g are the classifying maps of ξ and η, respectively, naturality of obstructions implies that $O^4(\xi, \eta)$ is contained in $O^4(f'f, f'g)$ which is contained in the operation $\delta Sq^2_{f'f,f'g}(\iota)$, where ι is the fundamental class, by 10.8 in [**7**]. Formula (2.3) now follows from naturality of functional operations ([**7**, 14.6]) and the fact that the indeterminacy of $\delta Sq^2_{\xi,\eta}(w_2)$ is $(P_1(\xi) - P_1(\eta))$. Formula (2.4) follows from (2.3), the defining diagram of $\delta Sq^2_{\xi,\eta}$ and the fact that Sq^3 is zero on 2-dimensional classes. Formula (2.5) follows from Theorem 3 in [**4**] and may be regarded as arising from naturality in the same way as (2.3).

We turn now to the proof of Theorem 1. If $w_2(\xi) = w_2(\eta)$, $P_1(\xi) = P_1(\eta)$, and $0 \in \delta Sq^2_{\xi,\eta}(w_2)$, it follows immediately from (2.2), (2.3), and the fact that $\pi_i(BSO) = 0$, $5 \leqq i \leqq 7$, that the restrictions of ξ and η to the 7-skeleton of X are stably isomorphic. If $P_2(\xi) = P_2(\eta)$, formula (2.5) reduces to the containments $2O^8(\xi, \eta) \equiv O^8(f'f, f'g) = \delta P^1_{f'f,f'g}(\iota) = \delta P^1_{\xi,\eta}(P(1))$, where f and g classify ξ and η and $f' : BSO \to K(\mathbf{Z}, 4; \mathbf{Z}, 8, \delta P^1)$ is a lifting of $P(1)$. Let $K = K(\mathbf{Z}, 4; \mathbf{Z}, 8, \delta P^1)$. The last of the three containments are equalities because it is easy to see that f' can be chosen in such a way that image $\{f'^* : H^8(K; \mathbf{Z}) \to H^8(BSO; \mathbf{Z})\}$ is contained in the kernel of the difference homomorphism $f^* - g^*$ and so the obstruction $O^8(f'f, f'g)$ is precisely $\delta P^1_{f'f,f'g}(\iota)$ by 10.8 in [**7**] and this functional operation has the same indeterminacy as $\delta P^1_{\xi,\eta}(P(1))$. The indeterminacy of $\delta P^1_{\xi,\eta}(P(1))$ is image δP^1 and the proof of Theorem 1 will be complete when we show that $2O^8(\xi, \eta)$ is not a proper subset of $\delta P^1_{\xi,\eta}(P(1))$. That is, we must show that $f_*'O^8(\xi, \eta) = O^8(f'f, f'g)$.

We view the problem of constructing a homotopy between f and g as the problem of extending the map on $X \times \dot{I}$ defined by f and g over $X \times I$. Let $h : (X \times I)^8 \to K$ be an extension of a homotopy of $f'f$ and $f'g$ over the 8-skeleton of $X \times I$. Regard the obstruction cohomology class $\{c^8(h)\}$ as an element in $H^8(X; \pi_8(K))$ and suppose that $\{c^8(\bar{h})\}$ is in $O^8(f, g)$. We assert that by altering h and $\bar{h}$ in such a way that $\{c^8(h)\}$ is unchanged, we may assume that $\{c^8(h)\}$ is in image f_*'. We begin proving this assertion by showing that we may assume that $O^3(f'\bar{h}, h)$ in $H^3(X; \pi_4(K))$ is zero. The map $f_\#' : \pi_4(BSO) \to \pi_4(K)$ is multiplication by 2 [**8**], and $H^3(X; \mathbf{Z})/\text{kernel } \delta P^1$ is a 3-torsion group, so the composite $H^3(X; \pi_4(BSO)) \to H^3(X; \mathbf{Z}) \to H^3(X; \mathbf{Z})/\text{kernel } \delta P^1$ is an epimorphism. Therefore, there is a class $\{\mu\}$ in $H^3(X; \pi_4(BSO))$ such that $f_*'\{\mu\} - O^3(f'\bar{h}, h) = \{\nu\}$, where $\{\nu\}$ is in kernel δP^1. Alter h by the cocycle ν to get a new homotopy of $f'f$ and $f'g$, h_ν, defined on the 8-skeleton of $X \times I$ such that $O^3(h, h_\nu) = \{\nu\}$ and hence $\{c^8(h)\} = \{c^8(h_\nu)\}$ since $\{c^8(h)\} - \{c^8(h_\nu)\} = \delta P^1 O^3(h, h_\nu)$ [**9**]. But $-O^3(f'\bar{h}, h) = O^3(h, h_\nu) + O^3(h_\nu, f'\bar{h})$ and so $f_*'\{\mu\} = O^3(f'\bar{h}, h_\nu)$. Altering $\bar{h}$ by μ, we obtain a homotopy of f and g, $\bar{h}_\mu$, defined over the 8-skeleton of $X \times I$ because $\pi_i(BSO) = 0$, $5 \leqq i \leqq 7$, such that $O^3(\bar{h}, \bar{h}_\mu) = \{\mu\}$. Since $O^3(f'\bar{h}_\mu, h_\nu) = O^3(f'\bar{h}_\mu, f'\bar{h}) + O^3(f'\bar{h}, h_\nu) = 0$, $f'\bar{h}_\mu \cong h_\nu$ over the 7-skeleton of $X \times I$ and the standard cocycle formula implies that $f_*'\{c^8(\bar{h}_\mu)\} = \{c^8(h_\nu)\} = \{c^8(h)\}$. The proof Theorem 1 is complete.

