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Infinite-dimensional Lie Algebras and Their Multivariable Generalizations

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The loop algebra $L\mathfrak{g} = \mathfrak{g}[z, z^{-1}]$, consisting of Laurent polynomials valued in a Lie algebra $\mathfrak{g}$, admits a nontrivial central extension $\widehat{\mathfrak{g}}$ for each choice of invariant pairing on $\mathfrak{g}$. This affine Lie algebra and its cousin the Virasoro algebra are foundational objects in representation theory and conformal field theory. A natural question then arises: do there exist multivariable, or higher dimensional, generalizations of the affine algebra?

Overview and Prerequisites

In §8.1 we give a rapid, biased review of the theory of infinite-dimensional Lie algebras. In §8.2 we introduce higher dimensional, multivariable generalizations of the Lie algebras from §8.1. In the final section we give a less formal discussion of further topics related to the theory of multivariable Lie algebras. The goal of §8.3.1 is to provide a geometric perspective on infinite-dimensional Lie algebras and modules based on the theory of factorization algebras. Finally, §8.3.2 introduces some modest enhancements of multivariable algebras analogous to the theory of affine algebras associated to super Lie algebras.

In these notes we assume some familiarity with Lie algebras, including the notion of a module. We assume that the reader is familiar with the notion of a cochain complex (especially in §8.2 and §8.3), and some other rudimentary notions in homological algebra. Some basic exposure to algebraic geometry will also be helpful for following the later sections.

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quantum field theories. Most importantly for these notes I'd like to acknowledge the work of Faonte, Hennion and Kapranov in [10], which develops the general theory of higher dimensional Kac–Moody algebras and whose presentation we closely follow in §8.2. I'm also extremely grateful for valuable collaborations with Owen Gwilliam and Ingmar Saberi whose results are discussed in §8.2 and §8.3.

8.1 Infinite-dimensional Lie algebras

The first infinite-dimensional Lie algebra we focus on arises from the algebra of Laurent polynomials $\mathbb{C}[z,z^{-1}]$, which is the algebra of functions on the punctured affine line $\mathbb{A}^{\times}$.

The space of derivations of any commutative algebra always forms a Lie algebra where the bracket is simply the *commutator* of the endomorphisms defining the derivations. The *Witt algebra* is the Lie algebra of derivations of $\mathbb{C}[z,z^{-1}]$. As a vector space, the Witt algebra admits a presentation in terms of vector fields on the punctured disk:

$$\mathsf{witt} \stackrel{\text{def}}{=} \left\{ f(z)\frac{\mathrm{d}}{\mathrm{d}z} \mid f(z) \in \mathbb{C}[z,z^{-1}] \right\} \cong \mathbb{C}[z,z^{-1}]\frac{\mathrm{d}}{\mathrm{d}z}.$$

The Lie bracket is simply the Lie bracket of vector fields

$$\left[f(z)\frac{\mathrm{d}}{\mathrm{d}z}, g(z)\frac{\mathrm{d}}{\mathrm{d}z} \right] = \left(f(z)g'(z) - f'(z)g(z) \right) \frac{\mathrm{d}}{\mathrm{d}z}.$$

It is standard to choose a basis $\{L_n \stackrel{\text{def}}{=} -z^{n+1}\frac{\mathrm{d}}{\mathrm{d}z}\}_{n\in\mathbb{Z}}$ for witt so that the commutator takes the form $[L_n, L_m] = (n-m)L_{n+m}$.

8.1.1 Central Extensions

We will turn to examples of modules for infinite-dimensional Lie algebras momentarily. The Witt algebra, for example, has a much richer theory of modules if instead of plain modules, one considers "projective" modules. This means that it is not the Witt algebra that acts but rather a certain *central extension* of the Witt algebra. We begin with some examples of central extensions.

The Heisenberg Algebra

There is another Lie algebra associated to the commutative algebra $\mathbb{C}[z,z^{-1}]$. In a trivial way, we can view it as an *abelian* Lie algebra. This is a rather boring infinite-dimensional Lie algebra, but there is a particular extension of it that will be characteristic of many of the constructions to come.

Suppose V is a vector space[1] equipped with a bilinear pairing $(\cdot,\cdot)\colon V\times V\to\ \mathbb{C}$. Let $\widehat{V}$ denote the vector space $V\oplus\mathbb{C}=V\oplus\mathrm{span}\{K\}$, and denote elements of this vector space by $v+\lambda K$, where $v\in V$ and $\lambda\in\mathbb{C}$. Then, we ask whether the formulas $[v,w]=(v,w)K$ and $[v,K]=0$ for $v,w\in V$ define the structure of a Lie algebra on $\widehat{V}$. The Jacobi identity for the bracket $[\cdot,\cdot]$ just defined is trivially satisfied. Thus, the only condition we must consider is the skew-symmetry of the bracket. Clearly, skew-symmetry holds if and only if the original bracket $(\cdot,\cdot)$ is anti-symmetric: we need $(v,w)=-(w,v)$ for all $v,w\in\mathbb{C}$.

Thus, these formulas define a Lie algebra structure on $\widehat{V}=V\oplus\mathbb{C}K$ if and only if the pairing $(\cdot,\cdot)$ is anti-symmetric. Let us return to the algebra $\mathbb{C}[z,z^{-1}]$. As a vector space $V=\mathbb{C}[z,z^{-1}]$ is equipped with an anti-symmetric pairing $(z^n,z^m)=m\delta_{n,-m}K$, where $\delta_{n,-m}$ is the function that is 1 when $n=-m$ and zero otherwise.

Definition 8.1 The *Heisenberg algebra* is the Lie algebra $\mathsf{h}=\mathbb{C}[z,z^{-1}]\oplus\mathbb{C}K$ with brackets defined by

$$[z^n,z^m]=m\delta_{n,-m}K$$

and $[K,z^n]=0$ for all $n,m\in\mathbb{Z}$.

The Heisenberg algebra is an example of the following more general notion (which is meaningful even for finite-dimensional algebras, of course):

Definition 8.2 A *central extension* of a Lie algebra $\mathfrak{g}$ by a Lie algebra $\mathfrak{c}$ is a Lie algebra $\widetilde{\mathfrak{g}}$ that sits in an exact sequence of Lie algebras

$$\mathfrak{c}\to\widetilde{\mathfrak{g}}\to\mathfrak{g}$$

with the property that $\mathfrak{c}$ lies in the center of $\widetilde{\mathfrak{g}}$.

In particular, $\mathfrak{c}$ must be an abelian Lie algebra. There is a classification of equivalences classes of central extensions based on the *Lie algebra cohomology* of $\mathfrak{g}$ with coefficients in $\mathfrak{c}$. We take a brief detour in this direction.

Lie Algebra Cohomology

We define the Lie algebra cohomology of a Lie algebra $\mathfrak{g}$ with coefficients in a $\mathfrak{g}$-module. Categorically speaking, Lie algebra cohomology of a module is a certain "derived" replacement for a very classical notion in Lie theory: the $\mathfrak{g}$-invariants of the module. There is a particular cochain complex modeling this replacement that we recall.

[1] We will work over $\mathbb{C}$ unless otherwise stated.

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The commutator in an associative algebra is a Lie bracket. Given any Lie algebra $\mathfrak{g}$, we let $U\mathfrak{g}$ denote its universal enveloping algebra. This is an associative algebra equipped with a canonical map $i\colon \mathfrak{g} \to U\mathfrak{g}$ of Lie algebras. It is universal in the sense that if A is any other algebra which admits a Lie algebra map $f : \mathfrak{g} \to A$, there is a unique map $\widetilde{f} : U\mathfrak{g} \to A$, for which $f = \widetilde{f} \circ i$. Explicitly, one obtains $U\mathfrak{g}$ as a quotient of the tensor algebra $T(\mathfrak{g}) = \oplus_{n\geqslant 0}\mathfrak{g}^{\otimes n}$. The famous Poincaré–Birkhoff–Witt theorem identifies $U\mathfrak{g}$ with $\mathrm{Sym}(\mathfrak{g})$ as vector spaces (but not algebras!).

Given a Lie algebra $\mathfrak{g}$ and a module M, the Lie algebra cohomology is the derived functor

$$H^n(\mathfrak{g};M) = \mathrm{Ext}^n_{U\mathfrak{g}}(\mathbb{C},M).$$

Here, we view $\mathbb{C}$ as a trivial $\mathfrak{g}$-module. One can use a particular resolution for the trivial $\mathfrak{g}$-module to come up with the following cochain model for Lie algebra cohomology.

Definition 8.3 Let $\mathfrak{g}$ be a Lie algebra and M a $\mathfrak{g}$-module. The *Chevalley–Eilenberg* cochain complex $\mathrm{C}^\bullet(\mathfrak{g};M)$ computing Lie algebra cohomology is the cochain complex whose underlying graded vector space is $\mathrm{Hom}(\mathrm{Sym}(\mathfrak{g}[1]), M) = \mathrm{Hom}\left(\oplus_{k\geqslant 0}(\wedge^k\mathfrak{g})[k]), M\right)$. The differential is defined as follows. Given a k-cochain $\varphi : \wedge^k\mathfrak{g} \to M$, the $(k+1)$-cochain $\mathrm{d}(\varphi)$ is defined by

$$\begin{aligned}\mathrm{d}(\varphi)(x_1,\dots,x_{k+1}) &= \sum_{i=1}^{k+1}(-1)^{i+1}x_i\cdot\varphi(x_1,\dots,\widehat{x_i},\dots,x_{k+1}) \\ &\quad + \sum_{1\leqslant i<j\leqslant k+1}(-1)^{i+j}\varphi([x_i,x_j],x_1,\dots,\widehat{x_i},\dots,\widehat{x_j},\dots,x_{k+1}). \qquad (8.1.1)\end{aligned}$$

It is a tedious but straightforward exercise to verify that $\mathrm{d}\circ\mathrm{d} = 0$, so that we have defined a cochain complex. The Chevalley–Eilenberg cochain complex computes the Lie algebra cohomology $H^\bullet(\mathfrak{g};M) = H^\bullet(\mathrm{C}^\bullet(\mathfrak{g};M))$.

Remark The linear dual cochain complex $\mathrm{C}_\bullet(\mathfrak{g};M)$ computes Lie algebra *homology*. As a graded vector space $\mathrm{C}_\bullet(\mathfrak{g};M)$ is $\mathrm{Sym}(\mathfrak{g}[1]) \otimes_{\mathbb{C}} M = \left(\oplus_{k\geqslant 0}(\wedge^k\mathfrak{g})[k])\right) \otimes_{\mathbb{C}} M$. We leave it as an exercise to write down the differential of this cochain complex which is linear dual to the one above.

Lemma 8.1 *Central extensions of* $\mathfrak{g}$ *by a* $\mathfrak{c}$ *are in one-to-one correspondence with the second cohomology group* $H^2(\mathfrak{g}\,;\,\mathfrak{c})$.

