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GENERATORS FOR $\mathcal{H}$-INVARIANT PRIME IDEALS IN $O_q(\mathcal{M}_{m,p}(\mathbb{C}))$

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Abstract It is known that, for generic q, the $\mathcal{H}$-invariant prime ideals in $O_q(\mathcal{M}_{m,p}(\mathbb{C}))$ are generated by quantum minors (see S. Launois, Les idéaux premiers invariants de $O_q(\mathcal{M}_{m,p}(\mathbb{C}))$, *J. Alg.*, in press). In this paper, m and p being given, we construct an algorithm which computes a generating set of quantum minors for each $\mathcal{H}$-invariant prime ideal in $O_q(\mathcal{M}_{m,p}(\mathbb{C}))$. We also describe, in the general case, an explicit generating set of quantum minors for some particular $\mathcal{H}$-invariant prime ideals in $O_q(\mathcal{M}_{m,p}(\mathbb{C}))$. In particular, if $(Y_{i,\alpha})_{(i,\alpha)\in[\![1,m]\!]\times[\![1,p]\!]}$ denotes the matrix of the canonical generators of $O_q(\mathcal{M}_{m,p}(\mathbb{C}))$, we prove that, if $u \geqslant 3$, the ideal in $O_q(\mathcal{M}_{m,p}(\mathbb{C}))$ generated by $Y_{1,p}$ and the $u \times u$ quantum minors is prime. This result allows Lenagan and Rigal to show that the quantum determinantal factor rings of $O_q(\mathcal{M}_{m,p}(\mathbb{C}))$ are maximal orders (see T. H. Lenagan and L. Rigal, *Proc. Edinb. Math. Soc.* **46** (2003), 513–529).

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1. Introduction

Fix two positive integers m and p with $m, p \geqslant 2$ and consider some complex number q which is transcendental over $\mathbb{Q}$. Denote by $R = O_q(\mathcal{M}_{m,p}(\mathbb{C}))$ the quantization of the ring of regular functions on $m \times p$ matrices with entries in $\mathbb{C}$ (the field of complex numbers) and let $(Y_{i,\alpha})_{(i,\alpha)\in[\![1,m]\!]\times[\![1,p]\!]}$ denote the matrix of its canonical generators. There is an action of the torus $\mathcal{H} = (\mathbb{C}^*)^{m+p}$ on R by $\mathbb{C}$-automorphisms via

$$(a_1, \dots, a_m, b_1, \dots, b_p).Y_{i,\alpha} = a_i b_\alpha Y_{i,\alpha} \quad ((i,\alpha) \in [\![1,m]\!] \times [\![1,p]\!]).$$

(If $m = p$, this action is induced by the bialgebra structure of R and, if $m \neq p$, it is easy to check that the relations which define R are preserved by the group $\mathcal{H}$.)

It is known from work of Goodearl and Letzter that R has only finitely many $\mathcal{H}$-invariant prime ideals (see [**8**]) and that, in order to calculate the prime and primitive spectra of R, it is enough to determine the $\mathcal{H}$-invariant prime ideals of R (see [**8**, Theorem 6.6]).

In [**10**], we proved that the $\mathcal{H}$-invariant prime ideals in R are generated by quantum minors, as conjectured by Goodearl and Lenagan (see [**5**] and [**6**]). In this paper, we use this result, together with Cauchon's description for the set of $\mathcal{H}$-invariant prime ideals of R (see [**3**, Théorème 3.2.1]), to construct an algorithm which provides an explicit generating set of quantum minors for each $\mathcal{H}$-invariant prime ideal in R (see § 4). (Of course, these generating sets can be computed with this algorithm only when m and p have fixed values.)

The last part of this paper is devoted to the general case. We construct certain sets of quantum minors which generate prime ideals of R. In order to do that, we consider a new deleting-derivations algorithm (see [**2**]) that we define in § 5. Using this new tool, we can prove that, if $u \geqslant 3$, the ideal in $O_q(\mathcal{M}_{m,p}(\mathbb{C}))$ generated by $Y_{1,p}$ and the $u \times u$ quantum minors is prime. This result allows Lenagan and Rigal [**11**] to show that the quantum determinantal factor rings of $O_q(\mathcal{M}_{m,p}(\mathbb{C}))$ are maximal orders.

2. $\mathcal{H}$-invariant prime ideals in $O_q(\mathcal{M}_{m,p}(\mathbb{C}))$

Throughout this paper, we use the following conventions.

(i) $\mathbb{N}$, $\mathbb{Q}$ and $\mathbb{C}$ denote, respectively, the set of natural numbers, the field of rational numbers and the field of complex numbers. We set $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

(ii) If I is any non-empty finite subset of $\mathbb{N}$, $|I|$ denotes its cardinality.

(iii) $q \in \mathbb{C}$ is transcendental over $\mathbb{Q}$.

(iv) m and p denote two positive integers with $m, p \geqslant 2$.

(v) If k is a positive integer, S_k denotes the group of permutations of $[\![1, k]\!]$.

(vi) $R = O_q(\mathcal{M}_{m,p}(\mathbb{C}))$ denotes the quantization of the ring of regular functions on $m \times p$ matrices with entries in $\mathbb{C}$; it is the $\mathbb{C}$-algebra generated by the $m \times p$ indeterminates $Y_{i,\alpha}$, $1 \leqslant i \leqslant m$ and $1 \leqslant \alpha \leqslant p$, subject to the following relations.

If

$$\begin{pmatrix} x & y \\ z & t \end{pmatrix}$$

is any 2×2 sub-matrix of $\mathcal{Y} = (Y_{i,\alpha})_{(i,\alpha) \in [\![1,m]\!] \times [\![1,p]\!]}$, then

(a) $yx = q^{-1}xy$, $zx = q^{-1}xz$, $zy = yz$, $ty = q^{-1}yt$, $tz = q^{-1}zt$;

(b) $tx = xt - (q - q^{-1})yz$.

These relations agree with the relations used in [**3**], [**5**], [**6**], [**10**] and [**11**], but they differ from those of [**12**] by an interchange of q and q^{-1}. It is well known that the ring R is a Noetherian domain. We denote by F its skew field of fractions. Moreover, since q is transcendental over $\mathbb{Q}$, it follows from [**7**, Theorem 2.3] that all prime ideals of R are completely prime.

(vii) As in [**3**, § 2.1], one can show that the group $\mathcal{H} = (\mathbb{C}^*)^{m+p}$ acts on R by $\mathbb{C}$-algebra automorphisms via

$$(a_1, \dots, a_m, b_1, \dots, b_p).Y_{i,\alpha} = a_i b_\alpha Y_{i,\alpha} \quad \forall (i,\alpha) \in [\![1,m]\!] \times [\![1,p]\!].$$

(viii) An $\mathcal{H}$-*eigenvector* x of R is a non-zero element $x \in R$ such that $h.x \in \mathbb{C}^* x$ for each $h \in \mathcal{H}$. An ideal I of R is said to be $\mathcal{H}$-*invariant* if $h.I = I$ for all $h \in \mathcal{H}$. Let $\mathcal{H}$- Spec(R) denote the set of $\mathcal{H}$-invariant prime ideals of R.

The aim of this section is to describe the set $\mathcal{H}$- Spec(R) by using the standard deleting-derivations algorithm (see [**10**, § 2.1]).

Notation 2.1.

(i) We denote by $\leqslant_{\mathrm{s}}$ the lexicographic ordering on $\mathbb{N}^2$. We often call it *the standard ordering on* $\mathbb{N}^2$. Recall that

$$(i,\alpha) \leqslant_{\mathrm{s}} (j,\beta) \quad \Longleftrightarrow \quad [(i<j) \text{ or } (i=j \text{ and } \alpha \leqslant \beta)].$$

(ii) We set $E_{\mathrm{s}} = ([\![1,m]\!] \times [\![1,p]\!] \cup \{(m,p+1)\}) \setminus \{(1,1)\}$.

(iii) Let $(j,\beta) \in E_{\mathrm{s}}$. If $(j,\beta) \neq (m,p+1)$, $(j,\beta)^{+_{\mathrm{s}}}$ denotes the smallest element (relative to $\leqslant_{\mathrm{s}}$) of the set $\{(i,\alpha) \in E_{\mathrm{s}} \mid (j,\beta) <_{\mathrm{s}} (i,\alpha)\}$.

Notation 2.2. If $r = (j,\beta)$ and $v = (i,\alpha)$ belong to $[\![1,m]\!] \times [\![1,p]\!]$, we define a complex number $\lambda_{r,v}$ by

$$\text{if } r \neq v, \text{ then } \lambda_{r,v} = \begin{cases} q^{-1} & \text{if } i=j \text{ or } \alpha=\beta, \\ 1 & \text{otherwise,} \end{cases}$$
$$\text{if } r = v, \text{ then } \lambda_{r,v} = q^{-2}.$$

Recall that R can be written as an iterated Ore extension

$$R = \mathbb{C}[Y_{1,1}] \cdots [Y_{m,p}; \sigma_{m,p}, \delta_{m,p}],$$

where the indices are increasing for $\leqslant_{\mathrm{s}}$ and where, for $(1,2) \leqslant_{\mathrm{s}} r = (j,\beta) \leqslant_{\mathrm{s}} (m,p)$, σ_r is a $\mathbb{C}$-algebra automorphism and δ_r a $\mathbb{C}$-linear σ_r-derivation such that, for $(1,1) \leqslant_{\mathrm{s}} v = (i,\alpha) <_{\mathrm{s}} r = (j,\beta)$,

$$\sigma_r(Y_v) = \lambda_{r,v} Y_v,$$
$$\delta_r(Y_v) = \begin{cases} -(q-q^{-1}) Y_{i,\beta} Y_{j,\alpha} & \text{if } i<j \text{ and } \alpha<\beta, \\ 0 & \text{otherwise.} \end{cases}$$

In [**10**, § 2.1], we have shown that the theory of deleting derivations (see [**2**]) can be applied to the iterated Ore extension $R = \mathbb{C}[Y_{1,1}] \cdots [Y_{m,p}; \sigma_{m,p}, \delta_{m,p}]$. The corresponding algorithm is called *the standard deleting-derivations algorithm*. It consists of the construction, for each $r \in E_{\mathrm{s}}$, of a family $(Y_{i,\alpha}^{(r)_{\mathrm{s}}})_{(i,\alpha) \in [\![1,m]\!] \times [\![1,p]\!]}$ of elements of $F = \mathrm{Fract}(R)$, defined as follows.

(i) If $r=(m,p+1)$, then $Y_{i,\alpha}^{(m,p+1)_{\mathrm{s}}}=Y_{i,\alpha}$ for all $(i,\alpha)\in [\![1,m]\!]\times[\![1,p]\!]$.

(ii) Assume that $r=(j,\beta)<_{\mathrm{s}}(m,p+1)$ and that the $Y_{i,\alpha}^{(r^{+_{\mathrm{s}}})_{\mathrm{s}}}$ $((i,\alpha)\in [\![1,m]\!]\times[\![1,p]\!])$ are already known. For convenience of notation, we set

$$Y_{i,\alpha}^{(r^{+})_{\mathrm{s}}}=Y_{i,\alpha}^{(r^{+_{\mathrm{s}}})_{\mathrm{s}}}\quad\text{for }(i,\alpha)\in [\![1,m]\!]\times[\![1,p]\!].$$

If $(i,\alpha)\in [\![1,m]\!]\times[\![1,p]\!]$, then

$$Y_{i,\alpha}^{(r)_{\mathrm{s}}}=\begin{cases}Y_{i,\alpha}^{(r^{+})_{\mathrm{s}}}-Y_{i,\beta}^{(r^{+})_{\mathrm{s}}}(Y_{j,\beta}^{(r^{+})_{\mathrm{s}}})^{-1}Y_{j,\alpha}^{(r^{+})_{\mathrm{s}}} & \text{if } i<j \text{ and } \alpha<\beta,\\ Y_{i,\alpha}^{(r^{+})_{\mathrm{s}}} & \text{otherwise.}\end{cases}$$

Notation 2.3. Let $r\in E_{\mathrm{s}}$. We denote by $R^{(r)_{\mathrm{s}}}$ the subalgebra of $F=\mathrm{Fract}(R)$ generated by the $Y_{i,\alpha}^{(r)_{\mathrm{s}}}$ $((i,\alpha)\in [\![1,m]\!]\times[\![1,p]\!])$, that is,

$$R^{(r)_{\mathrm{s}}}=\mathbb{C}\langle Y_{i,\alpha}^{(r)_{\mathrm{s}}}\mid (i,\alpha)\in [\![1,m]\!]\times[\![1,p]\!]\rangle.$$

Remark 2.4. Let $r\in E_{\mathrm{s}}$ with $r\neq(m,p+1)$. We will often drop a subscript and write $R^{(r^{+})_{\mathrm{s}}}$ for $R^{(r^{+_{\mathrm{s}}})_{\mathrm{s}}}$.

Notation 2.5. We set $\bar{R}_{\mathrm{s}}=R^{(1,2)_{\mathrm{s}}}$ and $T_{i,\alpha}=Y_{i,\alpha}^{(1,2)_{\mathrm{s}}}$ for all $(i,\alpha)\in [\![1,m]\!]\times[\![1,p]\!]$.

Let $(j,\beta)\in E_{\mathrm{s}}$ with $(j,\beta)\neq(m,p+1)$. The theory of deleting derivations allows us to construct embeddings $\varphi_{(j,\beta)_{\mathrm{s}}}:\mathrm{Spec}(R^{(j,\beta)_{\mathrm{s}}^{+}})\to\mathrm{Spec}(R^{(j,\beta)_{\mathrm{s}}})$ (see [**2**, § 4.3]). By composition, we obtain an embedding $\varphi_{\mathrm{s}}:\mathrm{Spec}(R)\to\mathrm{Spec}(\bar{R}_{\mathrm{s}})$, which is called *the canonical embedding*. Now to describe the set $\mathcal{H}\text{-}\mathrm{Spec}(R)$ we just have to determine its canonical image $\varphi_{\mathrm{s}}(\mathcal{H}\text{-}\mathrm{Spec}(R))$. To do this, as in [**3**, Conventions 3.2.1], we introduce some conventions and notation.