The proof of Theorem 2 is essentially the same as the proof of Theorem 1 and uses Theorem 2 in [**4**]. We need the fact that stable isomorphism and isomorphism are the same in the context of Theorem 2, that is, the map $[X; BU(n/2)] \to [X; BU]$ is a bijection when dimension $X \leqq n$. In this case, the map $f_{\#}' : \pi_4(BU) \to \pi_4(K)$ is the identity [**8**], and so there is a cocycle μ such that $f_*'\{\mu\} = O^3(f'\bar{h}, h)$. Since $\pi_6(BU) = \mathbf{Z}$, it is not clear that altering $\bar{h}$ by μ will produce a homotopy of f and g extendable over the 8-skeleton of $X \times I$. One alters h by ν where $\{\nu\} = -3O^3(f'\bar{h}, h)$. We then have $\{c^8(h)\} = \{c^8(h_\nu)\}$ and $O^3(f'\bar{h}, h_\nu) = -2O^3(f'\bar{h}, h)$ which is in kernel δSq^2 and so altering $\bar{h}$ by $\{\mu\} = -2O^3(f'\bar{h}, h)$ produces a homotopy of f and g defined over the 8-skeleton of $X \times I$.

The functional operations are non-trivial invariants of the classification problem. It is possible to give an example of a 7-manifold M and a 7-bundle over M, ξ, such that $w_2(\xi) = 0$, $P_1(\xi) = 0$ but ξ is not stably trivial. If 2-tor $H^4(M; \mathbf{Z})$ denotes the subgroup of $H^4(M; \mathbf{Z})$ of elements of order 2, it follows from Theorem 3.1 in [**10**] and Theorem 4.2 in [**11**] and Theorem 1, that $w_2(\xi) = 0$ and $P_1(\xi) = 0$ imply $\xi = 0$ for every stable orientable bundle if, and only if, the quotient group 2-tor $H^4(M; \mathbf{Z})$/image δSq^2 is zero. Take $M = L^7(m)$, a lens space of dimension 7 with fundamental group $\mathbf{Z}_m$, where m is even. Since Sq^2 is zero on 1-dimensional classes, the above quotient group is just 2-tor $H^4(L^7(m); \mathbf{Z})$ which is not zero since m is even, and so there is an orientable 7-bundle over $L^7(m)$ such that $w_2(\xi) = 0$ and $P_1(\xi) = 0$ but ξ is not stably trivial. If $m \equiv 0 \pmod 4$, there are elements of order 4 in $H^4(L^7(m); \mathbf{Z})$. In this case, (2.4) can be used to show that there is an orientable 7-bundle over $L^7(m)$ such that $w_i(\xi) = 0$, $i = 2$ and 4, $P_1(\xi) = 0$, but ξ is not stably trivial.

3. Applications. Let M be a connected, smooth n-manifold. A theorem of Whitney [**1**] says that if $n \geqq 1$, M immerses in $\mathbf{R}^{2n-1}$. Recall that M is called a *spin manifold* if M is closed, orientable and $w_2(M) = 0$. We will use Hirsch's theorem on immersions [**1**] together with Theorem 1 above to prove the two theorems below which represent improvements of Whitney's theorem in special cases.

THEOREM 3.1. *Every closed, orientable 5-manifold immerses in* $\mathbf{R}^8$.

THEOREM 3.2. *If* $n = 6$ *or* 7 *and* M *is a spin manifold, then* M *immerses in* $\mathbf{R}^{n+3}$.

Hirsch originally proved Theorem 3.1 by showing that the normal bundle of the Whitney immersion of M in $\mathbf{R}^9$ has a normal vector field and then applying his immersion theory. We prove this theorem in a different way, using a lemma about stable bundles and the Hirsch theory. Thomas has shown that if $n \equiv 3 \pmod 4$, then any spin n-manifold immerses in $\mathbf{R}^{2n-3}$, [**14**]. Theorem 3.2 sharpens Thomas' result by one dimension in the case $n = 7$.