Proof We will use our model for Lie algebra cohomology. Suppose φ is a 2-cocycle representing a class in $H^2(\mathfrak{g};\mathfrak{c})$. Then, define the Lie algebra $\widetilde{\mathfrak{g}}$, which as a vector space is $\mathfrak{g}\oplus\mathfrak{c}$ with Lie brackets $[x,x'] = [x,x']_{\mathfrak{g}} + \varphi(x,x')$,

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and $[x,c] = [c,c'] = 0$ for $x,x' \in \mathfrak{g}$ and $c,c' \in \mathfrak{c}$. Here $[\cdot,\cdot]_{\mathfrak{g}}$ denotes the original Lie bracket on $\mathfrak{g}$.

Conversely, suppose $\widetilde{\mathfrak{g}}$ is such a central extension with Lie bracket $[\cdot,\cdot]$. Then, define the 2-cochain $\varphi(x,x') = [x,x'] - [x,x']_{\mathfrak{g}}$. It is immediate to check that φ is a 2-cocycle. We leave it as an exercise to formulate the appropriate notion of "equivalence" of two central extensions and to show how cohomologous cocycles give rise to it. □

Example 8.4 Consider the algebra $\mathbb{C}[z,z^{-1}]$ thought of as an abelian Lie algebra. Then, the bilinear map $\mathbb{C}[z,z^{-1}] \times \mathbb{C}[z,z^{-1}] \to \mathbb{C}$ defined by $(z^n,z^m) = m\delta_{n,-m}$ defines a 2-cocycle in $\mathrm{C}^2(\mathbb{C}[z,z^{-1}])$. This is rather trivial in this abelian case, as the only condition to check is that it is antisymmetric.

Gelfand–Fuks Cohomology

We now turn to classifying central extensions of the Witt algebra. The Lie algebra witt is infinite dimensional, so one must use caution when defining the Chevalley–Eilenberg cochain complex.

The following result is well known and can be found in [14].

Proposition 8.2 *The cohomology $H^2(\mathsf{witt})$ is one-dimensional spanned by a single class. This class may be represented by the cocycle $\varphi_{\mathrm{Vir}}(L_m,L_n) = \frac{1}{12}\delta_{m,-n}(m^3-m)$. Then φ_{Vir}.*

Remark Let $\mathrm{Res}: \mathbb{C}((z))\mathrm{d}z \to \mathbb{C}$ be the formal residue map, which sends a Laurent 1-form $\sum a_n z^n \mathrm{d}z$ to a_{-1}. In terms of vector fields on the formal punctured disk, one can rewrite this cocycle as $\frac{1}{12}\mathrm{Res}_z\,(f'(z)\mathrm{d}g'(z))$. Here, d is the formal de Rham differential $\mathrm{d}(h(z)) = h'(z)\mathrm{d}z$. The factor of $\frac{1}{12}$ is conventional and can be traced back to the Grothendieck–Riemann–Roch theorem and string theory.

Definition 8.5 The *Virasoro algebra* vir is the one-dimensional central extension of the Witt algebra witt defined by the 2-cocycle φ_{Vir}.

Remark The Witt algebra describes the infinitesimal symmetries of the punctured disk. The Virasoro algebra is closely tied to the moduli space of Riemann surfaces. The Lie algebra of conformal transformations on a punctured domain in a Riemann surface is given by two copies of the Witt algebra

$$\mathsf{witt} \oplus \overline{\mathsf{witt}} = \mathbb{C}[z,z^{-1}]\partial_z \oplus \mathbb{C}[\bar{z},\bar{z}^{-1}]\partial_{\bar{z}}.$$

The Virasoro algebra (associated to the holomorphic copy of the Witt algebra) corresponds to a certain *line bundle* over the moduli space of Riemann surfaces.

8.1.2 Affine Algebras

Algebraically, affine algebras are constructed from generalized Cartan matrices of a particular type [22]. Geometrically, and closer to the perspective we take here, affine algebras arise infinitesimally from *loop groups* – smooth maps from a circle S^1 into a Lie group [31]. We will not work directly with loop groups in these notes, rather we pass straight to the level of the Lie algebra.

Suppose that $\mathfrak{g}$ is a Lie algebra. For applications in representation theory, one often restricts to the case that $\mathfrak{g}$ is simple, but for now it will make no difference. We can tensor the commutative algebra of Laurent polynomials $\mathbb{C}[z,z^{-1}]$ with the Lie algebra $\mathfrak{g}$ to obtain a new Lie algebra

$$\mathfrak{g}\otimes\mathbb{C}[z,z^{-1}] = \mathfrak{g}[z,z^{-1}].$$

Write elements of this Lie algebra as $x\otimes f(z)$. Explicitly, the Lie bracket is defined by

$$[x\otimes f(z), y\otimes g(z)] = [x,y]\otimes(f\cdot g)(z).$$

We refer to $\mathfrak{g}[z,z^{-1}]$ as the *current algebra* associated to $\mathfrak{g}$. Equivalently, if we think of $\mathfrak{g}$ as an affine variety, this Lie algebra is the same as maps from the punctured affine space $\mathbb{A}^{\times}$ to $\mathfrak{g}$.

Just as in the case of the Virasoro algebra, there is a one-dimensional central extension of this Lie algebra. According to Lemma 8.1 we know to look for such central extensions in the Lie algebra cohomology $H^2(\mathfrak{g}[z,z^{-1}])$. To describe the relevant piece of this cohomology we introduce some terminology.

The algebra of polynomials on a vector space V is $\mathbb{C}[V] = \mathrm{Sym}(V^*)$. When $V=\mathfrak{g}$ we note that $\mathfrak{g}$ acts on its polynomials $\mathbb{C}[\mathfrak{g}]$ by the adjoint representation, which we will denote on elements by ad_x. By definition, an *invariant polynomial* of $\mathfrak{g}$ is a polynomial P on $\mathfrak{g}$ such that $\mathrm{ad}_x(P)=0$ for all $x\in\mathfrak{g}$. One has the following class of 2-cocycles on the current algebra $\mathfrak{g}[z,z^{-1}]$.

Definition 8.6 Suppose κ is an invariant quadratic polynomial of $\mathfrak{g}$. Define the 2-cochain $\varphi_\kappa \in \mathrm{C}^2(\mathfrak{g}[z,z^{-1}])$ by the formula $\varphi_\kappa(f(z)\otimes x, g(z)\otimes y) = \mathrm{Res}_z(f\mathrm{d}g)\kappa(x,y)$.

Remark More concisely, if we view $x(z)\in\mathfrak{g}[z,z^{-1}]$ as a $\mathfrak{g}$-valued function on the punctured disk, then we can write the 2-cochain as $\mathrm{Res}_z\,\kappa(x(z)\mathrm{d}y(z))$.

One can calculate that φ_κ is a cocycle. In this way, there is an embedding from invariant quadratic polynomials into the second cohomology:

$$\mathrm{Sym}^2(\mathfrak{g}^*)^{\mathfrak{g}} \hookrightarrow H^2(\mathfrak{g}[z,z^{-1}]).$$

For every such invariant quadratic, we obtain a central extension.

Definition 8.7 The *Kac–Moody affine algebra*, or simply *affine algebra*, $\widehat{\mathfrak{g}}_\kappa$ associated to an invariant quadratic polynomial κ is the central extension of $\mathfrak{g}[z,z^{-1}]$ defined by the 2-cocycle φ_κ.

Example 8.8 There is a natural quadratic invariant polynomial associated to any Lie algebra. The *Killing form* is the invariant polynomial $\kappa_{\mathrm{Kill}}(x,y) = \mathrm{Tr}\left(\mathrm{ad}_x \circ \mathrm{ad}_y\right)$ where the trace on the right-hand side is in the adjoint representation. When $\mathfrak{g}$ is simple, this is the unique invariant quadratic polynomial up to scale. Furthermore, $\mathfrak{g}$ is semisimple if and only if κ_{Kill} is nondegenerate.

8.1.3 Modules

The theory of modules for the infinite-dimensional Lie algebras is extremely rich and we make no assertions of giving a complete account here. Instead, we opt to give the idea of the flavor of modules we are interested in generalizing to the multivariable case. Let us first examine an example called the *Fock module*. This is a module for the Heisenberg Lie algebra h.

Fock Module

As a vector space the Heisenberg Lie algebra admits a presentation

$$\mathrm{span}_{\mathbb{C}}\{b[n]\}_{n\in\mathbb{Z}} \oplus \mathbb{C}\cdot K,$$

where in our previous notation $b[n]$ corresponds to the homogenous Laurent polynomial z^n. The commutator is simply $[b[n],K] = 0$ and $[b[n],b[m]] = m\delta_{n,-m}K$.

Consider the following polynomial algebra on an infinite number of generators:

$$V = \mathbb{C}[x_{-1},x_{-2},\ldots].$$

We will describe the structure of an h-module on this space.

The definition of this module is somewhat motivated by a physical system: one views the Heisenberg algebra h as some algebra of operators and V as the space of "states" of the system. In this case, one should imagine V as describing the configurations of some number of particles. The unit element $|0\rangle \stackrel{\mathrm{def}}{=} 1 \in V$ plays the role of the "vacuum", whereby no particles are present.

The usual convention is that the operator $b[-n]$, for $n > 0$, plays the role of the "creation" operator. That is, from the vacuum $b[-n]$ creates the nth excited state $b[-n]|0\rangle = x_{-n}$. Similarly, if $F(x_1,x_2,\ldots) \in V$ is any combination of multiparticle states it is natural to define

$$b[-n]F(x_{-1},x_{-2}\ldots) \stackrel{\mathrm{def}}{=} x_{-n}F(x_{-1},x_{-2},\ldots).$$

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This determines part of the module structure. Now, how do the $b[n]$ act for $n \geqslant 0$?

In the Heisenberg algebra, $b[n]$ participates in the bracket $[b[n], b[-n]] = nK$. Since K is central, we can choose that it acts on V diagonally by some fixed scalar. For now, we will assume it acts by the identity.

In order to have an action, then, we see that $b[n]$, for $n > 0$, must act on linear homogenous polynomials by $b[n]x_m = n\delta_{m,-n}|0\rangle$. Similarly, if $F(x_1, x_2, \ldots) \in V$ is any combination of multiparticle states one defines

$$b[n]F(x_{-1}, x_{-2} \ldots) \stackrel{\text{def}}{=} n\frac{\partial}{\partial x_{-n}} F(x_{-1}, x_{-2} \ldots).$$

It remains to define the operator b_0. One obvious choice is to declare it acts by zero, but in fact we obtain a module structure by declaring that it acts diagonally by multiplication by an arbitrary complex number $\mu \in \mathbb{C}$. The μ is called the *weight*.

To summarize, we have postulated an action of h on V where K acts by the identity and $b[n]$ acts by

$$b[n] = \begin{cases} x_n, & n < 0 \\ n\dfrac{\partial}{\partial x_{-n}}, & n > 0 \\ \mu, & n = 0 \end{cases}.$$

It is immediate to check that this endows V with an h-module structure. We arrive at the following definition. From now on, we refer to $F(\mu) = \mathbb{C}[x_{-1}, x_{-2} \ldots]$, with this module structure, as the *Fock module* of weight $\mu \in \mathbb{C}$.