Conventions 2.6.

(i) Let $v=(l,\gamma)\in [\![1,m]\!]\times[\![1,p]\!]$.

   (a) The set $C_v=\{(i,\gamma)\mid 1\leqslant i\leqslant l\}\subset [\![1,m]\!]\times[\![1,p]\!]$ is called the *truncated column with extremity* v.

   (b) The set $L_v=\{(l,\alpha)\mid 1\leqslant \alpha\leqslant \gamma\}\subset [\![1,m]\!]\times[\![1,p]\!]$ is called the *truncated row with extremity* v.

(ii) W denotes the set of all the subsets in $[\![1,m]\!]\times[\![1,p]\!]$ which are a union of truncated rows and columns.

Notation 2.7. Given $w\in W$, K_w denotes the ideal in $\bar{R}_{\mathrm{s}}$ generated by the $T_{i,\alpha}$ such that $(i,\alpha)\in w$. (Recall that K_w is a completely prime ideal in the quantum affine space $\bar{R}_{\mathrm{s}}$ (see [**9**, § 2.1]).)

The following result is proved in the same manner as [**3**, Corollaire 3.2.1].

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Proposition 2.8.

(i) *Given $w \in W$, there exists a (unique) $\mathcal{H}$-invariant (completely) prime ideal J_w in R such that $\varphi_{\mathrm{s}}(J_w) = K_w$.*

(ii) $\mathcal{H}\text{-}\operatorname{Spec}(R) = \{J_w \mid w \in W\}$.

3. The factor ring R/J_w

In this section, K denotes a $\mathbb{C}$-algebra which is also a skew field. Except where stated otherwise, all matrices considered have their entries in K.

Definition 3.1 (see Chapter 4 in [12]).

(i) Let u and v be two positive integers and let $M = (x_{i,\alpha})_{(i,\alpha)\in[\![1,u]\!]\times[\![1,v]\!]}$ be an $u \times v$ matrix. We say that M is a *q-quantum matrix* if the following relations hold between the entries of M:

$$\begin{aligned}
x_{i,\beta}x_{i,\alpha} &= q^{-1}x_{i,\alpha}x_{i,\beta} && (1 \leqslant i \leqslant u,\ 1 \leqslant \alpha < \beta \leqslant v),\\
x_{j,\alpha}x_{i,\alpha} &= q^{-1}x_{i,\alpha}x_{j,\alpha} && (1 \leqslant i < j \leqslant u,\ 1 \leqslant \alpha \leqslant v),\\
x_{j,\beta}x_{i,\alpha} &= x_{i,\alpha}x_{j,\beta} && (1 \leqslant i < j \leqslant u,\ 1 \leqslant \beta < \alpha \leqslant v),\\
x_{j,\beta}x_{i,\alpha} &= x_{i,\alpha}x_{j,\beta} - (q-q^{-1})x_{i,\beta}x_{j,\alpha} && (1 \leqslant i < j \leqslant u,\ 1 \leqslant \alpha < \beta \leqslant v).
\end{aligned}$$

(ii) Let n be a positive integer and let $M = (x_{i,\alpha})_{(i,\alpha)\in[\![1,n]\!]^2}$ be a square q-quantum matrix. The *quantum determinant* of M is defined by

$$\det_q(M) = \textstyle\sum_{\sigma\in S_n}(-q)^{l(\sigma)}x_{1,\sigma(1)}\cdots x_{n,\sigma(n)},$$

where $l(\sigma)$ denotes the length of the n-permutation σ.

(iii) Let $M = (x_{i,\alpha})_{(i,\alpha)\in[\![1,m]\!]\times[\![1,p]\!]}$ be a q-quantum matrix. The quantum determinant of a square sub-matrix of M is called a *quantum minor* of M.

Definition 3.2. Let $M = (x_{i,\alpha})_{(i,\alpha)\in[\![1,m]\!]\times[\![1,p]\!]}$ be an $m \times p$ matrix and let $(j,\beta) \in E_{\mathrm{s}}$. We say that M is a *$(j,\beta)_{\mathrm{s}}$-q-quantum matrix* if the following relations hold between the entries of M.

If

$$\begin{pmatrix} x & y \\ z & t \end{pmatrix}$$

is any 2×2 sub-matrix of M, then

(i) $yx = q^{-1}xy$, $zx = q^{-1}xz$, $zy = yz$, $ty = q^{-1}yt$, $tz = q^{-1}zt$;

(ii) if $t = x_v$, then $\begin{cases} v \geqslant_{\mathrm{s}} (j,\beta) \implies tx = xt, \\ v <_{\mathrm{s}} (j,\beta) \implies tx = xt - (q-q^{-1})yz. \end{cases}$

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Conventions 3.3 (see Convention 4.1.1 in [3] and Conventions 2.2.3 in [10]). Let

$$M = (x_{i,\alpha})_{(i,\alpha)\in[\![1,m]\!]\times[\![1,p]\!]}$$

be a q-quantum matrix. As r runs over the set E_{s}, we define matrices

$$M^{(r)_{\mathrm{s}}} = (x_{i,\alpha}^{(r)_{\mathrm{s}}})_{(i,\alpha)\in[\![1,m]\!]\times[\![1,p]\!]}$$

as follows.

(i) If $r = (m, p+1)$, then the entries of the matrix $M^{(m,p+1)_{\mathrm{s}}}$ are defined by $x_{i,\alpha}^{(m,p+1)_{\mathrm{s}}} = x_{i,\alpha}$ for all $(i,\alpha) \in [\![1,m]\!] \times [\![1,p]\!]$.

(ii) Assume that $r = (j,\beta) \in E_{\mathrm{s}} \setminus \{(m,p+1)\}$ and that the matrix $M^{(r^{+_{\mathrm{s}}})_{\mathrm{s}}}$ is already known. *For convenience of notation, we set* $M^{(r^+)_{\mathrm{s}}} = M^{(r^{+_{\mathrm{s}}})_{\mathrm{s}}}$ *and* $x_{i,\alpha}^{(r^+)_{\mathrm{s}}} = x_{i,\alpha}^{(r^{+_{\mathrm{s}}})_{\mathrm{s}}}$ *for each* $(i,\alpha) \in [\![1,m]\!] \times [\![1,p]\!]$. The entries $x_{i,\alpha}^{(r)_{\mathrm{s}}}$ of the matrix $M^{(r)_{\mathrm{s}}}$ are defined as follows.

(a) If $x_{j,\beta}^{(r^+)_{\mathrm{s}}} = 0$, then $x_{i,\alpha}^{(r)_{\mathrm{s}}} = x_{i,\alpha}^{(r^+)_{\mathrm{s}}}$ for all $(i,\alpha) \in [\![1,m]\!] \times [\![1,p]\!]$.

(b) If $x_{j,\beta}^{(r^+)_{\mathrm{s}}} \neq 0$ and $(i,\alpha) \in [\![1,m]\!] \times [\![1,p]\!]$, then

$$x_{i,\alpha}^{(r)_{\mathrm{s}}} = \begin{cases} x_{i,\alpha}^{(r^+)_{\mathrm{s}}} - x_{i,\beta}^{(r^+)_{\mathrm{s}}}(x_{j,\beta}^{(r^+)_{\mathrm{s}}})^{-1}x_{j,\alpha}^{(r^+)_{\mathrm{s}}} & \text{if } i < j \text{ and } \alpha < \beta, \\ x_{i,\alpha}^{(r^+)_{\mathrm{s}}} & \text{otherwise.} \end{cases}$$

We say that $M^{(r)_{\mathrm{s}}}$ is the matrix obtained from M by applying the standard deleting-derivations algorithm *at step* r.

(iii) If $r = (1,2)$, we set $t_{i,\alpha} = x_{i,\alpha}^{(1,2)_{\mathrm{s}}}$ for all $(i,\alpha) \in [\![1,m]\!] \times [\![1,p]\!]$.

Note that our definitions of q-quantum matrix and $(j,\beta)_{\mathrm{s}}$-q-quantum matrix slightly differ from those of [**1**] (see [**1**, Definitions III.1.1 and III.1.3]). Because of this, we must interchange q and q^{-1} whenever carrying over a result of [**1**].

Lemma 3.4. *Let* $(j,\beta) \in E_{\mathrm{s}}$. *If* $M = (x_{i,\alpha})_{(i,\alpha)\in[\![1,m]\!]\times[\![1,p]\!]}$ *is a* q*-quantum matrix, then the matrix* $M^{(j,\beta)_{\mathrm{s}}}$ *is* $(j,\beta)_{\mathrm{s}}$*-*q*-quantum.*

Proof. This lemma is proved in the same manner as [**1**, Proposition III.2.3.1]. □

The formulae of Conventions 3.3 allow us to express the entries of $M^{(r^+)_{\mathrm{s}}}$ in terms of those of $M^{(r)_{\mathrm{s}}}$.

Proposition 3.5 (restoration algorithm). *Let* $M = (x_{i,\alpha})_{(i,\alpha)\in[\![1,m]\!]\times[\![1,p]\!]}$ *be a* q*-quantum matrix and let* $r = (j,\beta) \in E_{\mathrm{s}}$ *with* $r \neq (m,p+1)$.

(1) *If* $x_{j,\beta}^{(r)_s} = 0$, *then* $x_{i,\alpha}^{(r^+)_s} = x_{i,\alpha}^{(r)_s}$ *for all* $(i,\alpha) \in [\![1,m]\!] \times [\![1,p]\!]$.

(2) *If* $x_{j,\beta}^{(r)_s} \neq 0$ *and* $(i,\alpha) \in [\![1,m]\!] \times [\![1,p]\!]$, *then*

$$x_{i,\alpha}^{(r^+)_s} = \begin{cases} x_{i,\alpha}^{(r)_s} + x_{i,\beta}^{(r)_s}(x_{j,\beta}^{(r)_s})^{-1}x_{j,\alpha}^{(r)_s} & \text{if } i < j \text{ and } \alpha < \beta, \\ x_{i,\alpha}^{(r)_s} & \text{otherwise.} \end{cases}$$

We now come back to the $\mathcal{H}$-invariant prime ideals J_w of R (see the notation of § 2). The aim of the rest of this section is to study the effect of the standard deleting-derivations algorithm on the matrix whose entries are $y_{i,\alpha} = Y_{i,\alpha} + J_w$ $((i,\alpha) \in [\![1,m]\!] \times [\![1,p]\!])$.

Notation 3.6. Let $w \in W$.

(i) Set $R_w = R/J_w$. It follows from [**2**, Lemme 5.3.3] that R_w and $\bar{R}_s/K_w$ are two Noetherian algebras with no zero-divisors and which have the same skew field of fractions. We set $F_w = \mathrm{Fract}(R_w) = \mathrm{Fract}(\bar{R}_s/K_w)$.

(ii) If $(i,\alpha) \in [\![1,m]\!] \times [\![1,p]\!]$, then $y_{i,\alpha}$ denotes the element of R_w defined by $y_{i,\alpha} = Y_{i,\alpha} + J_w$.

(iii) We denote by M_w the matrix, with entries in the $\mathbb{C}$-algebra F_w, defined by

$$M_w = (y_{i,\alpha})_{(i,\alpha)\in[\![1,m]\!]\times[\![1,p]\!]}.$$

Let $w \in W$. Since $\mathcal{Y} = (Y_{i,\alpha})_{(i,\alpha)\in[\![1,m]\!]\times[\![1,p]\!]}$ is a q-quantum matrix, M_w is also a q-quantum matrix. Thus, we can apply the standard deleting-derivations algorithm to M_w (see Conventions 3.3 with $K = F_w$) and if we still denote $t_{i,\alpha} = y_{i,\alpha}^{(1,2)_s}$ for $(i,\alpha) \in [\![1,m]\!] \times [\![1,p]\!]$, we get the following theorem.

Theorem 3.7.

(i) $M_w^{(1,2)_s}$ *is* $(1,2)_s$-*q-quantum.*

(ii) $t_{i,\alpha} = 0$ *if and only if* $(i,\alpha) \in w$.

(iii) *There is a* $\mathbb{C}$-*algebra isomorphism from* $\mathbb{C}\langle t_{i,\alpha} \mid (i,\alpha) \notin w\rangle$ *onto the subalgebra* $\mathbb{C}\langle T_{i,\alpha} \mid (i,\alpha) \notin w\rangle$ *of* $\bar{R}_s$, *which sends* $t_{i,\alpha}$ *onto* $T_{i,\alpha}$ *for each* $(i,\alpha) \notin w$.