If ξ is a bundle, let (ξ) denote its stable equivalence class. The stable bundle (ξ) is said to have *geometric dimension* $\leqq k$ (for some positive integer k) if (ξ) contains a k-plane bundle. For a smooth manifold M, let τM denote the tangent bundle and νM the stable normal bundle; i.e. $\nu M = -(\tau M)$. Hirsch's theorem says that M immerses in Euclidean space with codimension k if, and only if, geometric dimension $\nu M \leqq k$ [**1**]. Theorems 3.1 and 3.2 will follow from Hirsch's theorem and the lemma below. In the proof of the lemma, we will use the following fact: if ξ is an orientable bundle over X such that $w_4(\xi) = 0$ and γ is an orientable 3-bundle over X such that $w_2(\xi) = w_2(\gamma)$, then there is a class e in $H^4(X; \mathbf{Z})$ such that $P_1(\xi) - P_1(\gamma) = 4e$ and $2e \in O^4(\xi, \gamma)$. This fact follows immediately from (2.2) and (2.4).

LEMMA 3.3. *Let ξ be a stable, orientable bundle over a closed, orientable n-manifold, $5 \leqq n \leqq 7$. If $n \neq 5$, assume that $w_2(\xi) = w_2(M) = 0$. Then geometric dimension $\xi \leqq 3$ if, and only if, $w_4(\xi) = 0$.*

Proof. The condition is clearly necessary. We prove sufficiency first in the case $n = 5$. The argument begins by observing that if M is a closed, orientable 5-manifold and x is a class in $H^2(M; \mathbf{Z}_2)$, there exists an orientable 3-bundle γ over M such that $w_2(\gamma) = x$. This is proved by viewing the construction of γ as the extension of a map into $BSO(3)$ over M. It is clearly possible to construct a map g from the 3-skeleton of M into $BSO(3)$ such that $g^*w_2 = x$. Arguments similar to those used in the proof of (2.3) and the homotopy properties of $BSO(3)$ [**2**], show that g extends over M if $\delta Sq^2(g^*w_2) = 0$, but this is true since $H^5(M; \mathbf{Z})$ has no torsion. If $w_4(\xi) = 0$, let γ be an orientable 3-bundle such that $w_2(\xi) = w_2(\gamma)$, and let e be a class in $H^4(M; \mathbf{Z})$ such that $P_1(\xi) - P_1(\eta) = 4e$ and $2e \in O^4(\xi, \gamma)$. It follows from the homotopy sequence of the fibration $V_2(\mathbf{R}^5) = SO(5)/SO(3)$ and the fact that $\pi_3(V_2(\mathbf{R}^5)) = \mathbf{Z}_2$ [**2**], that the homomorphism $\pi_4(BSO(3)) \to \pi_4(BSO)$ is multiplication by 2. This means that it is possible to alter γ by a cocycle representing $-e$ and obtain a 3-bundle over M, γ', such that $O^4(\gamma, \gamma') = -2e$. Since $O^4(\xi, \gamma') = O^4(\xi, \gamma) + O^4(\gamma, \gamma')$, we have $0 \in O^4(\xi, \gamma')$ and hence geometric dimension $\xi \leqq 3$ since $\pi_5(BSO) = 0$.

If $n = 6$ or 7 and $w_4(\xi) = 0$, let e be a class in $H^4(M; \mathbf{Z})$ such that $P_1(\xi) = 4e$ and $2e \in O^4(\xi, *)$, where $*$ is the trivial stable bundle. There is a 3-bundle over S^4, $\hat{\gamma}$, such that $P_1(\hat{\gamma}) = 4\iota$ and so by (2.2), $O^4(\hat{\gamma}, *) = 2\iota$ since $H^4(S^4; \mathbf{Z})$ is torsion free. Since $Sq^2 e = 0$ and $0 \in \Phi(e)$, where Φ is the secondary operation associated with the relation $Sq^2 Sq^2 = O(\mathbf{Z})$, classical obstruction theory tells us that there is a map $g : M \to S^4$ such that $g^*\iota = e$, and so $\gamma = g^*\hat{\gamma}$ is a spin 3-bundle satisfying the conditions $P_1(\gamma) = 4e$ and $2e \in O^4(\gamma, *)$. (See [**13**].) Therefore ξ is stably isomorphic to γ since $O^4(\xi, \gamma) = O^4(\xi, *) - O^4(\gamma, *)$ and $\pi_i(BSO) = 0$, $5 \leqq i \leqq 7$. We have established that geometric dimension $\xi \leqq 3$.

Massey has shown that $w_{n-1}(\nu M) = 0$ for any closed, orientable n-manifold [**5**] and so Theorem 3.1 follows from this fact, Lemma 3.3, and Hirsch's theorem on immersions. If M is a spin n-manifold, $n = 6$ or 7, it follows from Wu's formula that $w_4(M) = 0$ and hence $w_4(\nu M) = 0$. Therefore, Theorem 3.2 follows from Lemma 3.3. There is reason to believe that Theorem 3.2 is true without the spin hypothesis: $w_4(\nu M) = 0$ for any closed, orientable n-manifold, $n = 6$ or 7 [**5**].

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