In principle, we can generalize this definition slightly by declaring that K act diagonally by some arbitrary scalar $k \in \mathbb{C}$ rather than by the identity. There are essentially two cases. First, if $k \neq 0$ one can show that the reparametrization $b[n] \mapsto \frac{1}{\sqrt{k}} b[n]$ defines an isomorphism of the resulting module with $F(\mu)$ as defined above. Second, if $k = 0$ then $\mathbb{C}[x_{-1}, x_{-2}, \ldots]$ descends to a module for the abelian Lie algebra $\mathfrak{h}/(\mathbb{C}K)$ spanned by $\{b[n]\}_{n\in\mathbb{Z}}$.

One of the key properties of the module $F(\mu)$ is that it is irreducible, meaning it has no proper submodules. In fact, something stronger is true.

Proposition 8.3 *$F(\mu) \simeq F(\nu)$ if and only if $\mu = \nu$. Furthermore, if V is any irreducible h-module such that K acts by 1 and b_0 acts by μ, then $V \cong F(\mu)$.*

Here is a more invariant way to present this module. The Heisenberg algebra has an abelian subalgebra $\mathbb{C}[z] \oplus \mathbb{C} \cdot K \subset \mathsf{h}$ which is spanned by $\{b[n], K\}$ for $n \geqslant 0$. Define the one-dimensional module $\mathbb{C}(\mu) \simeq \mathbb{C}$ for this subalgebra by the rule that $b[0]$ acts by μ, K acts by 1 and $b[n]$ acts by zero for $n > 0$.

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The enveloping algebra $U(\mathsf{h})$ of the Heisenberg algebra is naturally an h-module. In particular, it is a module for the subalgebra $\mathbb{C}[z] \oplus \mathbb{C} \cdot K$.

Proposition 8.4 *There is an isomorphism of* h*-modules* $F(\mu) \cong U(h) \otimes_{U(\mathbb{C}[z] \oplus K)} \mathbb{C}(\mu)$. *In other words,* $F(\mu)$ *is the* h*-module* **induced** *from the* $\mathbb{C}[z] \oplus K$ *module* $\mathbb{C}(\mu)$.

Vacuum Modules

The other class of modules we introduce are modules over the affine algebra $\widehat{\mathfrak{g}}_\kappa$. Like the Fock module, it is induced from a representation of a subalgebra. Consider the subalgebra

$$\mathfrak{g}[z] \oplus \mathbb{C}K \subset \widehat{\mathfrak{g}}_\kappa.$$

Define the one-dimensional module $\mathbb{C}_1 \cong \mathbb{C}$ for this subalgebra where $\mathfrak{g}[z]$ acts trivially and K acts by the identity.

Definition 8.9 The *vacuum module* of level κ is the $\widehat{\mathfrak{g}}_\kappa$ module

$$V_\kappa(\mathfrak{g}) \overset{\text{def}}{=} U(\widehat{\mathfrak{g}}_\kappa) \otimes_{U(\mathfrak{g}[z] \oplus K)} \mathbb{C}_1. \tag{8.1.2}$$

While we do not touch on many details here, this class of modules for the affine Kac–Moody algebra appears in areas of physics, representation theory, and number theory. Perhaps most importantly, the vacuum module carries the structure of a *vertex algebra* which we will briefly touch on later in §8.3.1. The description as a vertex algebra has led to progress in the context of Langlands duality, specifically in the characterization of the center of the affine algebra in terms of “opers” on the Langlands dual group; see [11] and references therein for formulations of this perspective. Many of these applications rest on the relationship between the affine algebra and the Virasoro algebra. The famous Segal–Sugawara construction endows $V_\kappa(\mathfrak{g})$ with the structure of a $\mathsf{vir}_{c(\kappa)}$-module (for some value of $c(\kappa)$) for all but a single value of κ, namely when $\kappa_c = -\frac{1}{2}\kappa_{\text{Kill}}$ called the *critical level.*

8.2 Multivariable Generalizations

We return to the algebraic situation to discuss possible multivariable generalizations of the algebra of Laurent polynomials $\mathbb{C}[z, z^{-1}]$, which we recall is the algebra of functions on punctured affine space $\mathbb{A}^\times$. For $n > 1$, there are essentially two obvious candidates:

- Replace punctured affine space by the n-fold product

$$(\mathbb{A}^\times)^n = \mathbb{A}^\times \times \cdots \times \mathbb{A}^\times.$$

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The algebra of functions is generated by n invertible algebraic parameters

$$\mathcal{O}((\mathbb{A}^{\times})^n) = \mathbb{C}[z_1^{\pm}, \ldots, z_n^{\pm}].$$

- Replace punctured affine space $\mathbb{A}^{\times} = \mathbb{A}^1 \setminus 0$ by punctured n-space $\mathbb{A}^n \setminus 0$. By an algebraic version of Hartogs' theorem, one finds that the algebra of functions on $\mathbb{A}^n \setminus 0$ is the algebra of polynomials in n-variables

$$\mathcal{O}(\mathbb{A}^n \setminus 0) \simeq \mathbb{C}[z_1, \ldots, z_n].$$

In particular, $\mathcal{O}(\mathbb{A}^n \setminus 0) = \mathcal{O}(\mathbb{A}^n)$.

There are intermediate versions of these two extreme cases where one takes some number of punctured m-spaces for $1 \leqslant m \leqslant n$, e.g. $\mathbb{A}^{\times} \times (\mathbb{A}^2 \setminus 0)$ when $n = 3$.

Central extensions play a crucial role in the theory of infinite-dimensional Lie algebras. The most basic example is the Heisenberg algebra which is the one-dimensional central extension of the abelian Lie algebra of Laurent polynomials $\mathbb{C}[z, z^{-1}]$ defined by the 2-cocycle $(f, g) \mapsto \mathrm{Res}(f\mathrm{d}g)$. We ask for analogs of the Heisenberg algebra starting with the multivariable algebras above. In other words, do there exist analogs of this 2-cocycle defined in terms of some residue pairing?

The algebra $\mathbb{C}[z_1^{\pm}, \ldots, z_n^{\pm}]$ is the building block for the theory of toroidal algebras. There is no obvious analog of the one-dimensional central extension, as in the Heisenberg algebra. However, there is a "universal" central extension which takes values in the space $\Omega^1_B/\mathrm{d}_B\mathcal{O}$, of Kähler differentials modulo exact Kähler differentials. We will not be concerned with this extension.

Let us turn to punctured n-space. For $n > 1$, we recalled that $\mathcal{O}(\mathbb{A}^n \setminus 0) = \mathcal{O}(\mathbb{A}^n)$. This result might suggest that $\mathbb{A}^n \setminus 0$ is an unnatural place to seek a generalization of the loop algebra. Such pessimism is misplaced because of the fundamental difference with the one-dimensional case: $\mathbb{A}^n \setminus 0$ *is not an affine scheme* for $n > 1$. In other words, its sheaf of holomorphic functions has non-trivial cohomology. From the point of view of algebraic geometry it is more natural to use a *derived* algebra of functions modeling the space of derived global sections $\mathbb{R}\Gamma(\mathbb{A}^n \setminus 0, \mathcal{O})$.

It will be convenient to reference the cohomology of higher dimensional punctured affine space. A standard computation reveals the following description

$$H^i(\mathbb{A}^n \setminus 0, \mathcal{O}) = \begin{cases} 0, & i \neq 1, \ldots, n-2, n \\ \mathbb{C}[z_1, \ldots, z_n], & i = 0 \\ z_1^{-1} \cdots z_n^{-1} \mathbb{C}[z_1^{-1}, \ldots, z_n^{-1}], & i = n-1 \end{cases}.$$

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This description arises from the calculation of the Čech cohomology with respect to the cover by the affine open sets $\mathbb{A}^n \setminus \{z_i = 0\}$. The algebra structure on the cohomology is easy to describe. In degree zero, one finds the usual polynomial ring. The piece in degree $(n-1)$ is a module for this ring and the full graded algebra structure is a square-zero extension of this ring by this module.

In the first section we recall a particular model following [10] at the level of cochain complexes for the derived algebra of global sections of punctured affine space.

8.2.1 Differential Graded Algebras

The problem is to effectively probe the "non-affineness" of $\mathbb{A}^n \setminus 0$. A characterizing property is the existence of nontrivial cohomology classes in degrees other than zero. One of the core principles of "derived mathematics" is that one should keep track not of the cohomology, but of a particular *model* for it as a cochain complex.

As an example of a "model," consider the sheaf cohomology $H^\bullet(X, \mathcal{F})$ for $\mathcal{F}$ a sheaf on X. The Čech complex associated to an open cover $\mathcal{U}$ of X is a particular cochain complex $(\check{C}^\bullet(\mathcal{U}, \mathcal{F}), \delta)$ which computes the cohomology of $\mathcal{F}$:

$$H^\bullet(X, \mathcal{F}) \simeq H^\bullet\left(\check{C}^\bullet(\mathcal{U}, \mathcal{F}), \delta\right).$$

Any such model for sheaf cohomology of the pair $(X, \mathcal{F})$ is denoted $\mathbb{R}\Gamma(X, \mathcal{F})$ and is referred to as the "derived" global section of $\mathcal{F}$. This object is well defined only up to quasi-isomorphism of cochain complexes. For instance, if X is a smooth complex manifold and $\mathcal{F}$ is the global section of a holomorphic vector bundle F, another model is given by the so-called "Dolbeault complex"

$$\left(\Omega^{0,\bullet}(X, F)\;,\;\overline{\partial}\right).$$

The Dolbeault complex is analogous to the ordinary de Rham complex of differential forms. The Dolbeault complex is concentrated in degrees $0, \ldots, n = \dim_{\mathbb{C}}(X)$. In degree k, the space $\Omega^{0,k}(X, F)$ is the smooth sections of the vector bundle $\wedge^k \mathrm{T}_X^{*0,1} \otimes F$, where $\mathrm{T}_X^{*0,1}$ stands for the anti-holomorphic piece of the complexified cotangent bundle of X. The differential $\overline{\partial}$ is a component of the de Rham differential, which is extracted using the complex structure of both X and the bundle F. We refer to [21] for a more detailed exposition of Dolbeault cohomology.

Let us spell out what this looks like locally. We assume that $X = \mathbb{C}^n$ and that F is the trivial bundle, for simplicity. Introduce a holomorphic coordinate

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system $z = (z_1, \ldots, z_n)$. The zeroth piece of the Dolbeault complex $\Omega^{0,0}(\mathbb{C}^n)$ consists simply of smooth functions $f(z, \bar{z})$ on $\mathbb{C}^n$. In fact, the full Dolbeault complex is a graded module over smooth functions freely generated by n elements $\mathrm{d}\bar{z}_1, \ldots, \mathrm{d}\bar{z}_n$ providing a frame for $\mathrm{T}^{*0,1}$; they have cohomological degree $+1$. That is, as a graded vector space the Dolbeault complex is the graded polynomial algebra over smooth functions

$$\Omega^{0,\bullet}(\mathbb{C}^n) = \mathbb{C}^\infty(\mathbb{C}^n)[\mathrm{d}\bar{z}_1, \ldots, \mathrm{d}\bar{z}_n]. \tag{8.2.1}$$

Here, "graded" means that we impose the relations $\mathrm{d}\bar{z}_i \mathrm{d}\bar{z}_j = -\mathrm{d}\bar{z}_j \mathrm{d}\bar{z}_i$[2] since $\mathrm{d}\bar{z}_i$ have cohomological degree $+1$. The $\overline{\partial}$ operator is defined by the formula

$$\overline{\partial} \stackrel{\text{def}}{=} \sum_i \mathrm{d}\bar{z}_i \frac{\partial}{\partial \bar{z}_i}, \tag{8.2.2}$$

which is easily seen to satisfy $\overline{\partial}^2 = 0$.