Proof. The first point follows from Lemma 3.4. By [**2**, Propositions 5.4.1 and 5.4.2], there exists a $\mathbb{C}$-algebra homomorphism $f_{(1,2)} : \bar{R}_s \to F_w$ such that $f_{(1,2)}(T_{i,\alpha}) = t_{i,\alpha}$ for $(i,\alpha) \in [\![1,m]\!] \times [\![1,p]\!]$. Its kernel is K_w and its image is the subalgebra of F_w generated by the $t_{i,\alpha}$ with $(i,\alpha) \in [\![1,m]\!] \times [\![1,p]\!]$. Hence $t_{i,\alpha} = 0$ if and only if $T_{i,\alpha} \in K_w$, that is, if and only if $(i,\alpha) \in w$, and

$$\mathbb{C}\langle t_{i,\alpha} \mid (i,\alpha) \notin w\rangle \simeq \bar{R}_s/K_w \simeq \mathbb{C}\langle T_{i,\alpha} \mid (i,\alpha) \notin w\rangle.$$

□

4. An algorithm which computes a generating set for J_w

Input. Fix $w \in W$. Denote by $M_w = (y_{i,\alpha})_{(i,\alpha)\in[\![1,m]\!]\times[\![1,p]\!]}$ the matrix whose entries are

$$y_{i,\alpha} = Y_{i,\alpha} + J_w \quad ((i,\alpha) \in [\![1,m]\!] \times [\![1,p]\!]).$$

It follows from Theorem 3.7 that $M_w^{(1,2)_s} = (t_{i,\alpha})_{(i,\alpha)\in[\![1,m]\!]\times[\![1,p]\!]}$ is a $(1,2)_s$-q-quantum matrix whose entries have the following properties.

(i) $t_{i,\alpha} = 0$ if and only if $(i,\alpha) \in w$.

(ii) There is an isomorphism from $\mathbb{C}\langle t_{i,\alpha} \mid (i,\alpha) \notin w\rangle$ onto $\mathbb{C}\langle T_{i,\alpha} \mid (i,\alpha) \notin w\rangle$, which sends $t_{i,\alpha}$ onto $T_{i,\alpha}$ $((i,\alpha) \notin w)$.

Step 1: restoration of M_w. Starting with the matrix $M_w^{(1,2)_s}$, we compute the matrix

$$M_w = (y_{i,\alpha})_{(i,\alpha)\in[\![1,m]\!]\times[\![1,p]\!]}$$

by using the restoration algorithm (see Proposition 3.5). This is a q-quantum matrix with entries in the McConnell–Pettit algebra $\mathbb{C}\langle t_{i,\alpha}^{\pm 1} \mid (i,\alpha) \notin w\rangle$.

Step 2: we calculate all quantum minors of M_w.

Result. Let

$$X_w = \{(I,\Lambda) \mid I \subseteq [\![1,m]\!],\ \Lambda \subseteq [\![1,p]\!],\ |I| = |\Lambda| \text{ and } \det_q(y_{i,\alpha})_{(i,\alpha)\in I\times\Lambda} = 0\}.$$

Then J_w is generated, as right and left ideal, by the quantum minors $\det_q(Y_{i,\alpha})_{(i,\alpha)\in I\times\Lambda}$ with $(I,\Lambda) \in X_w$.

Proof. This is immediate from [**10**, Théorème 3.7.2]. □

Example 4.1. Assume that $m = p = 3$. If this algorithm is applied to $w = \{(1,1), (1,3), (2,1), (2,2)\}$, one can show that the two-sided ideal in $O_q(\mathcal{M}_3(\mathbb{C}))$ generated by

$$Y_{1,3},\quad \det_q\begin{pmatrix} Y_{1,1} & Y_{1,2} \\ Y_{2,1} & Y_{2,2}\end{pmatrix},\quad \det_q\begin{pmatrix} Y_{1,1} & Y_{1,2} \\ Y_{3,1} & Y_{3,2}\end{pmatrix},$$
$$\det_q\begin{pmatrix} Y_{2,1} & Y_{2,2} \\ Y_{3,1} & Y_{3,2}\end{pmatrix},\quad \det_q\begin{pmatrix} Y_{2,1} & Y_{2,3} \\ Y_{3,1} & Y_{3,3}\end{pmatrix}\quad \text{and}\quad \det_q\begin{pmatrix} Y_{2,2} & Y_{2,3} \\ Y_{3,2} & Y_{3,3}\end{pmatrix}$$

is (completely) prime. In the more general case where we just assume that $q \in \mathbb{C}^*$ is not a root of unity, this result was proved by Goodearl and Lenagan (see [**6**, § 7.2]) by using different methods.

5. The last-column deleting-derivations algorithm

The aim of this section is to define a new deleting-derivations algorithm which will allow us to show that certain sets of quantum minors generate prime ideals of R. We shall only use it when m and p are greater than or equal to 3, so for the remainder of this section, we assume that $\min(m,p) \geqslant 3$ (although most of the following results are still true when $m = 2$ or $p = 2$).

Definition 5.1. Define the relation $\leqslant_{\mathrm{dc}}$ by

$$(i,\alpha) \leqslant_{\mathrm{dc}} (j,\beta) \iff [(\alpha = \beta = p \text{ and } i \leqslant j) \text{ or } (\beta = p \text{ and } \alpha < p) \text{ or } (\alpha,\beta < p \text{ and } (i,\alpha) \leqslant_{\mathrm{s}} (j,\beta))].$$

This defines a total ordering on $[\![1,m]\!] \times [\![1,p]\!] \cup \{(m+1,p)\}$ that we will call *the last-column ordering on* $[\![1,m]\!] \times [\![1,p]\!] \cup \{(m+1,p)\}$.

Notation 5.2.

(i) We set $E_{\mathrm{dc}} = ([\![1,m]\!] \times [\![1,p]\!] \cup \{(m+1,p)\}) \setminus \{(1,1)\}$.

(ii) Let $(j,\beta) \in E_{\mathrm{dc}}$. If $(j,\beta) \neq (m+1,p)$, denote by $(j,\beta)^{+_{\mathrm{dc}}}$ the smallest element (relative to $\leqslant_{\mathrm{dc}}$) of the set $\{(i,\alpha) \in E_{\mathrm{dc}} \mid (j,\beta) <_{\mathrm{dc}} (i,\alpha)\}$.

Using [**2**, Propositions 6.1.1 and 6.1.2], we get the following theorem.

Theorem 5.3.

(1) *R can be written as an iterated Ore extension*

$$R = \mathbb{C}[Y_{1,1}]\cdots[Y_{m,p-1};\sigma'_{m,p-1},\delta'_{m,p-1}][Y_{1,p};\sigma'_{1,p},\delta'_{1,p}]\cdots[Y_{m,p};\sigma'_{m,p},\delta'_{m,p}],$$

where the indices are increasing for $\leqslant_{\mathrm{dc}}$ and where, for $(1,2) \leqslant_{\mathrm{dc}} r = (j,\beta) \leqslant_{\mathrm{dc}} (m,p)$, σ'_r is a $\mathbb{C}$-algebra automorphism and δ'_r a $\mathbb{C}$-linear σ'_r-derivation such that, for $(1,1) \leqslant_{\mathrm{dc}} v = (i,\alpha) <_{\mathrm{dc}} r = (j,\beta)$,

$$\sigma'_r(Y_v) = \lambda_{r,v} Y_v \quad (\lambda_{r,v} \text{ was defined in Notation 2.2});$$

$$\delta'_r(Y_v) = \begin{cases} -(q-q^{-1})Y_{i,\beta}Y_{j,\alpha} & \text{if } i < j \text{ and } \alpha < \beta, \\ 0 & \text{otherwise.} \end{cases}$$

(2) *R satisfies Conventions 3.1 of [**2**] with $q_r = q^{-2}$ for any $r \in [\![1,m]\!] \times [\![1,p]\!]$.*

(3) *If $r \in [\![1,m]\!] \times [\![1,p]\!] \setminus \{(1,1)\}$, there exists $h'_r \in \mathcal{H}$ such that $h'_r.Y_v = \lambda_{r,v} Y_v$ for $v \in [\![1,m]\!] \times [\![1,p]\!]$. Thus, R satisfies Hypotheses 4.1.2 of [**2**] with the group $\mathcal{H}$.*

It follows from the previous theorem that the theory of deleting derivations (see [**2**]) can be applied to the iterated Ore extension $R = \mathbb{C}[Y_{1,1}]\cdots[Y_{m,p};\sigma'_{m,p},\delta'_{m,p}]$. The corresponding algorithm is called *the last-column deleting-derivations algorithm.*

Let $r = (j,\beta) \in [\![1,m]\!] \times [\![1,p]\!]$ with $(1,1) <_{\mathrm{dc}} r$. Denote by B the subalgebra of R generated by the Y_v with $v \in [\![1,m]\!] \times [\![1,p]\!]$ and $(1,1) \leqslant_{\mathrm{dc}} v <_{\mathrm{dc}} r$, and let C be the subalgebra of R generated by B and Y_r. It follows from Theorem 5.3 that C is the (left) Ore extension $B[Y_r;\sigma'_r,\delta'_r]$, and that, in $F = \mathrm{Fract}(R)$, we have

$$\sum_{k=0}^{+\infty} \frac{(1-q^{-2})^{-k}}{[k]!_{q^{-2}}} \lambda_{r,v}^{-k} \delta_r'^{k}(Y_v) Y_r^{-k} = \begin{cases} Y_v - Y_{i,\beta} Y_{j,\beta}^{-1} Y_{j,\alpha} & \text{if } i<j \text{ and } \alpha<\beta, \\ Y_v & \text{otherwise,} \end{cases}$$

where $[k]!_{q^{-2}} = [0]_{q^{-2}} \times \cdots \times [k]_{q^{-2}}$ with $[0]_{q^{-2}} = 1$ and $[i]_{q^{-2}} = 1 + q^{-2} + \cdots + q^{-2(i-1)}$ if i is a positive integer.

Hence, the last-column deleting-derivations algorithm consists of the construction, for each $r \in E_{\mathrm{dc}}$, of a family $(Y_{i,\alpha}^{(r)_{\mathrm{dc}}})_{(i,\alpha)\in[\![1,m]\!]\times[\![1,p]\!]}$ of elements of $F = \mathrm{Fract}(R)$, defined as follows.

(1) If $r = (m+1,p)$, then $Y_{i,\alpha}^{(m+1,p)_{\mathrm{dc}}} = Y_{i,\alpha}$ for all $(i,\alpha) \in [\![1,m]\!] \times [\![1,p]\!]$.

(2) Assume that $r = (j,\beta) <_{\mathrm{dc}} (m+1,p)$ and that the

$$Y_{i,\alpha}^{(r^{+_{\mathrm{dc}}})_{\mathrm{dc}}} \quad ((i,\alpha) \in [\![1,m]\!] \times [\![1,p]\!])$$

are already known. For convenience of notation, we set $Y_{i,\alpha}^{(r^+)_{\mathrm{dc}}} = Y_{i,\alpha}^{(r^{+_{\mathrm{dc}}})_{\mathrm{dc}}}$ for $(i,\alpha) \in [\![1,m]\!] \times [\![1,p]\!]$.

If $(i,\alpha) \in [\![1,m]\!] \times [\![1,p]\!]$, then

$$Y_{i,\alpha}^{(r)_{\mathrm{dc}}} = \begin{cases} Y_{i,\alpha}^{(r^+)_{\mathrm{dc}}} - Y_{i,\beta}^{(r^+)_{\mathrm{dc}}} (Y_{j,\beta}^{(r^+)_{\mathrm{dc}}})^{-1} Y_{j,\alpha}^{(r^+)_{\mathrm{dc}}} & \text{if } i<j \text{ and } \alpha<\beta, \\ Y_{i,\alpha}^{(r^+)_{\mathrm{dc}}} & \text{otherwise.} \end{cases}$$

6. A link between the standard and last-column deleting-derivations algorithms

Throughout this section, we use the following conventions.

(i) We assume that $\min(m,p) \geqslant 3$.

(ii) K denotes a $\mathbb{C}$-algebra which is also a skew field. All the matrices considered have their entries in K.

Conventions 6.1. Let $M = (x_{i,\alpha})_{(i,\alpha)\in[\![1,m]\!]\times[\![1,p]\!]}$ be a q-quantum matrix. As r runs over the set E_{dc}, we define matrices

$$M^{(r)_{\mathrm{dc}}} = (x_{i,\alpha}^{(r)_{\mathrm{dc}}})_{(i,\alpha)\in[\![1,m]\!]\times[\![1,p]\!]}$$

as follows.

(1) If $r = (m+1,p)$, then the entries of the matrix $M^{(m+1,p)_{\mathrm{dc}}}$ are defined by $x_{i,\alpha}^{(m+1,p)_{\mathrm{dc}}} = x_{i,\alpha}$ for all $(i,\alpha) \in [\![1,m]\!] \times [\![1,p]\!]$.

(2) Assume that $r=(j,\beta)\in E_{\mathrm{dc}}\setminus\{(m+1,p)\}$ and that the matrix $M^{(r^{+\mathrm{dc}})_{\mathrm{dc}}}$ is already known. We set

$$M^{(r^+)_{\mathrm{dc}}}=M^{(r^{+\mathrm{dc}})_{\mathrm{dc}}}\quad\text{and}\quad x_{i,\alpha}^{(r^+)_{\mathrm{dc}}}=x_{i,\alpha}^{(r^{+\mathrm{dc}})_{\mathrm{dc}}}\quad\text{for each }(i,\alpha)\in[\![1,m]\!]\times[\![1,p]\!].$$

The entries $x_{i,\alpha}^{(r)_{\mathrm{dc}}}$ of the matrix $M^{(r)_{\mathrm{dc}}}$ are defined as follows.

(a) If $x_{j,\beta}^{(r^+)_{\mathrm{dc}}}=0$, then $x_{i,\alpha}^{(r)_{\mathrm{dc}}}=x_{i,\alpha}^{(r^+)_{\mathrm{dc}}}$ for all $(i,\alpha)\in[\![1,m]\!]\times[\![1,p]\!]$.

(b) If $x_{j,\beta}^{(r^+)_{\mathrm{dc}}}\neq 0$ and $(i,\alpha)\in[\![1,m]\!]\times[\![1,p]\!]$, then

$$x_{i,\alpha}^{(r)_{\mathrm{dc}}}=\begin{cases}x_{i,\alpha}^{(r^+)_{\mathrm{dc}}}-x_{i,\beta}^{(r^+)_{\mathrm{dc}}}(x_{j,\beta}^{(r^+)_{\mathrm{dc}}})^{-1}x_{j,\alpha}^{(r^+)_{\mathrm{dc}}} & \text{if } i<j \text{ and } \alpha<\beta,\\ x_{i,\alpha}^{(r^+)_{\mathrm{dc}}} & \text{otherwise.}\end{cases}$$

We say that $M^{(r)_{\mathrm{dc}}}$ is the matrix obtained from M by applying the last-column deleting-derivations algorithm *at step* r.

Let $M=(x_{i,\alpha})_{(i,\alpha)\in[\![1,m]\!]\times[\![1,p]\!]}$ be a q-quantum matrix. The following lemma is obvious.

Lemma 6.2.

(1) $M^{(m,p)_{\mathrm{s}}}=M^{(m,p)_{\mathrm{dc}}}$.

(2) *If* $(j,\beta)\in E_{\mathrm{s}}\cap E_{\mathrm{dc}}=[\![1,m]\!]\times[\![1,p]\!]\setminus\{(1,1)\}$, *then*

$$\begin{aligned}x_{m,\alpha}^{(j,\beta)_{\mathrm{s}}}&=x_{m,\alpha}^{(j,\beta)_{\mathrm{dc}}}=x_{m,\alpha}\quad\textit{for any }\alpha\in[\![1,p]\!],\\ x_{i,p}^{(j,\beta)_{\mathrm{s}}}&=x_{i,p}^{(j,\beta)_{\mathrm{dc}}}=x_{i,p}\quad\textit{for any }i\in[\![1,m]\!].\end{aligned}$$

Proposition 6.3. *If* $M=(x_{i,\alpha})_{(i,\alpha)\in[\![1,m]\!]\times[\![1,p]\!]}$ *is a q-quantum matrix, then* $M^{(1,2)_{\mathrm{s}}}=M^{(1,2)_{\mathrm{dc}}}$.

Since the proof of this result is very technical, we just give a sketch.

Proof. First, if $i=m$ or $\alpha=p$, it follows from Lemma 6.2 that $x_{i,\alpha}^{(1,2)_{\mathrm{dc}}}=t_{i,\alpha}$. Now we assume that $i\leqslant m-1$ and $\alpha\leqslant p-1$. A decreasing induction shows that, if $j\in[\![1,m+1]\!]$, then

$$x_{i,\alpha}=x_{i,\alpha}^{(j,p)_{\mathrm{dc}}}+\sum_{\substack{k=\max(i+1,j)\\ x_{k,p}\neq 0}}^{m}x_{i,p}x_{k,p}^{-1}x_{k,\alpha}^{(j,p)_{\mathrm{dc}}}.$$

In particular, for $j=1$, we obtain

$$x_{i,\alpha}=x_{i,\alpha}^{(1,p)_{\mathrm{dc}}}+\sum_{\substack{k=i+1\\ x_{k,p}\neq 0}}^{m}x_{i,p}x_{k,p}^{-1}x_{k,\alpha}^{(1,p)_{\mathrm{dc}}}.\tag{6.1}$$

Now, we easily deduce the following equalities from (6.1):

$$x_{i,\alpha}^{(m,p)_{\mathrm{s}}} = x_{i,\alpha}^{(1,p)_{\mathrm{dc}}} + \sum_{\substack{k=i+1 \\ x_{k,p}\neq 0}}^{m-1} x_{i,p} x_{k,p}^{-1} x_{k,\alpha}^{(1,p)_{\mathrm{dc}}}$$

and

$$x_{i,\alpha}^{(m,p-1)_{\mathrm{s}}} = x_{i,\alpha}^{(m,p-1)_{\mathrm{dc}}} + \sum_{\substack{k=i+1 \\ x_{k,p}\neq 0}}^{m-1} x_{i,p} x_{k,p}^{-1} x_{k,\alpha}^{(m,p-1)_{\mathrm{dc}}}.$$

Next, by a decreasing induction (with respect to $\leqslant_{\mathrm{dc}}$), we can show that, if $(j,\beta) \in E_{\mathrm{dc}}$ with $(j,\beta) \leqslant_{\mathrm{dc}} (m,p-1)$, then

$$x_{i,\alpha}^{(j,\beta)_{\mathrm{s}}} = x_{i,\alpha}^{(j,\beta)_{\mathrm{dc}}} + \sum_{\substack{k=i+1 \\ x_{k,p}\neq 0}}^{j-1} x_{i,p} x_{k,p}^{-1} x_{k,\alpha}^{(j,\beta)_{\mathrm{dc}}}. \tag{6.2}$$

For $(j,\beta) = (1,2)$, equality (6.2) becomes $t_{i,\alpha} = x_{i,\alpha}^{(1,2)_{\mathrm{dc}}}$, and Proposition 6.3 follows. $\square$

Corollary 6.4. *Let $M = (x_{i,\alpha})_{(i,\alpha)\in[\![1,m]\!]\times[\![1,p]\!]}$ be a q-quantum matrix. Then*

$$x_{j,\beta}^{(j,\beta)^+_{\mathrm{dc}}} = t_{j,\beta} \quad \textit{for any } (j,\beta) \in [\![1,m]\!] \times [\![1,p]\!].$$

Proof. Since

$$x_{j,\beta}^{(j,\beta)^+_{\mathrm{dc}}} = x_{j,\beta}^{(1,2)_{\mathrm{dc}}} \quad \text{for any } (j,\beta) \in [\![1,m]\!] \times [\![1,p]\!],$$

this corollary is an immediate consequence of Proposition 6.3. $\square$

7. The effect of the last-column deleting-derivations algorithm on quantum minors

Throughout this section, we keep the conventions and notation of § 6.

Definition 7.1. Let $M = (x_{i,\alpha})_{(i,\alpha)\in[\![1,m]\!]\times[\![1,p]\!]}$ be a matrix and let $(j,\beta) \in E_{\mathrm{dc}}$. We say that M is a $(j,\beta)_{\mathrm{dc}}$-*q-quantum matrix* if the following relations hold between the entries of M.

If

$$\begin{pmatrix} x & y \\ z & t \end{pmatrix}$$

is any 2×2 sub-matrix of M, then

(1) $yx = q^{-1}xy$, $zx = q^{-1}xz$, $zy = yz$, $ty = q^{-1}yt$, $tz = q^{-1}zt$;

(2) if $t = x_v$, then $\begin{cases} v \geqslant_{\mathrm{dc}} (j,\beta) & \Longrightarrow \quad tx = xt, \\ v <_{\mathrm{dc}} (j,\beta) & \Longrightarrow \quad tx = xt - (q-q^{-1})yz. \end{cases}$

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Lemma 7.2. *Let $(j,\beta) \in E_{\mathrm{dc}}$. If $M = (x_{i,\alpha})_{(i,\alpha)\in[\![1,m]\!]\times[\![1,p]\!]}$ is a q-quantum matrix, then $M^{(j,\beta)_{\mathrm{dc}}}$ is a $(j,\beta)_{\mathrm{dc}}$-q-quantum matrix.*

Proof. First, it follows from [**2**, Théorème 3.2.1] that the matrix

$$(Y_{i,\alpha}^{(j,\beta)_{\mathrm{dc}}})_{(i,\alpha)\in[\![1,m]\!]\times[\![1,p]\!]}$$

(see § 5) is a $(j,\beta)_{\mathrm{dc}}$-q-quantum matrix. The rest of the proof is similar to [**10**, Lemme 2.5.3]. □

The following result can be deduced easily from this lemma.

Corollary 7.3. *Let M be an $m \times p$ q-quantum matrix and let $(j,\beta) \in E_{\mathrm{dc}}$.*

(1) *If $\beta = p$, then*

  (a) *the matrix obtained from $M^{(j,\beta)_{\mathrm{dc}}}$ by deleting the last column is q-quantum;*

  (b) *the matrix obtained from $M^{(j,\beta)_{\mathrm{dc}}}$ by deleting the rows $j,\dots,m$ is q-quantum $(j>1)$.*

(2) *If $\beta < p$, then*

  (a) *the matrix obtained from $M^{(j,\beta)_{\mathrm{dc}}}$ by deleting the rows $j,\dots,m$ and the last column is q-quantum $(j>1)$;*

  (b) *the matrix obtained from $M^{(j,\beta)_{\mathrm{dc}}}$ by deleting the rows $j+1,\dots,m$ and the columns $\beta,\dots,p$ is q-quantum $(j,\beta>1)$.*

Let $M = (x_{i,\alpha})_{(i,\alpha)\in[\![1,m]\!]\times[\![1,p]\!]}$ be a q-quantum matrix. We now express the quantum minors of $M^{(j,\beta)^{+}_{\mathrm{dc}}}$ in terms of those of $M^{(j,\beta)_{\mathrm{dc}}}$.

Notation 7.4. Let $M = (x_{i,\alpha})_{(i,\alpha)\in[\![1,m]\!]\times[\![1,p]\!]}$ be a q-quantum matrix and let

$$\delta = \det_q(x_{i,\alpha})_{\substack{i=i_1,\dots,i_l\\ \alpha=\alpha_1,\dots,\alpha_l}}$$

be an $l \times l$ quantum minor of M ($1 \leqslant l \leqslant \min(m,p)$, $1 \leqslant i_1 < \cdots < i_l \leqslant m$, $1 \leqslant \alpha_1 < \cdots < \alpha_l \leqslant p$).

(1) (a) If I is a non-empty subset of $\{i_1,\dots,i_l\}$, we set $\hat{I} = \{i_1,\dots,i_l\} \setminus I$. In the particular case where $I = \{i_k\}$ ($k \in [\![1,l]\!]$), we set $\hat{i}_k = \hat{I}$.

  (b) If Λ is a non-empty subset of $\{\alpha_1,\dots,\alpha_l\}$, we set $\bar{\Lambda} = \{\alpha_1,\dots,\alpha_l\} \setminus \Lambda$. In the particular case where $\Lambda = \{\alpha_k\}$ ($k \in [\![1,l]\!]$), we set $\bar{\alpha}_k = \bar{\Lambda}$.

  (Observe that the set $\hat{I}$ (respectively, $\bar{\Lambda}$) depends on the set $\{i_1,\dots,i_l\}$ (respectively, $\{\alpha_1,\dots,\alpha_l\}$).)

(2) If $(j,\beta) \in E_{\mathrm{dc}}$ is greater (relative to $\leqslant_{\mathrm{dc}}$) than the elements of $\{i_1,\dots,i_l\} \times \{\alpha_1,\dots,\alpha_l\}$, it follows from Lemma 7.2 that the matrix

$$(x_{i,\alpha}^{(j,\beta)_{\mathrm{dc}}})_{(i,\alpha)\in\{i_1,\dots,i_l\}\times\{\alpha_1,\dots,\alpha_l\}}$$

is q-quantum. We set

$$\delta^{(j,\beta)_{\mathrm{dc}}} = \det_q(x_{i,\alpha}^{(j,\beta)_{\mathrm{dc}}})_{(i,\alpha)\in\{i_1,\dots,i_l\}\times\{\alpha_1,\dots,\alpha_l\}}.$$

(3) Let I be a non-empty subset of $\{i_1,\dots,i_l\}$ and let Λ be a non-empty subset of $\{\alpha_1,\dots,\alpha_l\}$ with $|I| = |\Lambda|$.

(a) We set $\delta_{\hat{I},\bar{\Lambda}} = \det_q(x_{i,\alpha})_{(i,\alpha)\in\hat{I}\times\bar{\Lambda}}$.

(b) If $(j,\beta) \in E_{\mathrm{dc}}$ is greater (relative to $\leqslant_{\mathrm{dc}}$) than the elements of $\hat{I}\times\bar{\Lambda}$, it follows from Lemma 7.2 that the matrix $(x_{i,\alpha}^{(j,\beta)_{\mathrm{dc}}})_{(i,\alpha)\in\hat{I}\times\bar{\Lambda}}$ is q-quantum. We set

$$\delta_{\hat{I},\bar{\Lambda}}^{(j,\beta)_{\mathrm{dc}}} = \det_q(x_{i,\alpha}^{(j,\beta)_{\mathrm{dc}}})_{(i,\alpha)\in\hat{I}\times\bar{\Lambda}}.$$

(4) Consider $\lambda' \in [\![1,p]\!]\backslash\{\alpha_1,\dots,\alpha_l\}$. If $(j,\beta) \in E_{\mathrm{dc}}$ is greater (relative to $\leqslant_{\mathrm{dc}}$) than the elements of $\{i_1,\dots,i_l\}\times\{\alpha_1,\dots,\alpha_{k-1},\alpha_{k+1},\dots,\alpha_l,\lambda'\}$, it follows from Lemma 7.2 that the matrix

$$(x_{i,\alpha}^{(j,\beta)_{\mathrm{dc}}})_{(i,\alpha)\in\{i_1,\dots,i_l\}\times\{\alpha_1,\dots,\alpha_{k-1},\alpha_{k+1},\dots,\alpha_l,\lambda'\}}$$

is q-quantum. We set

$$\delta_{\alpha_k\to\lambda'}^{(j,\beta)_{\mathrm{dc}}} = \det_q(x_{i,\alpha}^{(j,\beta)_{\mathrm{dc}}})_{(i,\alpha)\in\{i_1,\dots,i_l\}\times\{\alpha_1,\dots,\alpha_{k-1},\alpha_{k+1},\dots,\alpha_l,\lambda'\}}.$$

(5) Consider $i' \in [\![1,m]\!]\backslash\{i_1,\dots,i_l\}$. If $(j,\beta) \in E_{\mathrm{dc}}$ is greater (relative to $\leqslant_{\mathrm{dc}}$) than the elements of $\{i_1,\dots,i_{k-1},i_{k+1},\dots,i_l,i'\}\times\{\alpha_1,\dots,\alpha_l\}$, it follows from Lemma 7.2 that the matrix

$$(x_{i,\alpha}^{(j,\beta)_{\mathrm{dc}}})_{(i,\alpha)\in\{i_1,\dots,i_{k-1},i_{k+1},\dots,i_l,i'\}\times\{\alpha_1,\dots,\alpha_l\}}$$

is q-quantum. We set

$$\delta_{i_k\to i'}^{(j,\beta)_{\mathrm{dc}}} = \det_q(x_{i,\alpha}^{(j,\beta)_{\mathrm{dc}}})_{(i,\alpha)\in\{i_1,\dots,i_{k-1},i_{k+1},\dots,i_l,i'\}\times\{\alpha_1,\dots,\alpha_l\}}.$$

Using [**10**, Proposition 2.2.8], one can prove the following proposition.