There are two very special properties of the Dolbeault complex that we'd like to emphasize:

- First, the Dolbeault complex extends to a *sheaf* of cochain complexes. In particular, if $U \subset \mathbb{C}^n$ is an open set then the Dolbeault complex $\Omega^{0,\bullet}(U)$ is defined and there is a natural restriction map $\Omega^{0,\bullet}(\mathbb{C}^n) \to \Omega^{0,\bullet}(U)$. Explicitly, $\Omega^{0,\bullet}(U)$ is the free polynomial algebra over $\mathbb{C}^\infty(U)$ on the same generators $\mathrm{d}\bar{z}_i$ as above. Restriction is simply the restriction of smooth functions.
- At least for the trivial bundle, the Dolbeault complex has the structure of a *differential graded algebra*.

Definition 8.10 A *differential graded (dg) algebra* is a cochain complex (A, d) together with a multiplication $\cdot : A \times A \to A$ such that

$$\mathrm{d}(a \cdot b) = (\mathrm{d}a) \cdot b + (-1)^{|a|} a \cdot (\mathrm{d}b).$$

A commutative dg algebra is one that additionally satisfies $a \cdot b = (-1)^{|a||b|} b \cdot a$ for all a, b.

In particular, for any open $U \subset \mathbb{C}^n$, the Dolbeault complex $\Omega^{0,\bullet}(U)$ is a commutative dg algebra model for the derived sections of the structure sheaf $\mathbb{R}\Gamma(U, \mathcal{O}_U)$, where $\mathcal{O}_U$ stands for the sheaf of holomorphic functions on U.

Take $U = \mathbb{C}^n \setminus 0$. There is a small difference between the cohomology of the Dolbeault complex of $\mathbb{C}^n \setminus 0$ and the cohomology of $\mathbb{A}^n \setminus 0$, as we recalled in the introduction to this section. When we write $\mathbb{C}^n$, we are working in the category

[2] We drop the conventional $\wedge$ symbol in these notes.

of *complex manifolds*, whereas $\mathbb{A}^n$ is an *algebraic* variety. In this section we want to work algebraically as an attempt to follow the setting in the first chapter as closely as possible.

An Algebraic Model for Punctured Affine Space

All polynomials are holomorphic functions, so there is an embedding of algebras

$$\mathcal{O}(\mathbb{A}^n) = \mathbb{C}[z_1, \ldots, z_n] \hookrightarrow \mathcal{O}(\mathbb{C}^n).$$

We are interested in an *algebraic* analog of the Dolbeault complex which gives a particular model for $\mathbb{R}\Gamma\left(\mathbb{A}^n \setminus 0, \mathcal{O}_{\mathbb{A}^n \setminus 0}\right)$.

For affine space, one way to do this is to formally replace $C^\infty(\mathbb{C}^n)$ in (8.2.1) with polynomials that have both holomorphic and anti-holomorphic dependence

$$\mathbb{C}[z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n][\mathrm{d}\bar{z}_1, \ldots, \mathrm{d}\bar{z}_n].$$

This is, in fact, a subalgebra of $\Omega^{0,\bullet}(\mathbb{C}^n)$ and the differential $\overline{\partial}$ clearly preserves it. So, this gives us an algebraic version of $\Omega^{0,\bullet}(\mathbb{C}^n)$. A useful way to characterize this subalgebra is as the sum of eigenspaces for the natural action by the group $\mathrm{U}(1)^{\times n}$, which rotates each coordinate independently.

What about an algebraic version of $\Omega^{0,\bullet}(\mathbb{C}^n \setminus 0)$? The model we review here was proposed first in [10] motivated by the Jouanolou torsor. First, consider the localized ring

$$\mathsf{R}_n \stackrel{\text{def}}{=} \mathbb{C}[z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n]\left[\frac{1}{z\bar{z}}\right],$$

where $z\bar{z} = z_1\bar{z}_1 + \cdots + z_n\bar{z}_n$. One should think of R_n as the ring of smooth polynomial functions on punctured affine space.

Next, consider the graded algebra freely generated over R_n by degree $+1$ elements $\mathrm{d}\bar{z}_i$

$$\mathsf{R}_n[\mathrm{d}\bar{z}_1, \ldots, \mathrm{d}\bar{z}_n]. \tag{8.2.3}$$

Assign a "$\bar{z}$-weight" to this ring by declaring that $\bar{z}_i$ and $\mathrm{d}\bar{z}_i$ have $\bar{z}$-weight $+1$ and $(z\bar{z})^{-1}$ has $\bar{z}$-weight -1.

Definition 8.11 Let A_n be the graded subalgebra of (8.2.3) consisting of elements $\alpha = \alpha(z, \bar{z}, \mathrm{d}\bar{z}, (z\bar{z})^{-1})$ satisfying the following two conditions:

- the $\bar{z}$-weight of α is zero;
- the contraction of α with the anti-holomorphic Euler vector field $\overline{E} = \sum_i \bar{z}_i \partial_{\bar{z}_i}$ vanishes.

The graded algebra A_n equipped with the $\overline{\partial}$ operator (8.2.2) provides a commutative dg algebra model for $\mathbb{R}\Gamma\left(\mathbb{A}^n \setminus 0, \mathcal{O}_{\mathbb{A}^n \setminus 0}\right)$.

Proposition 8.5 ([10]) *The $\overline{\partial}$ differential acting on the graded ring* (8.2.3) *restricts to a differential on A_n. Furthermore, this endows $(A_n, \overline{\partial})$ with the structure of a commutative dg algebra whose cohomology is isomorphic to $H^\bullet\left(\mathbb{A}^n \setminus 0, \mathcal{O}_{\mathbb{A}^n \setminus 0}\right)$.*

We will unpack this model in a few low-dimensional cases. First, notice that A_1 is concentrated in degree zero. Indeed $\bar{z}(z\bar{z})^{-1} = z^{-1}$ thus $\mathsf{R}_1 = \mathbb{C}[z, z^{-1}, \bar{z}, \bar{z}^{-1}]$ and $\mathsf{A}_1 = \mathbb{C}[z, z^{-1}]$ as well.

Let's look at the case $n = 2$. The graded algebra A_2 is only concentrated in degrees zero and one:

$$\mathsf{A}_2 : \quad \left(\mathsf{A}_2^0 \xrightarrow{\overline{\partial}} \mathsf{A}_2^1\right).$$

The degree zero piece A_2^0 is the algebra on generators z_1, z_2 and $\bar{z}_1/(z\bar{z}), \bar{z}_2/(z\bar{z})$ subject to the relation

$$z_1(\bar{z}_1/(z\bar{z})) + z_2(\bar{z}_2/(z\bar{z})) = 1.^3$$

In particular, an element can be written as a linear combination of monomials of the form

$$z_1^{k_1} z_2^{k_2} \frac{\bar{z}_1^{\ell_1} \bar{z}_2^{\ell_2}}{(z\bar{z})^{\ell_1 + \ell_2}}, \quad k_i, \ell_i \geqslant 0,$$

subject to the relation. Of course, the differential $\overline{\partial}$ annihilates z_1, z_2. Moreover,

$$\overline{\partial} \frac{\bar{z}_1^{\ell_1} \bar{z}_2^{\ell_2}}{(z\bar{z})^{\ell_1+\ell_2}} = \mathrm{d}\bar{z}_1 \frac{\bar{z}_1^{\ell_1 - 1} \bar{z}_2^{\ell_2}}{(z\bar{z})^{\ell_1+\ell_2+1}} (\ell_1 z_2 \bar{z}_2 - \ell_2 z_1 \bar{z}_1) + \mathrm{d}\bar{z}_2 \frac{\bar{z}_1^{\ell_1} \bar{z}_2^{\ell_2 - 1}}{(z\bar{z})^{\ell_1+\ell_2+1}} (\ell_2 z_1 \bar{z}_1 - \ell_1 z_2 \bar{z}_2). \tag{8.2.4}$$

This is zero if and only if $\ell_1 = \ell_2 = 0$. Thus, we see that the cohomology in degree zero is simply polynomials in z_1, z_2: $H^0(\mathsf{A}_2) = \mathbb{C}[z_1, z_2]$.

In degree one, A_2 consists of elements of the form $g_1\, \mathrm{d}\bar{z}_1 + g_2\, \mathrm{d}\bar{z}_2, g_i \in \mathsf{R}_2$ subject to the condition that the $\bar{z}$-weight of g_i is -1 and $g_1 \bar{z}_1 + g_2 \bar{z}_2 = 0$. Similarly as in the previous paragraph, we find that A_2^1 is equal to $\mathsf{A}_2^0 \cdot \omega$ where ω is

$$\omega = \frac{\bar{z}_2\, \mathrm{d}\bar{z}_1 - \bar{z}_1\, \mathrm{d}\bar{z}_2}{(z\bar{z})^2}. \tag{8.2.5}$$

[3] Generally, these equations describe a quadric in weighted projective space.

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This is the *Bochner–Martinelli* kernel in complex dimension two, which we will discuss in more detail in the next subsection.

To identify the cohomology in degree one, we can use (8.2.4) to show that non-exact elements can be identified with $h\omega$ where h is a polynomial in $\overline{z}_1/(z\overline{z}), \overline{z}_2/(z\overline{z})$. Thus

$$H^1(\mathsf{A}_2) \simeq \mathbb{C}\left[\frac{\overline{z}_1}{z\overline{z}}, \frac{\overline{z}_2}{z\overline{z}}\right]\omega.$$

There is an abstract isomorphism between this space and the presentation we gave for $H^1(\mathbb{A}^2 \setminus 0, \mathcal{O})$ as $z_1^{-1}z_2^{-1}\mathbb{C}[z_1^{-1}, z_2^{-1}]$ in the introduction of this section. We will see momentarily how the higher residue implements this isomorphism. We turn to that in the next section.

Before moving on, we want to observe that there is an embedding of cochain complexes from the algebraic model A_n of punctured affine space to the analytic one provided by the Dolbeault complex of $\mathbb{C}^n \setminus 0$

$$\mathsf{A}_n \hookrightarrow \Omega^{0,\bullet}(\mathbb{C}^n \setminus 0), \tag{8.2.6}$$

which is simply the inclusion $z, \overline{z}, \mathrm{d}\overline{z} \mapsto z, \overline{z}, \mathrm{d}\overline{z}$. In fact, this defines an embedding into the Dolbeault complex of $D^n \setminus 0$ for any punctured disk.

8.2.2 Interlude: Residues

Cauchy's integral formula for a disk $D \subset \mathbb{C}$ states that for any smooth function $f\colon D \to \mathbb{C}$ and point $z \in \mathbb{C}$,

$$2\pi\sqrt{-1}f(w) = \oint_{\partial D} \frac{\mathrm{d}z}{z-w} f(z) + \mathsf{int}_D \frac{\mathrm{d}z}{z-w} \wedge \overline{\partial} f.$$

If f is holomorphic the last term drops out and we obtain the more familiar formula $f(w) = \frac{1}{2\pi\sqrt{-1}} \oint \frac{\mathrm{d}z}{z-w} f(z)$. One uses the integral formula to prove the famous residue formula: for any holomorphic one-form ω defined on a punctured disk $D \setminus \{w\}$ one has $\mathrm{Res}_w(\omega) = \frac{1}{2\pi\sqrt{-1}} \oint_C \omega$ where C is a closed Jordan curve contained in $D \setminus \{w\}$. Note that a special role in the Cauchy integral formula is played by the "Cauchy kernel" $\frac{\mathrm{d}z}{z-w}$, which is a holomorphic one-form defined on $D \setminus \{w\}$ satisfying $\mathrm{Res}_w\left(\frac{\mathrm{d}z}{z-w}\right) = \frac{1}{2\pi\sqrt{-1}}$.

There are higher dimensional versions of the Cauchy kernel, which involve integrals over products of punctured disks. We will be more interested in integrals over the once punctured n-disk.

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Let $w = (w_1, \dots, w_n) \in \mathbb{C}^n$. Define the *Bochner–Martinelli* kernel to be the differential form of type $(0, n-1)$ defined on $\mathbb{C}^n \setminus w$ by

$$\omega_{\mathrm{BM}}(z,w) = (-1)^{n-1} \frac{(n-1)!}{(2\pi\sqrt{-1})^n} \frac{1}{|z-w|^{2n}} \sum_{1 \leqslant i \leqslant n} (\overline{z}_i - \overline{w}_i) \wedge \mathrm{d}\overline{z}_1 \wedge \cdots \wedge \widehat{\mathrm{d}\overline{z}_i} \wedge \cdots \wedge \mathrm{d}\overline{z}_n.$$

What will often appear in integral expressions is the differential form of type $(n, n-1)$ given by $\mathrm{d}^n z \wedge \omega_{\mathrm{BM}}(z,w)$.

The Bochner–Martinelli kernel appears in the higher dimensional generalization of the Cauchy integral formula. Consider an n-disk D^n around a point $w \in \mathbb{C}^n$ and suppose f is a smooth function $f\colon D^n \to \mathbb{C}$. Then

$$f(w) = \oint_{\partial D^n} \mathrm{d}^n z \wedge \omega_{\mathrm{BM}}(z,w) f(z) + \mathsf{int}_{D^n} \mathrm{d}^n z \wedge \omega_{\mathrm{BM}}(z,w) \wedge \overline{\partial} f(z). \qquad (8.2.7)$$

In (8.2.6) we have seen how the dg algebra A_n embeds in $\Omega^{0,\bullet}(D^n \setminus 0)$ for any disk D^n centered at the origin. Use this embedding to define the following "higher residue."

Definition 8.12 The n-dimensional *residue* at $z = 0$ is the linear map $\mathrm{Res}_{z=0}\colon \mathsf{A}_n[n-1] \to \mathbb{C}$ defined by the formula

$$\mathrm{Res}_{z=0}(\alpha) = \oint_{\partial D^n} \mathrm{d}^n z \wedge \alpha,$$

where D^n is any closed n-disk centered at zero.

The residue can be extended to points other than $z = 0$ in a standard way. We want to point out the peculiar cohomological shift in the definition of the residue. The reason for this is that the integral in question is over a $2n-1$-dimensional sphere and so $\mathrm{d}^n z \wedge \alpha$ must be a form of total degree $2n-1$, which means that α must be a Dolbeault form of type $(0, n-1)$. Also, we observe that the residue is trivially a cochain map: A_n is concentrated in degrees $[0, n-1]$ and the residue is only nonzero on the piece in top degree. Slightly more nontrivial is the following lemma.

Lemma 8.6 *For any* $\alpha \in \mathsf{A}_n$ *one has* $\mathrm{Res}_{z=0}(\overline{\partial}\alpha) = 0$.

Proof By Stokes' theorem $\mathrm{Res}_{z=0}(\overline{\partial}\alpha) = \oint \mathrm{d}^n z \wedge \overline{\partial}\alpha = \oint \overline{\partial}(\mathrm{d}^n z \wedge \alpha) = 0$. □

This lemma implies the residue descends to a map on cohomology $\mathrm{Res}_{z=0}\colon H^{n-1}(\mathsf{A}_n) \to \mathbb{C}$. The residue provides a useful characterization of the cohomology. Consider the Bochner–Martinelli kernel $\omega \stackrel{\mathrm{def}}{=} \omega_{\mathrm{BM}}(z,0)$. First, we observe

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that ω is an element in A_n of degree $(n-1)$. Also, notice that when $n=1$ this is simply the function $\frac{1}{z}$. When $n=2$ this is the $(0,1)$ form defined in (8.2.5). Notice that we verified part of this proposition in the previous section, by hand, without explicitly using the residue.

Proposition 8.7 *The class $[\omega] \in H^{n-1}(A_n)$ is nontrivial in cohomology. In fact, every element in the $(n-1)$st cohomology can be written as linear combinations of classes of elements of the form*

$$\partial_{z_1}^{k_1} \cdots \partial_{z_n}^{k_n} \omega \in A_n^{n-1}.$$

Proof By the integral formula (8.2.7) we have $\mathrm{Res}_{z=0}(\omega) = 1$. To see that the remaining classes in the proposition are nontrivial, we simply apply integration by parts in the integral formula. □

By this proposition we have the following description of the cohomology of A_n: in degree zero it is $\mathbb{C}[z_1, \ldots, z_n]$ and in degree $(n-1)$ it is

$$H^{n-1}(\mathsf{A}_n) \simeq \mathbb{C}[\partial_{z_1}, \ldots, \partial_{z_n}]\omega.$$

8.2.3 Higher Current Algebras

A familiar generalization of the notion of a Lie algebra in the context of supersymmetry and certain topics in representation theory is a *super* Lie algebra. A super Lie algebra is a $\mathbb{Z}/2$-graded vector space $\mathfrak{h} = \mathfrak{h}^{\mathrm{even}} \oplus \Pi\mathfrak{h}^{\mathrm{odd}}$ equipped with a bracket that preserved the parity. Here, the $\Pi(-)$ denotes parity shift. This bracket is required to satisfy $\mathbb{Z}/2$ graded versions of the antisymmetry and Jacobi relations as well. A natural further lift of a super Lie algebra is the notion of a $\mathbb{Z}$-graded Lie algebra, henceforth referred to as just a graded Lie algebra. We will be interested in graded Lie algebras equipped with the further data of a differential.

Definition 8.13 A *dg Lie algebra* is the data of a graded vector space $\mathfrak{h} = \oplus \mathfrak{h}^i[-i]$, a linear map $\mathrm{d}\colon \mathfrak{h}^\bullet \to \mathfrak{h}^{\bullet+1}$ and a bracket $[\cdot,\cdot]\colon \mathfrak{h}^\bullet \times \mathfrak{h}^\bullet \to \mathfrak{h}^\bullet$ such that the following conditions hold.

- d turns $\mathfrak{h} = \oplus \mathfrak{h}^i[-i]$ into a cochain complex (so $\mathrm{d}^2 = 0$) and the graded Leibniz rule is satisfied

$$\mathrm{d}[x,y] = [\mathrm{d}x, y] + (-1)^{|x|}[x, \mathrm{d}y].$$

- Graded anti-symmetry $[x,y] = (-1)^{|x||y|+1}[y,x]$.

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- Graded Jacobi identity

$$(-1)^{|x||z|}[x,[y,z]]+(-1)^{|x||y|}[y,[z,x]]+(-1)^{|y||z|}[z[x,y]]=0$$

is satisfied.

The starting point for affine algebras is the loop algebra $\mathfrak{g}[z,z^{-1}]=\mathfrak{g}\otimes_{\mathbb{C}}\mathbb{C}[z,z^{-1}]$. The Lie algebra structure on this loop algebra is induced from the Lie bracket on $\mathfrak{g}$ together with the commutative product on Laurent polynomials. A similar construction holds in the setting of dg Lie algebras.

Lemma 8.8 *Suppose (A,d) is a commutative dg algebra and $(\mathfrak{h},[\cdot,\cdot]_{\mathfrak{h}})$ is an ordinary Lie algebra. Then $A\otimes\mathfrak{h}$ is endowed with the natural structure of a dg Lie algebra. The differential is $d\otimes\mathrm{id}_{\mathfrak{h}}$ and the bracket is $[a\otimes X,b\otimes Y]=(ab)\otimes[X,Y]_{\mathfrak{h}}$.*

We arrive at the precise notion of the "higher current algebras" we referenced in the introduction. Recall the dg model A_n for the derived global sections of punctured affine space $\mathbb{A}^n\setminus 0$.

Definition 8.14 The *n-current algebra* of a Lie algebra $\mathfrak{g}$ is the dg Lie algebra $\mathfrak{g}\otimes\mathsf{A}_n$. This dg Lie algebra is concentrated in degrees $[0,n-1]$ and the differential is given by the $\overline{\partial}$ operator.

If $(\mathfrak{h},\mathrm{d},[\cdot,\cdot])$ is any dg Lie algebra then the d-cohomology $H^\bullet(\mathfrak{h})$ inherits the structure of a graded Lie algebra from the original bracket.

Proposition 8.9 *The graded Lie algebra $H^\bullet(\mathfrak{g}\otimes A_n)$ is concentrated in degrees zero and $n-1$ and admits the presentation:*

- *In degree zero the cohomology is $\mathfrak{g}\otimes\mathbb{C}[z_1,\dots,z_n]$. We use the notation for elements:*

$$X[k_1,\dots,k_n]\overset{\mathrm{def}}{=}X\otimes z_1^{k_1}\cdots z_n^{k_n}\in\mathfrak{g}\otimes\mathbb{C}[z_1,\dots,z_n],$$

where $X\in\mathfrak{g}$, $k_1,\dots,k_n\geqslant 0$.

- *In degree $n-1$ the cohomology is $\mathfrak{g}\otimes\mathbb{C}[\partial_{z_1},\dots,\partial_{z_n}]\omega$. We use the notation for elements:*

$$X[\ell_1,\dots,\ell_n]\overset{\mathrm{def}}{=}X\otimes\left(\prod_{j=1}^{n}\frac{(-1)^{-\ell_j-1}}{(-\ell_j-1)!}\partial_{z_j}^{-\ell_j-1}\right)\omega,$$

where $X\in\mathfrak{g}$, $\ell_1,\dots,\ell_n<0$.[4]

[4] The complicated-looking normalization makes for simpler formulas, which will appear later on.