Proposition 7.5. *Let $M = (x_{i,\alpha})_{(i,\alpha)\in[\![1,m]\!]\times[\![1,p]\!]}$ be a q-quantum matrix, let $(j,\beta) \in E_{\mathrm{dc}}$ with $(j,\beta) \leqslant_{\mathrm{dc}} (m,p-1)$ and let*

$$\delta = \det_q(x_{i,\alpha})_{\substack{i=i_1,\dots,i_l\\ \alpha=\alpha_1,\dots,\alpha_l}}$$

be an $l\times l$ quantum minor of M with $(i_l,\alpha_l) <_{\mathrm{dc}} (j,\beta)$.

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(1) *If* $t_{j,\beta} = 0$, *then* $\delta^{(j,\beta)^+_{\mathrm{dc}}} = \delta^{(j,\beta)_{\mathrm{dc}}}$.

(2) *Assume that* $t_{j,\beta} \neq 0$.

If $i_l = j$, *or if there exists* $k \in [\![1, l]\!]$ *such that* $\beta = \alpha_k$, *or if* $\beta < \alpha_1$, *then* $\delta^{(j,\beta)^+_{\mathrm{dc}}} = \delta^{(j,\beta)_{\mathrm{dc}}}$.

(3) *Assume that* $t_{j,\beta} \neq 0$ *and that* $i_l < j$.

(a) *If* $\alpha_l < \beta$, *then*

$$\delta^{(j,\beta)^+_{\mathrm{dc}}} = \delta^{(j,\beta)_{\mathrm{dc}}} - \sum_{k=1}^{l} (-q)^{k-(l+1)} t_{j,\beta}^{-1} x_{j,\alpha_k}^{(j,\beta)_{\mathrm{dc}}} \delta_{\alpha_k \to \beta}^{(j,\beta)_{\mathrm{dc}}} \tag{7.1}$$

and

$$\delta^{(j,\beta)^+_{\mathrm{dc}}} = \delta^{(j,\beta)_{\mathrm{dc}}} - \sum_{k=1}^{l} (-q)^{(l+1)-k} \delta_{i_k \to j}^{(j,\beta)_{\mathrm{dc}}} x_{i_k,\beta}^{(j,\beta)_{\mathrm{dc}}} t_{j,\beta}^{-1}. \tag{7.2}$$

(b) *If there exists* $h \in [\![1, l-1]\!]$ *such that* $\alpha_h < \beta < \alpha_{h+1}$, *then*

$$\delta^{(j,\beta)^+_{\mathrm{dc}}} = \delta^{(j,\beta)_{\mathrm{dc}}} - \sum_{k=1}^{h} (-q)^{k-(h+1)} t_{j,\beta}^{-1} x_{j,\alpha_k}^{(j,\beta)_{\mathrm{dc}}} \delta_{\alpha_k \to \beta}^{(j,\beta)_{\mathrm{dc}}}. \tag{7.3}$$

Proposition 7.6. *Let* $M = (x_{i,\alpha})_{(i,\alpha)\in[\![1,m]\!]\times[\![1,p]\!]}$ *be a* q*-quantum matrix, let* $j \in [\![1, m]\!]$ *and let*

$$\delta = \det_q(x_{i,\alpha})_{\substack{i=i_1,\dots,i_l \\ \alpha=\alpha_1,\dots,\alpha_l}}$$

be an $l \times l$ *quantum minor of* M *with* $(i_l, \alpha_l) <_{\mathrm{dc}} (j, p)$.

(1) *If* $t_{j,p} = 0$, *then* $\delta^{(j,p)^+_{\mathrm{dc}}} = \delta^{(j,p)_{\mathrm{dc}}}$.

(2) *Assume that* $t_{j,p} \neq 0$.

If $\alpha_l = p$, *or if there exists* $k \in [\![1, l]\!]$ *such that* $j = i_k$, *or if* $j < i_1$, *then* $\delta^{(j,p)^+_{\mathrm{dc}}} = \delta^{(j,p)_{\mathrm{dc}}}$.

(3) *Assume that* $t_{j,p} \neq 0$ *and that* $\alpha_l < p$.

(a) *If* $i_l < j$, *then*

$$\delta^{(j,p)^+_{\mathrm{dc}}} = \delta^{(j,p)_{\mathrm{dc}}} - \sum_{k=1}^{l} (-q)^{k-(l+1)} t_{j,p}^{-1} x_{i_k,p}^{(j,p)_{\mathrm{dc}}} \delta_{i_k \to j}^{(j,p)_{\mathrm{dc}}} \tag{7.4}$$

and

$$\delta^{(j,p)^+_{\mathrm{dc}}} = \delta^{(j,p)_{\mathrm{dc}}} - \sum_{k=1}^{l} (-q)^{(l+1)-k} \delta_{\alpha_k \to p}^{(j,p)_{\mathrm{dc}}} x_{j,\alpha_k}^{(j,p)_{\mathrm{dc}}} t_{j,p}^{-1}. \tag{7.5}$$

(b) *If there exists $h \in [\![1, l-1]\!]$ such that $i_h < j < i_{h+1}$, then*

$$\delta^{(j,p)^+_{\mathrm{dc}}} = \delta^{(j,p)_{\mathrm{dc}}} - \sum_{k=1}^{h} (-q)^{k-(h+1)} t_{j,p}^{-1} x_{i_k,p}^{(j,p)_{\mathrm{dc}}} \delta_{i_k \to j}^{(j,p)_{\mathrm{dc}}}. \tag{7.6}$$

Proof. We observe that the standard algorithm performed along the last row of a q-quantum matrix coincides with the last-column algorithm applied to the last column of its transpose. Since the algebras generated by a generic q-quantum matrix and by its transpose are isomorphic, this allows us to apply [**10**, Proposition 2.2.8] in order to obtain the result. □

An immediate corollary of Propositions 7.5 and 7.6 is the following result.

Corollary 7.7. *Let $M = (x_{i,\alpha})_{(i,\alpha) \in [\![1,m]\!] \times [\![1,p]\!]}$ be a q-quantum matrix, let $(j, \beta) \in E_{\mathrm{dc}} \setminus \{(m+1, p)\}$ and let*

$$\delta = \det_q(x_{i,\alpha})_{\substack{i=i_1,\dots,i_l \\ \alpha=\alpha_1,\dots,\alpha_l}}$$

be an $l \times l$ quantum minor of M with $(i_l, \alpha_l) <_{\mathrm{dc}} (j, \beta)$.

If either $i_l = j$ or $\alpha_l = \beta$, then $\delta^{(j,\beta)^+_{\mathrm{dc}}} = \delta^{(j,\beta)_{\mathrm{dc}}}$.

We finish this section by computing the quantum minors of $M^{(j,\beta)^+_{\mathrm{dc}}}$ that involve $x_{j,\beta}^{(j,\beta)^+_{\mathrm{dc}}}$ in terms of quantum minors of $M^{(j,\beta)_{\mathrm{dc}}}$.

Proposition 7.8. *Let $M = (x_{i,\alpha})_{(i,\alpha) \in [\![1,m]\!] \times [\![1,p]\!]}$ be a q-quantum matrix and let*

$$\delta = \det_q(x_{i,\alpha})_{\substack{i=i_1,\dots,i_l \\ \alpha=\alpha_1,\dots,\alpha_l}}$$

be an $l \times l$ quantum minor of M with $l \geqslant 2$. Then

$$\delta^{(i_l,\alpha_l)^+_{\mathrm{dc}}} = \delta_{\hat{i}_l,\hat{\alpha}_l}^{(i_l,\alpha_l)_{\mathrm{dc}}} t_{i_l,\alpha_l}.$$

Proof. If $t_{i_l,\alpha_l} = x_{i_l,\alpha_l}^{(i_l,\alpha_l)^+_{\mathrm{dc}}} = 0$, it follows from [**3**, Proposition 4.1.1] that $\delta^{(i_l,\alpha_l)^+_{\mathrm{dc}}} = 0$. Thus,

$$\delta^{(i_l,\alpha_l)^+_{\mathrm{dc}}} = 0 = \delta_{\hat{i}_l,\hat{\alpha}_l}^{(i_l,\alpha_l)_{\mathrm{dc}}} t_{i_l,\alpha_l}.$$

Assume now that $t_{i_l,\alpha_l} \neq 0$ and set

$$c_{i,\alpha} = x_{i,\alpha}^{(i_l,\alpha_l)^+_{\mathrm{dc}}} \quad \text{for } (i,\alpha) \in \{i_1,\dots,i_l\} \times \{\alpha_1,\dots,\alpha_l\}.$$

By Lemma 7.2, the matrix $C = (c_{i,\alpha})_{(i,\alpha) \in \{i_1,\dots,i_l\} \times \{\alpha_1,\dots,\alpha_l\}}$ is q-quantum. Hence, we can apply the standard deleting-derivations algorithm to C (see Conventions 3.3) and it is obvious that $c_{i,\alpha}^{(l,l)_{\mathrm{s}}} = x_{i,\alpha}^{(i_l,\alpha_l)_{\mathrm{dc}}}$ for all $(i,\alpha) \in \{i_1,\dots,i_l\} \times \{\alpha_1,\dots,\alpha_l\}$. So, we deduce from [**3**, Proposition 4.1.2] that

$$\delta^{(i_l,\alpha_l)^+_{\mathrm{dc}}} = \det_q(C) = \det_q(c_{i,\alpha}^{(l,l)_{\mathrm{s}}})_{\substack{i=i_1,\dots,i_{l-1} \\ \alpha=\alpha_1,\dots,\alpha_{l-1}}} c_{i_l,\alpha_l} = \delta_{\hat{i}_l,\hat{\alpha}_l}^{(i_l,\alpha_l)_{\mathrm{dc}}} t_{i_l,\alpha_l}.$$

□

8. Some vanishing criteria for quantum minors

Throughout this section we use the following conventions.

(1) K denotes a $\mathbb{C}$-algebra which is also a skew field.

(2) m and p are greater than or equal to 3.

(3) $M = (x_{i,\alpha})_{(i,\alpha)\in[\![1,m]\!]\times[\![1,p]\!]}$ is a q-quantum matrix with entries in K and we set

$$t_{i,\alpha} = x_{i,\alpha}^{(1,2)_{\mathrm{s}}} = x_{i,\alpha}^{(1,2)_{\mathrm{dc}}} \quad \text{for any } (i,\alpha)\in[\![1,m]\!]\times[\![1,p]\!]$$

(see Proposition 6.3).

(4) We assume that the following property holds for the matrix M: the non-zero monomials $t_{1,1}^{k_{1,1}}\cdots t_{m,p}^{k_{m,p}}$ (where the indices are increasing for $\leqslant_{\mathrm{s}}$) ($k_{i,\alpha}\in\mathbb{N}$) are linearly independent over $\mathbb{C}$, so that $\mathbb{C}\langle t_{i,\alpha} \mid (i,\alpha)\in[\![1,m]\!]\times[\![1,p]\!]$ and $t_{i,\alpha}\neq 0\rangle$ can be viewed as a quantum affine space.

Notation 8.1.

(i) L denotes the matrix obtained from M by deleting the last row and the last column, that is

$$L = (x_{i,\alpha})_{(i,\alpha)\in[\![1,m-1]\!]\times[\![1,p-1]\!]}.$$

(ii) If $(j,\beta)\in E_{\mathrm{dc}}$, we denote by $L^{(j,\beta)_{\mathrm{dc}}}$ the matrix obtained from $M^{(j,\beta)_{\mathrm{dc}}}$ by deleting the last row and the last column, that is

$$L^{(j,\beta)_{\mathrm{dc}}} = (x_{i,\alpha}^{(j,\beta)_{\mathrm{dc}}})_{(i,\alpha)\in[\![1,m-1]\!]\times[\![1,p-1]\!]}.$$

Observe that $L^{(m+1,p)_{\mathrm{dc}}} = L$.

(iii) We set $N = L^{(m,1)_{\mathrm{dc}}}$.

By Lemma 7.2, N is a q-quantum matrix. Hence, the standard deleting-derivations algorithm (see Conventions 3.3) can be applied to N, and from Proposition 6.3 we deduce the following lemma.

Lemma 8.2. *We have $N^{(1,2)_{\mathrm{s}}} = (t_{i,\alpha})_{(i,\alpha)\in[\![1,m-1]\!]\times[\![1,p-1]\!]}$, so that the matrix N satisfies convention (4) at the beginning of this section.*

Lemma 8.3. *Let $l\in[\![1,\inf(m-1,p-1)]\!]$ and assume that all $l\times l$ quantum minors of N are equal to 0. If $(j,\beta)\in E_{\mathrm{dc}}$ with $(m,1)\leqslant_{\mathrm{dc}}(j,\beta)\leqslant_{\mathrm{dc}}(m,p)$ and if k is an integer such that $k\geqslant l$, then all $k\times k$ quantum minors of the q-quantum matrix $L^{(j,\beta)_{\mathrm{dc}}}$ are equal to 0.*

Proof. If $k\geqslant l$, the $k\times k$ quantum minors of $L^{(j,\beta)_{\mathrm{dc}}}$ are right linear combinations (with coefficients in K) of $l\times l$ quantum minors of $L^{(j,\beta)_{\mathrm{dc}}}$. So, it is enough to prove that all $l\times l$ quantum minors of $L^{(j,\beta)_{\mathrm{dc}}}$ are zero. To achieve this aim, we proceed by iteration (for $\leqslant_{\mathrm{dc}}$) on (j,β).