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The (graded) Lie bracket is described by

$$\Big[[X[k_1,\dots,k_n],Y[k'_1,\dots,k'_n]\Big] = [X,Y]_{\mathfrak{g}}[k_1+k'_1,\dots,k_n+k'_n], \quad k_i,k'_j \geqslant 0,$$

$$\Big[[X[k_1,\dots,k_n],Y[\ell_1,\dots,\ell'_n]\Big] = [X,Y]_{\mathfrak{g}}[k_1+\ell_1,\dots,k_n+\ell_n], \quad 0 < -\ell_j \leqslant k_j.$$

Remaining brackets are determined by graded skew symmetry.

We point out a property of the graded Lie bracket in cohomology which is different than the ordinary (one-dimensional) current algebra. Consider the case $n=2$, and for simplicity take the Lie algebra $\mathfrak{g} = \mathfrak{sl}(2)$ with standard basis $\{e,f,h\}$. The cohomology $\mathfrak{sl}(2) \otimes H^\bullet(\mathsf{A}_2)$ is spanned by elements $\{e[k,\ell], f[k,\ell], h[k,\ell]\}$ for $k,\ell \in \mathbb{Z}$.

Consider, for instance, the elements $e[k,\ell]$ and $f[r,s]$. The elements with $k=\ell=r=s \geqslant 0$ satisfy the $\mathfrak{sl}(2)$-type relation $[e[k,\ell], f[r,s]] = h[k+r,\ell+s]$. However, notice that

$$\big[e[1,0], f[-1,-1]\big] = 0.$$

This property of the bracket is quite different than in the $n=1$ case. Indeed, if $e[k], f[r] \in \mathfrak{sl}(2) \otimes \mathsf{A}_1$ then

$$\big[e[k], f[r]\big] = h[k+r]$$

for *all* $k,r \in \mathbb{Z}$. In other words, it appears that for $n>1$, taking cohomology seems to result in the loss of some amount of structure that one might expect to be present in the current algebra. This reflects the existence of "higher order" operations present in the level of cohomology, reminiscent of Massey products in the de Rham cohomology of a non-formal space. It leads us to the theory of L_∞ algebras, which we will also need in order to describe the centrally extended algebras.

8.2.4 Interlude: L_∞ Algebras

The most explicit description of central extensions of higher dimensional current algebras will force us to leave the world (ever so slightly) of dg Lie algebras. An L_∞ *algebra* is a precise weakening of the notion of a dg Lie algebra. We give a brief synopsis here, but refer to [20, 26, 27] for further background and further explanations of the formulas given below. These algebras are a lot like dg Lie algebras: there is a differential d and bracket $[\cdot,\cdot]$, and d is required to be a derivation for the bracket. The key difference is: *the Jacobi identity may not hold*. But, it is required to hold *up to homotopy*. Precisely, this means that we have to prescribe the data of a new 3-ary "bracket"

$$[\cdot,\cdot,\cdot]_3\colon \mathfrak{g}^\bullet \times \mathfrak{g}^\bullet \times \mathfrak{g}^\bullet \to \mathfrak{g}^\bullet[-1]$$

of cohomological degree -1 which satisfies an identity of the form

$$\begin{aligned}&[[x_1,x_2],x_3] \pm [[x_2,x_3],x_1] \pm [[x_3,x_1],x_2]\\ &\quad = \mathrm{d}[x_1,x_2,x_3]_3 \pm [\mathrm{d}x_1,x_2,x_3]_3 \pm [x_1,\mathrm{d}x_2,x_3]_3 \pm [x_1,\mathrm{d}x_2,x_3]_3. \end{aligned} \tag{8.2.8}$$

This identity says that the left-hand side, while not zero, is trivial up to homotopy – the 3-bracket provides this homotopy. In particular, the Jacobi identity holds in cohomology. In addition, there are higher compatibilities that the 3-linear bracket $[\cdot]_3$ must satisfy which also potentially involve 4-ary brackets, and so on.

Before giving the precise definition, we set up some notation which is known as the "Koszul sign rule." Consider a collection of homogenous elements $x_1,\dots,x_n$. The *Koszul sign* $\varepsilon(x_1,\dots,x_n;\sigma)$ of a permutation $\sigma\in\Sigma_n$ is defined by the following relation

$$x_1\cdots x_n = \varepsilon(x_1,\dots,x_n;\sigma)x_{\sigma(1)}\cdots x_{\sigma(n)}.$$

We will abbreviate this by $\varepsilon(\sigma)$ in the definition below.

Definition 8.15 Let $\mathfrak{g}^\bullet = \oplus_{j\in\mathbb{Z}}\mathfrak{g}^j[-j]$ be a $\mathbb{Z}$-graded vector space. An L_∞ *algebra* structure on V is a collection of multilinear maps

$$[\cdot]_k\colon \mathfrak{g}^{\times k} \to \mathfrak{g}[2-k]$$

for $k\geqslant 1$ such that the following conditions hold.

(i) Graded skew symmetry. For all $\sigma\in S_k$, $x_i\in\mathfrak{g}$ one has

$$[x_{\sigma(1)},\dots,x_{\sigma(k)}]_k = (-1)^\sigma\varepsilon(\sigma)[x_1,\dots,x_k]_k.$$

(ii) Higher Jacobi identities. For all $x_i\in\mathfrak{g}$ one has

$$\sum_{i+j=k+1}\sum_{\sigma}(-1)^{i(j-1)}(-1)^\sigma\varepsilon(\sigma)\Big[\big[x_{\sigma(1)},\dots,x_{\sigma(i)}\big]_i,x_{\sigma(i+1)},\dots,x_{\sigma(k)}\Big]_j,$$

where σ ranges over $(i,k-i)$ unshuffles.

We point out that Lie, graded Lie and dg Lie algebras are special cases: when $g^j=0$ for $j\neq 0$ this returns the definition of an ordinary Lie algebra; a graded Lie algebra is an L_∞ algebra with $[\cdot]_k=0$ for $k\neq 2$; a dg Lie algebra is an L_∞ algebra with $[\cdot]_k=0$ for $k>2$.

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Transferred Structure

Before moving towards central extensions, we follow up the discussion following Proposition 8.9 about the “lack” of structure present in the cohomology of the current algebra $\mathfrak{g} \otimes \mathsf{A}_n$.

Given any dg Lie algebra $(\mathfrak{g}, \mathrm{d}, [\cdot,\cdot])$ its d-cohomology $H^\bullet(\mathfrak{g}, \mathrm{d})$ has the structure of a graded Lie algebra (a dg Lie algebra with zero differential). This theorem follows from a general result about operads [28, 10.3.15].

Theorem 8.16 *Suppose that $(\mathfrak{g}, d, [\cdot,\cdot])$ is a dg Lie algebra. Then, there exists an L_∞ structure $\{[\cdot]_k\}$ on $H = H^\bullet(\mathfrak{g}, d)$ such that $H \simeq \mathfrak{g}$ as L_∞ algebras.*

Generally speaking, the transferred L_∞ structure on $H^\bullet(\mathfrak{g} \otimes \mathsf{A}_n)$ has nontrivial higher operations. We will not fully characterize the transferred L_∞ algebra in these notes. We state a nontrivial higher bracket for the example $n = 2$ and $\mathfrak{g} = \mathfrak{sl}(2)$. Recall that the 2-ary bracket of both $e[1,0]$ and $e[0,1]$ with $f[-1,-1]$ is trivial in $\mathfrak{sl}(2) \otimes H^\bullet(\mathsf{A}_2)$. However, there is a higher 3-ary operation of the form

$$\big[e[1,0], e[0,1], f[-1,-1]\big]_3 = e[0,0].$$

The existence of these higher operations implies that $\mathfrak{sl}(2) \otimes \mathsf{A}_2$ is not formal as a dg Lie algebra.

Remark In general, the L_∞ structure on $H^\bullet(\mathfrak{g} \otimes \mathsf{A}_n) \simeq \mathfrak{g} \otimes H^\bullet(\mathsf{A}_n)$ is obtained from an A_∞ structure on $H^\bullet(\mathsf{A}_n)$. This A_∞ algebra has higher operations for $n > 1$.

8.2.5 Central Extensions, Redux

Recall that one-dimensional central extensions of an ordinary Lie algebra $\mathfrak{h}$ are on bijective correspondence with the cohomology group $H^2(\mathfrak{h}\,;\,\mathbb{C})$. There is a very similar correspondence for dg Lie algebras (in fact, all L_∞ algebras).

The Lie algebra cohomology of a dg Lie algebra $(\mathfrak{h}, \mathrm{d}, [\cdot,\cdot])$ is defined very similarly as in the ordinary case. Explicitly, it is cohomology of the cochain complex

$$\mathrm{C}^\bullet(\mathfrak{h}) \stackrel{\text{def}}{=} \Big(\mathrm{Sym}(\mathfrak{h}^*[-1])\,,\, \mathrm{d}_1 + \mathrm{d}_2\Big),$$

where

- d_1 stands for the “internal” differential of the cochain complex $(\mathfrak{h}, \mathrm{d})$. On $\mathfrak{h}^*[-1]$ it is the linear dual of d. It is extended to the full symmetric algebra by the graded Leibniz rule. In particular, it preserves symmetric degree
$$\mathrm{d}_1 : \mathrm{Sym}^k(\mathfrak{h}^*[-1]) \to \mathrm{Sym}^k(\mathfrak{h}^*[-1]).$$

- d_2 is the familiar Chevalley–Eilenberg differential associated to the bracket $[\cdot,\cdot]$. It is defined exactly as in the case of Lie algebra cohomology of an ordinary Lie algebra; see Definition 8.3.

Proposition 8.10 *Let $(\mathfrak{h}, d, [\cdot,\cdot])$ be a dg Lie algebra. Suppose that $\varphi \in C^\bullet(\mathfrak{h}\,;\mathbb{C})$ is a cocycle of total cohomological degree N. Moreover, assume that φ admits a decomposition of the form $\varphi = \varphi^{(1)} + \cdots + \varphi^{(k)} + \cdots$, where $\varphi^{(k)} \colon \mathrm{Sym}^k(\mathfrak{g}[1]) \to \mathbb{C}[N]$ is the kth Taylor component of φ. Then, φ defines an L_∞ algebra $\widetilde{\mathfrak{h}}_\varphi$ which as a vector space is*

$$\widetilde{\mathfrak{h}}_\varphi = \mathfrak{h} \oplus \mathbb{C}[N-2] \cdot K$$

and whose brackets are defined by the rules:

- *the element K is central;*
- *the remaining brackets $[\cdot]_k \colon \mathfrak{h}^{\times k} \to \widetilde{\mathfrak{h}}_\varphi$ are defined by*

$$\begin{aligned} [\cdot]_1 &= d + K\varphi^{(1)} \\ [\cdot]_2 &= [\cdot,\cdot] + K\varphi^{(2)} \\ [\cdot]_3 &= K\varphi^{(3)}, \quad \textit{etc.} \end{aligned}$$

This result generalizes Lemma 8.1 – *every* cocycle in $\mathrm{C}^\bullet(\mathfrak{h})$ represents an L_∞ central extension. We turn to the simplest central extension in the context of multivariable infinite-dimensional Lie algebras – the higher dimensional Heisenberg algebra.