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Since $N = L^{(m,1)_{\mathrm{dc}}}$, the case $(j,\beta) = (m,1)$ is done. Assume now that $(m,1) \leqslant_{\mathrm{dc}} (j,\beta) \leqslant_{\mathrm{dc}} (m-1,p)$ and that all $l \times l$ quantum minors of $L^{(j,\beta)_{\mathrm{dc}}}$ are equal to 0. In order to prove that the same property holds for all $l \times l$ quantum minors of $L^{(j,\beta)^{+}_{\mathrm{dc}}}$, two cases may be distinguished.

(i) If $j = m$, then $\beta < p$. Thus, by Proposition 7.5, every $l \times l$ quantum minor of $L^{(j,\beta)^{+}_{\mathrm{dc}}}$ is a left linear combination (with coefficients in K) of $l \times l$ quantum minors of $L^{(j,\beta)_{\mathrm{dc}}}$. The desired result follows from the induction hypothesis.

(ii) If $j \neq m$, then $\beta = p$. Thus, by Proposition 7.6, every $l \times l$ quantum minor of $L^{(j,\beta)^{+}_{\mathrm{dc}}}$ is a left linear combination (with coefficients in K) of $l \times l$ quantum minors of $L^{(j,\beta)_{\mathrm{dc}}}$. The desired result follows from the induction hypothesis.

□

Proposition 8.4. *Let l and s be two integers such that $l \in [\![1, \inf(m-1, p-1)]\!]$ and $s \in [\![1, m-1]\!]$, and assume that all $l \times l$ quantum minors of N are equal to 0. Then*

(1) *all $(l+1) \times (l+1)$ quantum minors of M are equal to 0;*

(2) *if, moreover, we suppose that $x_{i,p} = t_{i,p} = 0$ for $i \in [\![1, s]\!]$, then all $l \times l$ quantum minors of the matrix obtained from M by deleting the rows $s+1, \dots, m$ are equal to 0.*

Proof.

(1) Let

$$\delta = \det_q(x_{i,\alpha})_{\substack{i=i_1,\dots,i_{l+1} \\ \alpha=\alpha_1,\dots,\alpha_{l+1}}}$$

be an $(l+1) \times (l+1)$ quantum minor of M. In order to establish that $\delta = 0$, four cases are distinguished.

(i) If $\alpha_{l+1} = p$ and $i_{l+1} = m$, then, by Proposition 7.8, we have $\delta = \delta^{(m,p)_{\mathrm{dc}}}_{\hat{m},\bar{p}} t_{m,p}$. Since $\delta^{(m,p)_{\mathrm{dc}}}_{\hat{m},\bar{p}}$ is an $l \times l$ quantum minor of $L^{(m,p)_{\mathrm{dc}}}$, we deduce from Lemma 8.3 that

$$\delta^{(m,p)_{\mathrm{dc}}}_{\hat{m},\bar{p}} = 0,$$

so that

$$\delta = \delta^{(m,p)_{\mathrm{dc}}}_{\hat{m},\bar{p}} t_{m,p} = 0.$$

(ii) If $\alpha_{l+1} = p$ and $i_{l+1} < m$, then, by Corollary 7.7, we have $\delta = \delta^{(i_{l+1}+1,p)_{\mathrm{dc}}}$. Thus, it follows from Proposition 7.8 that

$$\delta = \delta^{(i_{l+1},p)_{\mathrm{dc}}}_{\hat{i}_{l+1},\bar{p}} t_{i_{l+1},p}.$$

Since $\delta^{(i_{l+1},p)_{\mathrm{dc}}}_{\hat{i}_{l+1},\bar{p}}$ is an $l \times l$ quantum minor of $L^{(i_{l+1},p)_{\mathrm{dc}}}$, we deduce from Lemma 8.3 that $\delta^{(i_{l+1},p)_{\mathrm{dc}}}_{\hat{i}_{l+1},\bar{p}} = 0$. Hence

$$\delta = \delta^{(i_{l+1},p)_{\mathrm{dc}}}_{\hat{i}_{l+1},\bar{p}} t_{i_{l+1},p} = 0.$$

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(iii) If $\alpha_{l+1} < p$ and $i_{l+1} = m$, then, by Corollary 7.7, we have $\delta = \delta^{(m,p)_{\mathrm{dc}}}$. Expanding this last quantum minor along the last row (see [**12**, Corollary 4.4.4]), we get

$$\delta = \sum_{k=1}^{l+1} (-q)^{k-(l+1)} x_{m,\alpha_k}^{(m,p)_{\mathrm{dc}}} \delta_{\hat{m},\bar{\alpha}_k}^{(m,p)_{\mathrm{dc}}}.$$

Since each $\delta_{\hat{m},\bar{\alpha}_k}^{(m,p)_{\mathrm{dc}}}$ is an $l \times l$ quantum minor of $L^{(m,p)_{\mathrm{dc}}}$, we deduce from Lemma 8.3 that each $\delta_{\hat{m},\bar{\alpha}_k}^{(m,p)_{\mathrm{dc}}}$ is equal to 0, so that $\delta = 0$.

(iv) If $\alpha_{l+1} < p$ and $i_{l+1} < m$, we have the following.

(a) If $t_{m,p} = 0$, then it follows from Proposition 7.6 that $\delta = \delta^{(m,p)_{\mathrm{dc}}}$. Since $\delta^{(m,p)_{\mathrm{dc}}}$ is an $(l+1) \times (l+1)$ quantum minor of $L^{(m,p)_{\mathrm{dc}}}$, we deduce from Lemma 8.3 that $\delta^{(m,p)_{\mathrm{dc}}} = 0$.

(b) Assume now that $t_{m,p} \neq 0$. By Proposition 7.6, we have

$$t_{m,p}\delta = t_{m,p}\delta^{(m,p)_{\mathrm{dc}}} - \sum_{k=1}^{l+1} (-q)^{k-(l+2)} x_{i_k,p}^{(m,p)_{\mathrm{dc}}} \delta_{i_k \to m}^{(m,p)_{\mathrm{dc}}}. \tag{8.1}$$

Since $\delta^{(m,p)_{\mathrm{dc}}}$ is an $(l+1) \times (l+1)$ quantum minor of $L^{(m,p)_{\mathrm{dc}}}$, we deduce from Lemma 8.3 that $\delta^{(m,p)_{\mathrm{dc}}} = 0$. Next, let $k \in [\![1, l+1]\!]$. Expanding $\delta_{i_k \to m}^{(m,p)_{\mathrm{dc}}}$ along the last row (see [**12**, Corollary 4.4.4]), we get

$$\delta_{i_k \to m}^{(m,p)_{\mathrm{dc}}} = \sum_{i=1}^{l+1} (-q)^{i-(l+1)} x_{m,\alpha_i}^{(m,p)_{\mathrm{dc}}} \delta_{\hat{i}_k,\bar{\alpha}_i}^{(m,p)_{\mathrm{dc}}}.$$

Since each $\delta_{\hat{i}_k,\bar{\alpha}_i}^{(m,p)_{\mathrm{dc}}}$ is an $l \times l$ quantum minor of $L^{(m,p)_{\mathrm{dc}}}$, we deduce from Lemma 8.3 that each $\delta_{\hat{i}_k,\bar{\alpha}_i}^{(m,p)_{\mathrm{dc}}}$ is equal to 0. Thus $\delta_{i_k \to m}^{(m,p)_{\mathrm{dc}}} = 0$.

Equation (8.1) and the above results show that $t_{m,p}\delta = 0$, so that $\delta = 0$. The proof of the first assertion is now complete.

(2) Let

$$\delta = \det_q(x_{i,\alpha})_{\substack{i=i_1,\dots,i_l \\ \alpha=\alpha_1,\dots,\alpha_l}}$$

be an $l \times l$ quantum minor of M with $i_l \leqslant s$. In order to show that $\delta = 0$, two cases may be distinguished.

(i) Assume that $\alpha_l = p$. Thus, the last column of δ is 0, so that $\delta = 0$.

(ii) Assume that $\alpha_l < p$.

(a) If $t_{m,p} = 0$, then $\delta = \delta^{(m,p)_{\mathrm{dc}}}$. Since $i_l \leqslant s < m$ and $\alpha_l < p$, $\delta^{(m,p)_{\mathrm{dc}}}$ is an $l \times l$ quantum minor of $L^{(m,p)_{\mathrm{dc}}}$. Thus, it follows from Lemma 8.3 that $\delta^{(m,p)_{\mathrm{dc}}} = 0$. Hence $\delta = \delta^{(m,p)_{\mathrm{dc}}} = 0$.

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(b) Assume now that $t_{m,p} \neq 0$. Since $i_l \leqslant s < m$ and $\alpha_l < p$, it follows from Proposition 7.6 that

$$\delta t_{m,p} = \delta^{(m,p)_{\mathrm{dc}}} t_{m,p} - \sum_{k=1}^{l} (-q)^{l+1-k} \delta^{(m,p)_{\mathrm{dc}}}_{\alpha_k \to p} x^{(m,p)_{\mathrm{dc}}}_{m,\alpha_k}. \tag{8.2}$$

Since $\delta^{(m,p)_{\mathrm{dc}}}$ is an $l \times l$ quantum minor of $L^{(m,p)_{\mathrm{dc}}}$, we deduce from Lemma 8.3 that $\delta^{(m,p)_{\mathrm{dc}}} = 0$. Next, let $k \in [\![1, l]\!]$. By Lemma 6.2, we have $x^{(m,p)_{\mathrm{dc}}}_{i,p} = x_{i,p}$ for $i \in [\![1, m]\!]$. Thus, since $i_l \leqslant s$, the last column of $\delta^{(m,p)_{\mathrm{dc}}}_{\alpha_k \to p}$ is 0, so that $\delta^{(m,p)_{\mathrm{dc}}}_{\alpha_k \to p} = 0$.

Equation (8.2) and the above results show that $\delta t_{m,p} = 0$, so that $\delta = 0$. The proof of the second assertion is now complete.

□

9. Some non-vanishing criteria for quantum minors

Throughout this section, we assume that the four conventions of § 8 are satisfied and we retain the notation of that section.

9.1. A criterion for 1 × 1 quantum minors

Let $(i, \alpha) \in [\![1, m]\!] \times [\![1, p]\!]$ and assume that $t_{i,p} \times t_{m,\alpha} \neq 0$.

(i) If $i = m$, it follows from Lemma 6.2 that $x_{i,\alpha} = x_{m,\alpha} = t_{m,\alpha} \neq 0$.

(ii) If $\alpha = p$, it follows again from Lemma 6.2 that $x_{i,\alpha} = x_{i,p} = t_{i,p} \neq 0$.

(iii) If $i < m$ and $\alpha < p$, we have $x_{m,p} x_{i,\alpha} - x_{i,\alpha} x_{m,p} = -(q - q^{-1}) x_{i,p} x_{m,\alpha}$. So we deduce from Lemma 6.2 that $x_{m,p} x_{i,\alpha} - x_{i,\alpha} x_{m,p} = -(q - q^{-1}) t_{i,p} t_{m,\alpha} \neq 0$. This implies that $x_{i,\alpha} \neq 0$.

So we can conclude with the following proposition.

Proposition 9.1. *Let $(i, \alpha) \in [\![1, m]\!] \times [\![1, p]\!]$ and assume that $t_{i,p} \times t_{m,\alpha} \neq 0$. Then $x_{i,\alpha} \neq 0$.*

Remark 9.2. The above result is still true if $m = 2$ or $p = 2$.

9.2. A criterion for quantum minors of L

Notation 9.3. Let $(j, \beta) \in E_{\mathrm{dc}}$.

(i) We denote by $B^{(j,\beta)_{\mathrm{dc}}}$ the subalgebra of K generated by the $x^{(j,\beta)_{\mathrm{dc}}}_{i,\alpha}$ ($(i, \alpha) \in [\![1, m]\!] \times [\![1, p]\!]$), that is

$$B^{(j,\beta)_{\mathrm{dc}}} = \mathbb{C}\langle x^{(j,\beta)_{\mathrm{dc}}}_{i,\alpha} \mid (i, \alpha) \in [\![1, m]\!] \times [\![1, p]\!] \rangle.$$

(ii) We denote by $C^{(j,\beta)_{\rm dc}}$ the subalgebra of $B^{(j,\beta)_{\rm dc}}$ defined by

$$C^{(j,\beta)_{\rm dc}} = \mathbb{C}\langle x_{i,\alpha}^{(j,\beta)_{\rm dc}} \mid (1,1) \leqslant_{\rm dc} (i,\alpha) <_{\rm dc} (j,\beta)\rangle.$$

The following result is proved in the same manner as [**10**, Corollaire 3.5.5].

Lemma 9.4. *Let $(j,\beta) \in E_{\rm dc}$. If $t_{j,\beta} = x_{j,\beta}^{(j,\beta)_{\rm dc}} \neq 0$, then the monomials $t_{j,\beta}^k$ $(k \in \mathbb{N})$ are linearly independent in $B^{(j,\beta)_{\rm dc}}$ viewed as a right (respectively, left) $C^{(j,\beta)_{\rm dc}}$-module.*

Proof. First, an induction (with respect to $\leqslant_{\rm dc}$) shows that, if $(j,\beta) \in E_{\rm dc}$ and $(i,\alpha) \in [\![1,m]\!] \times [\![1,p]\!]$, then $x_{i,\alpha}^{(j,\beta)_{\rm dc}} = t_{i,\alpha} + Q$, where Q is a Laurent polynomial (with coefficients in $\mathbb{C}$) in the non-zero $t_{u,\lambda}$ with $(1,1) \leqslant_{\rm dc} (u,\lambda) <_{\rm dc} (j,\beta)$. Hence, $C^{(j,\beta)_{\rm dc}}$ is contained in $D = \mathbb{C}\langle t_{i,\alpha}^{\pm 1} \mid (1,1) \leqslant_{\rm dc} (i,\alpha) <_{\rm dc} (j,\beta) \text{ and } t_{i,\alpha} \neq 0\rangle$. So it is enough to prove that the monomials $t_{j,\beta}^k$ $(k \in \mathbb{N})$ are linearly independent in K viewed as a left (respectively, right) D-module. This follows immediately from convention (4) of § 8. □

Proposition 9.5. *Let $(j,\beta) \in E_{\rm dc} \setminus \{(m+1,p)\}$ and let*

$$\delta = \det_q(x_{i,\alpha})_{\substack{i=i_1,\dots,i_l \\ \alpha=\alpha_1,\dots,\alpha_l}}$$

be an $l \times l$ quantum minor of M. Assume that $(i_l,\alpha_l) <_{\rm dc} (j,\beta)$. If $\delta^{(j,\beta)_{\rm dc}^+} = 0$, then $\delta^{(j,\beta)_{\rm dc}} = 0$.

Proof. If $t_{j,\beta} = 0$, we have $\delta^{(j,\beta)_{\rm dc}} = \delta^{(j,\beta)_{\rm dc}^+} = 0$, as required. Assume now that $t_{j,\beta} \neq 0$. We distinguish two cases.

(i) If $\beta = p$, then, since $\delta^{(j,\beta)_{\rm dc}^+} = 0$, it follows from Proposition 7.6 that $t_{j,\beta}\delta^{(j,\beta)_{\rm dc}} \in C^{(j,\beta)_{\rm dc}}$. On the other hand, it is clear that $\delta^{(j,\beta)_{\rm dc}} \in C^{(j,\beta)_{\rm dc}}$. Thus, we deduce from Lemma 9.4 that $\delta^{(j,\beta)_{\rm dc}} = 0$.

(ii) If $\beta < p$, then, since $\delta^{(j,\beta)_{\rm dc}^+} = 0$, it follows from Proposition 7.5 that $t_{j,\beta}\delta^{(j,\beta)_{\rm dc}} \in C^{(j,\beta)_{\rm dc}}$. On the other hand, it is clear that $\delta^{(j,\beta)_{\rm dc}} \in C^{(j,\beta)_{\rm dc}}$. Thus, we deduce from Lemma 9.4 that $\delta^{(j,\beta)_{\rm dc}} = 0$. The proof is now complete.

□

The following non-vanishing criterion can be easily deduced from Proposition 9.5.

Proposition 9.6. *Let*

$$\delta = \det_q(x_{i,\alpha})_{\substack{i=i_1,\dots,i_l \\ \alpha=\alpha_1,\dots,\alpha_l}}$$

be an $l \times l$ quantum minor of L. If $\delta^{(m,1)_{\rm dc}} \neq 0$, then $\delta \neq 0$.

9.3. A criterion for quantum minors of M

Proposition 9.7. *Let*

$$\delta = \det_q(x_{i,\alpha})_{\substack{i=i_1,\dots,i_l\\ \alpha=\alpha_1,\dots,\alpha_l}}$$

be an $l \times l$ quantum minor of M with $l \geqslant 2$, and assume that $t_{i_l,p} \times t_{m,\alpha_l} \neq 0$. If $\delta^{(m,1)_{\mathrm{dc}}}_{\hat{i}_l,\bar{\alpha}_l} \neq 0$, then $\delta \neq 0$.

Proof. Assume that $\delta = 0$. In order to prove that $\delta^{(m,1)_{\mathrm{dc}}}_{\hat{i}_l,\bar{\alpha}_l} = 0$, three cases may be distinguished.

(i) If $\alpha_l = p$, then Corollary 7.7 shows that $\delta^{(i_l+1,p)_{\mathrm{dc}}} = \delta = 0$. Since $t_{i_l,p} \neq 0$, it follows from Proposition 7.8 that $\delta^{(i_l,p)_{\mathrm{dc}}}_{\hat{i}_l,\bar{p}} = 0$. Now, since $i_{l-1} < m$ and $\alpha_{l-1} < p$, we have $(i_{l-1},\alpha_{l-1}) <_{\mathrm{dc}} (m,1)$, and so, we deduce from Proposition 9.5 that $\delta^{(m,1)_{\mathrm{dc}}}_{\hat{i}_l,\bar{\alpha}_l} = 0$, as desired.

(ii) If $i_l = m$ and $\alpha_l < p$, we deduce from Proposition 9.5 that $\delta^{(m,\alpha_l)^+_{\mathrm{dc}}} = 0$. Thus, since $t_{m,\alpha_l} \neq 0$, Proposition 7.8 shows that $\delta^{(m,\alpha_l)_{\mathrm{dc}}}_{\hat{m},\bar{\alpha}_l} = 0$. Now, since $i_{l-1} < m$ and $\alpha_{l-1} < p$, we have $(i_{l-1},\alpha_{l-1}) <_{\mathrm{dc}} (m,1)$, and so, we deduce from Proposition 9.5 that $\delta^{(m,1)_{\mathrm{dc}}}_{\hat{i}_l,\bar{\alpha}_l} = 0$, as required.

(iii) If $i_l < m$ and $\alpha_l < p$, we observe that since M is a q-quantum matrix, $x_{m,p} \neq 0$. Further, since $\alpha_l < p$, we have $(i_l,\alpha_l) <_{\mathrm{dc}} (m,p)$. It then follows from Proposition 9.5 that $\delta^{(m,p)_{\mathrm{dc}}} = 0$. Thus, by Lemma 6.2, formula (2) of [**10**, Proposition 2.2.8] gives us the equation

$$0 = \sum_{k=1}^{l} (-q)^{k-l-1} x_{m,\alpha_k} \delta^{(m,p)_{\mathrm{dc}}}_{\alpha_k \to p}.$$

By Corollary 7.7, we have

$$\delta^{(m,p)_{\mathrm{dc}}}_{\alpha_k \to p} = \delta^{(i_l+1,p)_{\mathrm{dc}}}_{\alpha_k \to p}.$$

Hence,

$$\sum_{k=1}^{l} (-q)^{k-l-1} x_{m,\alpha_k} \delta^{(i_l+1,p)_{\mathrm{dc}}}_{\alpha_k \to p} = 0.$$

Now, we deduce from Proposition 7.8 that

$$\sum_{k=1}^{l} (-q)^{k-l-1} x_{m,\alpha_k} \delta^{(i_l,p)_{\mathrm{dc}}}_{\hat{i}_l,\bar{\alpha}_k} t_{i_l,p} = 0.$$

Since $t_{i_l,p} \neq 0$, we conclude that

$$\sum_{k=1}^{l} (-q)^{k-l-1} x_{m,\alpha_k} \delta^{(i_l,p)_{\mathrm{dc}}}_{\hat{i}_l,\bar{\alpha}_k} = 0. \tag{9.1}$$

On the other hand, by expanding

$$\delta_{i_l\to m}^{(i_l,p)_{\mathrm{dc}}} = \det_q \begin{pmatrix} x_{i_1,\alpha_1}^{(i_l,p)_{\mathrm{dc}}} & \cdots & x_{i_1,\alpha_l}^{(i_l,p)_{\mathrm{dc}}} \\ \vdots & \ddots & \vdots \\ x_{i_{l-1},\alpha_1}^{(i_l,p)_{\mathrm{dc}}} & \cdots & x_{i_{l-1},\alpha_l}^{(i_l,p)_{\mathrm{dc}}} \\ x_{m,\alpha_1}^{(i_l,p)_{\mathrm{dc}}} & \cdots & x_{m,\alpha_l}^{(i_l,p)_{\mathrm{dc}}} \end{pmatrix}$$

along the last row (see [**12**, Corollary 4.4.4]), we obtain, by Lemma 6.2,

$$\delta_{i_l\to m}^{(i_l,p)_{\mathrm{dc}}} = \sum_{k=1}^{l} (-q)^{k-l} x_{m,\alpha_k}^{(i_l,p)_{\mathrm{dc}}} \delta_{\hat{i}_l,\bar{\alpha}_k}^{(i_l,p)_{\mathrm{dc}}} = \sum_{k=1}^{l} (-q)^{k-l} x_{m,\alpha_k} \delta_{\hat{i}_l,\bar{\alpha}_k}^{(i_l,p)_{\mathrm{dc}}}.$$

From (9.1), it follows that $\delta_{i_l\to m}^{(i_l,p)_{\mathrm{dc}}} = 0$. Hence, by using Proposition 9.5, we get $\delta_{i_l\to m}^{(m,\alpha_l)_{\mathrm{dc}}^{+}} = 0$. Thus, since $t_{m,\alpha_l} \neq 0$, it follows from Proposition 7.8 that $\delta_{\hat{i}_l,\bar{\alpha}_l}^{(m,\alpha_l)_{\mathrm{dc}}} = 0$. Since $i_{l-1} < m$ and $\alpha_{l-1} < p$, we have $(i_{l-1}, \alpha_{l-1}) <_{\mathrm{dc}} (m, 1)$; so, Proposition 9.5 shows that $\delta_{\hat{i}_l,\bar{\alpha}_l}^{(m,1)_{\mathrm{dc}}} = 0$. The proof is now complete.

□

10. A generating set for some $\mathcal{H}$-invariant prime ideals in R

The aim of this last section is to construct a generating set of quantum minors for some J_w. To do this, we use the notation of §§ 2 and 3. Let $w \in W$. Recall that J_w denotes the corresponding $\mathcal{H}$-invariant prime ideal in R (see Proposition 2.8), that F_w denotes the skew field of fractions of $R_w = R/J_w$, and that $M_w = (y_{i,\alpha})_{(i,\alpha)\in[\![1,m]\!]\times[\![1,p]\!]}$, where $y_{i,\alpha} = Y_{i,\alpha} + J_w$ (see Notation 3.6). If $(i,\alpha) \in [\![1,m]\!] \times [\![1,p]\!]$, we still set $t_{i,\alpha} = y_{i,\alpha}^{(1,2)_{\mathrm{s}}}$. We have shown (see Theorem 3.7) that

(i) M_w is a q-quantum matrix with entries in F_w;

(ii) $t_{i,\alpha} = 0$ if and only if $(i,\alpha) \in w$;

(iii) there exists an isomorphism from $\mathbb{C}\langle t_{i,\alpha} \mid (i,\alpha) \notin w\rangle$ onto the subalgebra $\mathbb{C}\langle T_{i,\alpha} \mid (i,\alpha) \notin w\rangle$ of $\bar{R}_{\mathrm{s}}$ which sends $t_{i,\alpha}$ onto $T_{i,\alpha}$ for $(i,\alpha) \notin w$.

Thus, the conventions 1, 3 and 4 of §§ 8 and 9 are satisfied if we replace K by F_w, M by M_w and $x_{i,\alpha}$ by $y_{i,\alpha}$ ($(i,\alpha) \in [\![1,m]\!] \times [\![1,p]\!]$).

10.1. The case $w = [\![1, m-u]\!] \times [\![1, p-u]\!]$ ($u \geqslant 0$)

By [**10**, Théorème 3.7.2], J_w is generated by the quantum minors of $\mathcal{Y}$ which belong to J_w. So, in order to find a generating set for J_w, we just have to determine the quantum minors of M_w which are equal to zero. To do this, we first establish the following result.

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Theorem 10.1. *Let K be a $\mathbb{C}$-algebra which is also a skew field, let $M = (x_{i,\alpha})_{(i,\alpha)\in[\![1,m]\!]\times[\![1,p]\!]}$ be a q-quantum matrix with entries in K and let $u \in [\![0, \inf(m-1, p-1)]\!]$. For $(i,\alpha) \in [\![1,m]\!] \times [\![1,p]\!]$, we set $t_{i,\alpha} = x_{i,\alpha}^{(1,2)_\mathrm{s}}$. Assume that the non-zero monomials $t_{1,1}^{k_{1,1}} \cdots t_{m,p}^{k_{m,p}}$ (where the indices are increasing for $\leqslant_\mathrm{s}$) with $k_{i,\alpha} \in \mathbb{N}$ are linearly independent over $\mathbb{C}$. We also assume that $t_{i,\alpha} = 0$ if and only if $(i,\alpha) \in [\![1, m-u]\!] \times [\![1, p-u]\!]$. Then*

(1) *the $v \times v$ quantum minors of M with $v \geqslant u+1$ are zero;*

(2) *the $v \times v$ quantum minors of M with $1 \leqslant v \leqslant u$ are non-zero.*

Proof. If $u = 0$, it is obvious that $x_{i,\alpha} = 0$ for all $(i,\alpha) \in [\![1,m]\!] \times [\![1,p]\!]$, as required.