The Higher Dimensional Heisenberg Algebra

Consider the cochain complex A_n modeling punctured affine space considered as an abelian dg Lie algebra. The following lemma is a rephrasing of observations made in §8.2.2.

Lemma 8.11 *The multilinear map $\varphi_n \colon (A_n)^{\times n+1} \to \mathbb{C}$ defined by*

$$\varphi_n(\alpha_0, \ldots, \alpha_n) \overset{\text{def}}{=} \mathrm{Res}_{z=0}(\alpha_0\, \partial\alpha_1 \cdots \partial\alpha_n)$$

defines a nontrivial degree $+2$ cocycle $\varphi_n \in C^\bullet(A_n\,;\mathbb{C})$.

Using Proposition 8.10 φ_n determines a one-dimensional L_∞ central extension of A_n.

Definition 8.17 The *n-Heisenberg algebra* is the L_∞ central extension of the abelian dg Lie algebra A_n by the degree $+2$ cocycle φ_n.

The underlying cochain complex of h_n is

$$\left(\mathsf{A}_n \oplus \mathbb{C}K \,,\, \overline{\partial}\right),$$

where $\overline{\partial}$ acts on A_n in the usual way and $\overline{\partial}(K) = 0$. The only nontrivial higher bracket is $(n+1)$-ary and is given in terms of the higher residue

$$[\alpha_1, \dots, \alpha_{n+1}]_{n+1} = \mathrm{Res}_{z=0}\left(\alpha_1 \partial \alpha_2 \cdots \partial \alpha_{n+1}\right).$$

The cohomology of the Heisenberg algebra admits a simple presentation. Recall, by Theorem 8.16 that the cohomology of any L_∞ algebra has the inherited structure of an L_∞ algebra with $[\cdot]_1 = 0$.

Proposition 8.12 *The transferred L_∞ structure present in the cohomology $H^\bullet = H^\bullet(\mathsf{h}_n)$ has trivial k-ary operation $[\cdot]_k$ for $k \neq n+1$. The $(n+1)$-ary operation is given by the residue*

$$\begin{array}{llll} [\cdot]_{n+1}\colon & (H^0)^{\times n} \times H^{n-1} & \to & \mathbb{C}\cdot K, \\ & (f_1, \dots, f_n; \alpha) & \mapsto & K \cdot \mathrm{Res}_{z=0}\left(f_1 \partial f_2 \cdots \partial f_n \partial \alpha\right). \end{array}$$

Let's unravel this proposition in the case $n = 2$. We will use the holomorphic coordinates $(z,w) = (z_1, z_2)$ for $\mathbb{C}^2$ in this section to avoid clutter. The cohomology of h_2 is nontrivial only in degrees zero and one. There is the following presentation for the degree zero cohomology $\mathrm{span}_{\mathbb{C}}\left\{b[n,m]\,,\, K \mid n,m \in \mathbb{Z}_{\geqslant 0}\right\}$. In terms of holomorphic polynomials in two variables, $b[n,m]$ corresponds to $b[n,m] = z^n w^m$ for $n,m \geqslant 0$, just as in Proposition 8.9 (where $\mathfrak{g}$ is the one-dimensional Lie algebra $\mathbb{C}\cdot b$.) In degree one, the cohomology is $\mathrm{span}_{\mathbb{C}}\left\{b[n,m] \mid n,m \in \mathbb{Z}_{<0}\right\}$. The generators $b[n,m]$ for $n,m < 0$ are given by the $\overline{\partial}$-class of derivatives of the Bochner–Martinelli kernel, see the formulas in Proposition 8.9.

The only nontrivial higher bracket is a 3-ary bracket on $H^\bullet(\mathsf{h}_2)$ of the form

$$[\cdot,\cdot,\cdot]_3 \colon H^1(\mathsf{h}_2) \times H^0(\mathsf{h}_2) \times H^0(\mathsf{h}_2) \to \mathbb{C}K$$

and permutations thereof. In terms of these generators, one can read off this 3-ary bracket using the explicit formula involving the residue class

$$\left[b[n,m]\,,\, b[k,\ell]\,,\, b[r,s]\right]_3 = (ks - \ell r)\delta_{-n,k+r}\delta_{-m,\ell+s}K. \tag{8.2.9}$$

It is instructive to check that the higher Jacobi identity is satisfied using this combinatorial description of the higher bracket. (One may find the identity useful: $(ks - \ell r)\delta_{-n,k+r}\delta_{-m,\ell+s} = (mr - ns)\delta_{-n,k+r}\delta_{-m,\ell+s}$.)

Higher Dimensional Kac–Moody Algebra

We now consider central extensions of the n-current algebra $\mathfrak{g} \otimes \mathsf{A}_n$, where $\mathfrak{g}$ is an ordinary Lie algebra. Recall that in the case $n = 1$, we considered central extensions built from invariant quadratic polynomials on $\mathfrak{g}$. One of the main results of [10] is a concrete relationship of higher order invariant polynomials with central extensions of the n-current algebra.

Theorem 8.18 ([10]) *Let $n \geqslant 1$ and suppose that $\mathfrak{g}$ is semisimple. Then there is an embedding of vector spaces*

$$\mathrm{Sym}^{n+1}(\mathfrak{g}^*)^{\mathfrak{g}} \hookrightarrow H^2(\mathfrak{g} \otimes A_n),$$

which sends an invariant polynomial θ to a class $[\varphi_\theta]$.

A particular representative for the class φ_θ can be constructed as follows. First, like the n-Heisenberg algebra the representative φ_θ is $(n+1)$-linear. If $x_0 \otimes \alpha_0, \ldots, x_n \otimes \alpha_n$ are n-currents then we can consider the differential form $\theta(x_0, \ldots, x_n)\alpha_0 \partial \alpha_1 \cdots \partial \alpha_n$ which is necessarily of type $(n, \bullet)$. The cochain φ_θ is the residue of this class

$$\varphi_\theta(x_0 \otimes \alpha_0, \ldots, x_n \otimes \alpha_n) = \theta(x_0, \ldots, x_n)\mathrm{Res}_{z=0}\big(\alpha_0 \partial \alpha_1 \cdots \partial \alpha_n\big).$$

It is an instructive exercise to verify that φ_θ is indeed of degree $+2$ and is closed for the Chevalley–Eilenberg differential on $\mathrm{C}^\bullet(\mathfrak{g} \otimes \mathsf{A}_n)$.

Definition 8.19 Let θ be a degree $(n+1)$ invariant polynomial of $\mathfrak{g}$. The *n-Kac–Moody algebra* associated to θ is the central extension $\widehat{\mathfrak{g}}_{n,\theta}$ of the n-current algebra associated to the class $[\varphi_\theta]$.

8.2.6 Derived Fock Modules

To begin our discussion of modules, we attempt to parallel the construction of the Fock module $F(\mu)$ for the ordinary Heisenberg algebra h (see §8.1.3) to modules for the higher dimensional Heisenberg algebra. We will find that already in the case $n = 2$, such modules are inherently *derived* in the sense that they are "L_∞ modules" for the 2-Heisenberg algebra h_2.

We try to mimic the definition of the ordinary Fock module as closely as possible. We immediately run into a discrepancy with the ordinary situation: the "creation operators" $b[n,m]$, $n, m < 0$ lie in a nontrivial cohomological degree. Thus, any generalization of the Fock module to higher dimensions must have components in nontrivial cohomological degree.

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Consider the following $\mathbb{Z}$-graded vector space given by a polynomial algebra on a (doubly) infinite number of generators:

$$V = \mathbb{C}[x_{i,j}]_{i,j \leqslant -1}.$$

The cohomological $\mathbb{Z}$-grading is determined by declaring that $x_{i,j}$ has cohomological degree $+1$ for all i, j.

We define the action of the "creation operators" by

$$b[n,m]|0\rangle \stackrel{\text{def}}{=} x_{n,m}, \quad n,m < 0.$$

More generally, if $F(x_{i,j}) \in V$ is any state, define

$$b[n,m]F(x_{i,j}) \stackrel{\text{def}}{=} x_{n,m}F(x_{i,j}), \quad n,m < 0.$$

We will assume that the central element K acts by the identity on V, just as in the case of the ordinary Fock module. Analogous to the ordinary Fock module, for $n,m > 0$ the elements $b[n,m]$ will be "annihilation operators". However, their action on V is more subtle. Notice that there are no elements in h_2 of cohomological degree -1. So, the naive definition of annihilation

$$b[k,\ell]x_{n,m} \stackrel{?}{=} C(k,\ell,m,n)\delta_{k,n}\delta_{\ell,m}|0\rangle,$$

where $C(k,\ell,n,m)$ is some constant, is not sensible. One, perhaps disappointing, definition is to declare that $b[n,m]$ act trivially for $n,m > 0$. This would certainly be compatible with the fact that h_2 has no nontrivial 2-ary brackets. But, what about compatibility with the 3-ary bracket $[\cdot,\cdot,\cdot]_3$ which we know is present in cohomology?

The key is that we do not obtain the structure of a *strict* $H^\bullet(\mathsf{h}_2)$-module on V. Instead, one finds the structure of an L_∞*-module*.

Interlude: L_∞ Modules

The notion of an L_∞ module is a weakening of the notion of a module for a Lie algebra in the same way that an L_∞ algebra weakens the notion of a Lie algebra. Let $\mathfrak{h}$ be a Lie algebra. An $\mathfrak{h}$-module is prescribed by the data of a vector space M and a map $\rho\colon \mathfrak{h} \times M \to M$, $(x,y) \mapsto \rho(x;y)$ which satisfies $\rho([x,x'];y) = \rho(x;\rho(x';y)) - \rho(x';\rho(x;y))$.

Suppose that $\mathfrak{h} = (\mathfrak{h}, \mathrm{d}_{\mathfrak{h}})$ is a dg Lie algebra. A dg $\mathfrak{h}$-module is a cochain complex (M, d_M) together with a grading-preserving map ρ as above which additionally satisfies $\rho(\mathrm{d}_{\mathfrak{h}}x;y) = \mathrm{d}_M\rho(x;y) - \rho(x;\mathrm{d}_M y)$. In the case of an L_∞ algebra, we have the following generalization of this definition.

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Definition 8.20 Let $(\mathfrak{h}, [\cdot]_k)$ be an L_∞ algebra and (M, d_M) a cochain complex. An L_∞ $\mathfrak{h}$*-module structure* on M is a collection of graded skew-symmetric multilinear maps

$$\rho^{(k)} \colon \mathfrak{h}^{\times k} \times M \to M[1-k]$$

such that

$$0 = \sum_{i+j=k+2} \sum_{\sigma} \chi(\sigma)(-1)^{i(j-1)} \rho^{(j)}(\rho^{(i)}(x_{\sigma(1)}, \dots, x_{\sigma(i)}), x_{\sigma(i+1)}, \dots, x_{\sigma(k)}),$$

where σ ranges over $(i, n-i)$-unshuffles, $x_1, \dots, x_{n-1} \in \mathfrak{h}$, $x_n \in M$. In the formulas above we set $\rho^{(0)} \stackrel{\text{def}}{=} \mathrm{d}_M$ and $\rho^{(n)}(y_1, \dots, y_m) = [x_1, \dots, x_m]_n$ if $y_1, \dots, y_m \in \mathfrak{h}$.