We now establish Theorem 10.1 when $m = 2$ and $u = 1$. In this case, it follows from Proposition 3.5 that we have

$$x_{i,\alpha} = \begin{cases} t_{i,\alpha} & \text{if } i = 2 \text{ or } \alpha = p, \\ t_{i,p} t_{2,p}^{-1} t_{2,\alpha} & \text{otherwise.} \end{cases}$$

Thus, all 1×1 quantum minors of M are non-zero, as desired. Next, let

$$\delta = \det_q \begin{pmatrix} x_{1,\alpha} & x_{1,\beta} \\ x_{2,\alpha} & x_{2,\beta} \end{pmatrix} \quad (1 \leqslant \alpha < \beta \leqslant p)$$

be a 2×2 quantum minor of M. In order to show that $\delta = 0$, we distinguish two cases.

(i) If $\beta < p$, then $\delta = t_{1,p} t_{2,p}^{-1} t_{2,\alpha} t_{2,\beta} - q t_{1,p} t_{2,p}^{-1} t_{2,\beta} t_{2,\alpha}$. Now, since $M^{(1,2)_\mathrm{s}}$ is a $(1,2)_\mathrm{s}$-q-quantum matrix, we have $t_{2,\beta} t_{2,\alpha} = q^{-1} t_{2,\alpha} t_{2,\beta}$, so that

$$\delta = t_{1,p} t_{2,p}^{-1} t_{2,\alpha} t_{2,\beta} - t_{1,p} t_{2,p}^{-1} t_{2,\alpha} t_{2,\beta} = 0.$$

(ii) If $\beta = p$, then $\delta = t_{1,p} t_{2,p}^{-1} t_{2,\alpha} t_{2,p} - q t_{1,p} t_{2,\alpha}$. Now, since $M^{(1,2)_\mathrm{s}}$ is a $(1,2)_\mathrm{s}$-q-quantum matrix, we have $t_{2,\alpha} t_{2,p} = q t_{2,p} t_{2,\alpha}$, so that

$$\delta = q t_{1,p} t_{2,p}^{-1} t_{2,p} t_{2,\alpha} - q t_{1,p} t_{2,\alpha} = q t_{1,p} t_{2,\alpha} - q t_{1,p} t_{2,\alpha} = 0.$$

Thus, all 2×2 quantum minors of M are zero and Theorem 10.1 is now established when $m = 2$ and $u = 1$.

By a similar argument, we establish Theorem 10.1 when $p = 2$ and $u = 1$, and this proves Theorem 10.1 when $m = 2$ or $p = 2$.

We now assume that $m, p \geqslant 3$ and that the result is true for any $m' \times p'$ q-quantum matrix with $(m', p') <_s (m, p)$. If $u = 0$, we have already proved the desired result.

Assume now that $u \geqslant 1$. Since m and p are greater than or equal to 3, the four conventions of §§ 8 and 9 are satisfied. Hence, we can use the notation and results of these two sections. In particular, we still denote by N the matrix obtained from $M^{(m,1)_\mathrm{dc}}$ by deleting the last row and the last column. By Lemma 8.2, the induction hypothesis can be applied to the q-quantum matrix N. This leads to the following properties.

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(1) The $v \times v$ quantum minors of N with $v \geqslant u$ are equal to zero.

(2) The $v \times v$ quantum minors of N with $1 \leqslant v \leqslant u-1$ are non-zero.

It then follows from Assertion 1 and Proposition 8.4 (with $l = v \geqslant u$) that the $(v+1) \times (v+1)$ quantum minors of M with $v \geqslant u$ are zero. On the other hand, since $t_{i,p} \times t_{m,\alpha} \neq 0$ for all $(i,\alpha) \in [\![1,m]\!] \times [\![1,p]\!]$, it follows from Proposition 9.1 that all 1×1 quantum minors of M are non-zero, and it follows from Assertion 2 and Proposition 9.7 (with $l = v+1$ when $1 \leqslant v \leqslant u-1$) that the $(v+1) \times (v+1)$ quantum minors of M with $1 \leqslant v \leqslant u-1$ are non-zero. All this together shows that the $v \times v$ quantum minors of M with $v \geqslant u+1$ are equal to zero and that the $v \times v$ quantum minors of M with $1 \leqslant v \leqslant u$ are non-zero. This completes the inductive step and the result follows. □

Let $u \in [\![0, \inf(m-1,p-1)]\!]$ and set $w = [\![1,m-u]\!] \times [\![1,p-u]\!]$. Then w is an element of W and the matrix M_w satisfies the hypotheses of Theorem 10.1. Since the $v \times v$ quantum minors with $v \geqslant u+1$ are right linear combinations (with coefficients in R) of $(u+1) \times (u+1)$ quantum minors, the following theorem results from Theorem 10.1 (and [**10**, Théorème 3.7.2]).

Theorem 10.2. *Let u be an integer such that $0 \leqslant u \leqslant \inf(m-1,p-1)$, and set $w = [\![1,m-u]\!] \times [\![1,p-u]\!]$. Then w belongs to W and J_w is generated by the $(u+1) \times (u+1)$ quantum minors of $\mathcal{Y}$.*

Remark 10.3. Let u be an integer such that $0 \leqslant u \leqslant \inf(m-1,p-1)$. It follows from Theorem 10.2 that the ideal generated by the $(u+1) \times (u+1)$ quantum minors of $\mathcal{Y}$ is (completely) prime. So, we have just established that the quantum determinantal ideals are (completely) prime. In the more general case where we only assume that q is a non-zero element of any base field, this result was proved by Goodearl and Lenagan (see [**4**, Corollary 2.6]) by using different methods.

10.2. The case $w = ([\![1,m-u]\!] \times [\![1,p-u]\!]) \cup ([\![1,s]\!] \times \{p\})$ ($u \geqslant 1$, $s \geqslant 1$)

Theorem 10.4. *Assume that m and p are greater than or equal to 3, and let u and s be two integers such that $u \in [\![1, \inf(m-1,p-1)]\!]$ and $1 \leqslant s \leqslant m-1$. Set $w = ([\![1,m-u]\!] \times [\![1,p-u]\!]) \cup ([\![1,s]\!] \times \{p\})$. Then w belongs to W and J_w is generated by*

(1) *the $(u+1) \times (u+1)$ quantum minors of $\mathcal{Y}$;*

(2) *the $u \times u$ quantum minors of the matrix obtained from $\mathcal{Y}$ by deleting the rows $s+1, \dots, m$;*

(3) $Y_{1,p}, \dots, Y_{s,p}$.

Proof. By [**10**, Théorème 3.7.2], J_w is generated by the quantum minors of $\mathcal{Y}$ which belong to J_w. So, in order to find a generating set for J_w, we just have to find the quantum minors of M_w which are equal to 0. This is what we do now.

Since m and p are greater than or equal to 3, the four conventions of §§ 8 and 9 are satisfied if we replace K by F_w, M by M_w and $x_{i,\alpha}$ by $y_{i,\alpha}$ ($(i,\alpha) \in [\![1,m]\!] \times [\![1,p]\!]$). So, we can use the notation and results of these two sections. In particular, N still denotes the matrix obtained from $M_w^{(m,1)_{\mathrm{dc}}}$ by deleting the last row and the last column.

Now if $(i,\alpha) \in [\![1,m-1]\!] \times [\![1,p-1]\!]$, it follows from Theorem 3.7 that $t_{i,\alpha} = 0$ if and only if $(i,\alpha) \in [\![1,m-u]\!] \times [\![1,p-u]\!]$. Hence, it follows from Lemma 8.2 that N satisfies the hypotheses of Theorem 10.1 if we replace m by $m-1$, p by $p-1$ and u by $u-1$. Thus, the $v \times v$ quantum minors of N with $v \geqslant u$ are zero and the $v \times v$ quantum minors of N with $v \leqslant u-1$ are non-zero. It then follows from the first assertion of Proposition 8.4 (with $l = v \geqslant u$) that the $(v+1) \times (v+1)$ quantum minors of M_w with $v \geqslant u$ are zero. Hence, the quantum minors

$$\det_q(Y_{i,\alpha})_{\substack{i=i_1,\dots,i_v\\ \alpha=\alpha_1,\dots,\alpha_v}}$$

with $v \geqslant u+1$ belong to J_w.

It remains to deal with the $v \times v$ quantum minors of M_w such that $1 \leqslant v \leqslant u$. Let

$$\delta = \det_q(y_{i,\alpha})_{\substack{i=i_1,\dots,i_v\\ \alpha=\alpha_1,\dots,\alpha_v}}$$

be such a quantum minor. We consider four cases.

(i) Assume that $v = u$ and that $i_u \leqslant s$. It follows from the second assertion of Proposition 8.4 (with $l = u$) that $\delta = 0$. Hence, the quantum minors

$$\det_q(Y_{i,\alpha})_{\substack{i=i_1,\dots,i_u\\ \alpha=\alpha_1,\dots,\alpha_u}}$$

with $i_u \leqslant s$ belong to J_w.

(ii) Assume that $1 < v \leqslant u$ and that $i_v > s$. Recall that, if $1 \leqslant k \leqslant u-1$, the $k \times k$ quantum minors of N are non-zero. In particular, $\delta_{\hat{i}_v,\hat{\alpha}_v}^{(m,1)_{\mathrm{dc}}} \neq 0$. Thus, since $t_{i_v,p} \times t_{m,\alpha_v} \neq 0$ (remember that $i_v > s$), it follows from Proposition 9.7 that δ is non-zero. Hence, the quantum minors

$$\det_q(Y_{i,\alpha})_{\substack{i=i_1,\dots,i_v\\ \alpha=\alpha_1,\dots,\alpha_v}}$$

with $1 < v \leqslant u$ and $i_v > s$ do not belong to J_w.

(iii) Assume that $1 \leqslant v < u$ and that $i_v \leqslant s$. If $\alpha_v = p$, then the last column of δ is zero, so that $\delta = 0$. Hence, the quantum minors

$$\det_q(Y_{i,\alpha})_{\substack{i=i_1,\dots,i_v\\ \alpha=\alpha_1,\dots,\alpha_v}}$$

with $1 \leqslant v < u$, $i_v \leqslant s$ and $\alpha_v = p$ belong to J_w.

If $\alpha_v < p$, then, since $i_v \leqslant s < m$, $\delta^{(m,1)_{\mathrm{dc}}}$ is a $v \times v$ quantum minor of N. Thus, since $v < u$, we have $\delta^{(m,1)_{\mathrm{dc}}} \neq 0$. It then follows from Proposition 9.6 that $\delta \neq 0$. Hence, the quantum minors

$$\det_q(Y_{i,\alpha})_{\substack{i=i_1,\dots,i_v\\ \alpha=\alpha_1,\dots,\alpha_v}}$$

with $1 \leqslant v < u$, $i_v \leqslant s$ and $\alpha_v < p$ do not belong to J_w.

(iv) Assume that $v = 1$ and that $i_v > s$. Observe that $t_{i_v,p} \times t_{m,\alpha_v} \neq 0$ (since $i_v > s$). Thus, it follows from Proposition 9.1 that $\delta = y_{i_v,\alpha_v} \neq 0$. Hence, the quantum minors

$$\det_q(Y_{i,\alpha})_{\substack{i=i_1,\dots,i_v \\ \alpha=\alpha_1,\dots,\alpha_v}}$$

with $v = 1$ and $i_v > s$ do not belong to J_w.

We deduce from the above results that J_w is generated by

(i) the $v \times v$ quantum minors with $v \geqslant u + 1$;

(ii) the $u \times u$ quantum minors of the matrix obtained from $\mathcal{Y}$ by deleting the rows $s + 1, \dots, m$;

(iii) the $v \times v$ quantum minors with $1 \leqslant v < u$, $\alpha_v = p$ and $i_v \leqslant s$.

Denote by L_w the two-sided ideal in R generated by the $(u + 1) \times (u + 1)$ quantum minors of $\mathcal{Y}$, by the $u \times u$ quantum minors of the matrix obtained from $\mathcal{Y}$ by deleting the rows $s + 1, \dots, m$ and by $Y_{1,p}, \dots, Y_{s,p}$. The above results show that $L_w \subseteq J_w$. Since the $v \times v$ quantum minors with $v \geqslant u+1$ are left linear combinations (with coefficients in R) of $(u+1) \times (u+1)$ quantum minors, and since the $v \times v$ quantum minors with $1 \leqslant v < u$, $\alpha_v = p$ and $i_v \leqslant s$ are left linear combinations of $Y_{1,p}, \dots, Y_{s,p}$, we have $J_w \subseteq L_w$. Hence $J_w = L_w$ and the proof is complete. □

10.3. The case $w = (\llbracket 1, m - u\rrbracket \times \llbracket 1, p - u\rrbracket) \cup (\llbracket 1, s\rrbracket \times \{p\})$ $(u > s \geqslant 1)$

An immediate corollary of Theorem 10.4 is the following result.

Corollary 10.5. *Assume that m and p are greater than or equal to 3 and let u and s be two integers such that $1 \leqslant s < u \leqslant \inf(m - 1, p - 1)$. Set $w = (\llbracket 1, m - u\rrbracket \times \llbracket 1, p - u\rrbracket) \cup (\llbracket 1, s\rrbracket \times \{p\})$. Then w belongs to W and J_w is generated by*

(1) *the $(u + 1) \times (u + 1)$ quantum minors of $\mathcal{Y}$;*

(2) $Y_{1,p}, \dots, Y_{s,p}$.

The following result can be easily deduced from Corollary 10.5 (with $u \geqslant 2$ and $s = 1$).

Corollary 10.6. *Assume that m and p are greater than or equal to 3 and let $u \in \llbracket 2, \inf(m-1, p-1)\rrbracket$. The two-sided ideal in R generated by the $(u+1) \times (u+1)$ quantum minors of $\mathcal{Y}$ and by $Y_{1,p}$ is (completely) prime.*

Remark 10.7. Corollary 10.6 allowed Lenagan and Rigal [**11**] to show that the quantum determinantal factor rings of $O_q(\mathcal{M}_{m,p}(\mathbb{C}))$ are maximal orders.

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