Let's unpack this in the simple case where $\mathfrak{h}$ is a graded Lie algebra. Then, we see that $\rho^{(1)}$ does *not* define an $\mathfrak{h}$-module structure, but rather

$$\rho^{(1)}(x; \rho^{(1)}(y; m)) - \rho^{(1)}(y; \rho^{(1)}(x; m)) = \mathrm{d}_M \rho^{(2)}(x, y; m), \quad x, y \in \mathfrak{h}, \quad m \in M.$$

In other words, the failure for $\rho^{(1)}$ to be an $\mathfrak{h}$-module is homotopically trivializable, with homotopy given by $\rho^{(2)}$.

Example 8.21 To any Lie algebra $\mathfrak{h}$, one can associate the module $\mathrm{Sym}(\mathfrak{h})$. Similarly, if $\mathfrak{h}$ is an L_∞ algebra, then $\mathrm{Sym}(\mathfrak{h})$ has the canonical structure of an L_∞ $\mathfrak{h}$-module. If $[\cdot]_k \colon \mathfrak{h}^{\times k} \to \mathfrak{h}$ is the k-ary operation of $\mathfrak{h}$ then this module structure is defined on generators $\mathfrak{h} \subset \mathrm{Sym}(\mathfrak{h})$ by the collection of maps $\rho^{(k)}$ with

$$\rho^{(k)}(x_1, \dots, x_k; y) = [x_1, \dots, x_k, y]_{k+1}.$$

The L_∞ Fock Module

The Fock module $V = \mathbb{C}[x_{i,j}]$ has the structure of an L_∞ module for the L_∞ algebra $H^\bullet(\mathsf{h}_2)$. This module structure has a nontrivial linear $\rho^{(1)}$ and 2-linear $\rho^{(2)}$ component which we now define.

For $n, m > 0$ the operators $b[n,m]$ participate in the L_∞-module structure as follows. First, the linear action of $b[n,m]$ on V is trivial

$$\rho^{(1)}\big(b[n,m]; F(x_{i,j})\big) = 0, \quad n, m > 0$$

for all $F(x_{i,j}) \in V$. Next, there is a 2-ary component to the L_∞-module structure defined by

$$\rho^{(2)}(b[k,\ell], b[r,s]; x_{n,m}) = (ks - \ell r)\delta_{-n,k+r}\delta_{-m,\ell+s}|0\rangle.$$

More generally, for any $F(x_{i,j}) \in V$ we define

$$\rho^{(2)}\big(b[k,\ell], b[r,s]; F(x_{i,j})\big) = (ks - \ell r)\frac{\partial}{\partial x_{-k-r,-\ell-s}} F(x_{i,j}).$$

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Notice that this formula is sensible for all k,ℓ,r,s such that the elements $b[k,\ell],b[r,s]$ are defined.

Proposition 8.13 *These formulas for* $\{\rho^{(1)},\rho^{(2)}\}$ *endow* V *with the structure of an* L_∞ $H^\bullet(\mathsf{h}_2)$*-module.*

Proof Let $F=F(x_{i,j})\in\mathbb{C}[x_{i,j}]$ and suppose that $n,m<0$ and $r,s,k,\ell\geqslant 0$. We check the following ∞-module relation:

$$\begin{aligned}\rho^{(2)}\big(b[k,\ell],b[r,s];\rho^{(1)}(b[n,m];F(x_{i,j}))\big)&\\ +\rho^{(1)}\big(b[n,m];\rho^{(2)}(b[k,\ell],b[r,s];F(x_{i,j}))\big)&\\ \stackrel{?}{=}\rho^{(1)}\big(\big[b[k,\ell],b[r,s],b[n,m]\big]_3;F(x_{i,j})\big).&\end{aligned}\tag{8.2.10}$$

Since $n,m<0$, the first term on the left-hand side is

$$\rho^{(2)}\big(b[k,\ell],b[r,s];x_{n,m}F(x_{i,j})\big)=(ks-\ell r)\left(F-x_{n,m}\frac{\partial F}{\partial x_{-k-r,-\ell-s}}\right).$$

The second term on the left-hand side is

$$\rho^{(1)}\left(b[n,m];(ks-\ell r)\frac{\partial F}{\partial x_{-k-r,-\ell-s}}\right)=(ks-\ell r)x_{n,m}\frac{\partial F}{\partial x_{-k-r,-\ell-s}}.$$

Using Equation (8.2.9) we see that the right-hand side is

$$(ks-\ell r)\rho^{(1)}(K;F(x_{i,j})=(ks-\ell r)F(x_{i,j})$$

thus verifying the relation. The proofs for other values of indices are analogous and we leave them to the interested reader. □

Derived Creation/Annihilation

We have just seen how to construct a class of modules for the cohomology of the higher dimensional Heisenberg algebra. In this section, we show how this can be lifted to the cochain level. We return to the case of a general dimension $n>1$.

Recall that the complex A_n is concentrated in degrees $0,\dots,n-1$. In particular, there is a quotient map

$$\mathsf{A}_n[n-1]\to H^{n-1}(\mathsf{A}_n),$$

which sends a class $\alpha\in\mathsf{A}_n^{n-1}$ (which is automatically a cocycle) to its cohomology class $[\alpha]$.

Definition 8.22 Define the vector space $\mathsf{A}_{n,-}$ to be the top cohomology $H^{n-1}(\mathsf{A}_n)$. Define $\mathsf{A}_{n,+}$ to be the short exact sequence of dg algebras

$$\mathsf{A}_{n,+}\hookrightarrow\mathsf{A}_n\to\mathsf{A}_{n,-}.$$

Notice that like A_n, the algebra $\mathsf{A}_{n,+}$ is concentrated in degrees $[0, n-1]$. However, its cohomology is concentrated in degree zero.

Let's return to the n-dimensional Heisenberg L_∞ algebra h_n. Notice that the higher residue pairing is trivial when restricted to the subalgebra $\mathsf{A}_{n,+}$. In particular, the *abelian* graded Lie algebra $\mathsf{A}_{n,+} \oplus \mathbb{C}K$ is a subalgebra of the full Heisenberg algebra

$$\mathsf{A}_{n,+} \oplus \mathbb{C}K \subset \mathsf{h}_n.$$

With this, we can proceed to a family of modules for h_n by induction just as in the ordinary case.

Fix complex numbers $\mu, k \in \mathbb{C}$. Let $\mathbb{C}(\mu, k)$ denote the one-dimensional $(\mathsf{A}_{n,+} \oplus \mathbb{C}K)$-module where:

- K acts by k;
- $b[0,0] = 1 \in \mathsf{A}^0_{n,+}$ acts by μ;
- the remaining elements act trivially.

The universal enveloping construction can be extended to L_∞ algebras (see [3]). In this way, any L_∞ algebra $\mathfrak{g}$ determines an A_∞ algebra $U(\mathfrak{g})$. In the definition below, recall that $U(\mathsf{h}_n) \simeq \mathrm{Sym}(\mathsf{h}_n)$ as complexes, and hence has the natural structure of a left L_∞ h_n-module.

Definition 8.23 The *derived Fock module* for the n-Heisenberg algebra h_n associated to the pair of numbers (μ, k) is the L_∞ module

$$\mathsf{F}_n(\mu, k) \overset{\text{def}}{=} U(\mathsf{h}_n) \otimes_{U(\mathsf{A}_{n,+} \oplus \mathbb{C}K)} \mathbb{C}(\mu, k).$$

The reader will recognize the similarities of this definition with formula (8.1.2). The cohomology of the derived Fock module is simple to describe based on previous calculations.

Proposition 8.14 *As a graded vector space, the higher residue identifies $H^\bullet(F_n(\mu, k))$ with the graded symmetric algebra on $A_{n,-}[1-n]$. In particular:*

- *When n is even, the cohomology is a symmetric algebra*

$$H^\bullet(F_n(\mu, k)) \simeq \bigoplus_{k \geqslant 0} \mathrm{Sym}^k(\mathsf{A}_{n,-})\,[k(1-n)].$$

- *When n is odd, the cohomology is an exterior algebra*

$$H^\bullet(F_n(\mu, k)) \simeq \bigoplus_{k \geqslant 0} \wedge^k(A_{n,-})\,[k(1-n)].$$

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Remark This statement is valid even when $n = 1$. Indeed, we recover the usual presentation of the Fock module since $\mathsf{A}_{1,-} \cong z^{-1}\mathbb{C}[z^{-1}]$ by convention. For any n, the cohomology of the Fock module is the graded symmetric algebra of a vector space concentrated in degree $n-1$. In particular, when $n > 1$, the Fock module always carries nontrivial cohomological degree.

When $n = 2$, we see that the cohomology of the derived Fock module is the graded symmetric algebra on a vector space in concentrated in degree $+1$. Thus, there is some chance for $H^\bullet(\mathsf{F}_2(\mu,k))$ to agree with the module we constructed by hand in the introduction to this section. This is indeed the case. The transferred L_∞ $H^\bullet(\mathsf{h}_2)$-module structure on $H^\bullet(\mathsf{F}_2(\mu,1))$ is equivalent to the one we described in Proposition 8.13.

Just like for the ordinary Fock module, the behavior of $\mathsf{F}_n(\mu,k)$ is not very sensitive to the number $k \in \mathbb{C}$. Indeed, for $k, k' \neq 0$ there is an equivalence of modules $\mathsf{F}_n(\mu,k) \simeq \mathsf{F}_n(\mu,k')$. Furthermore, $\mathsf{F}(\mu,0)$ is a trivial h_n-module.

We leave further questions pertaining to the higher dimensional Fock modules that we do not attempt to answer in these lectures.

- Does $\mathsf{F}_n(\mu,k) \simeq \mathsf{F}_n(\mu',k)$ imply $\mu = \mu'$?
- In what sense (if at all) is $\mathsf{F}_n(\mu,k)$ "irreducible" as an L_∞-module for h_n? In what sense (if at all) is $H^\bullet\mathsf{F}_n(\mu,k)$ "irreducible" as an L_∞-module for $H^\bullet\mathsf{h}_n$?

Vacuum Modules

We define a class of modules for the higher dimensional Kac–Moody modules. Fix an invariant polynomial θ of degree $n+1$ and consider the associated higher Kac–Moody algebra $\widehat{\mathfrak{g}}_{n,\theta}$. Then, there is the subalgebra

$$\mathfrak{g} \otimes \mathsf{A}_{n,+} \oplus \mathbb{C}K \subset \widehat{\mathfrak{g}}_{n,\theta}.$$

One can see this is a subalgebra by checking that the restriction of the higher residue vanishes. Define the module $\mathbb{C}_1 \cong \mathbb{C}$ for this subalgebra where $\mathfrak{g} \otimes \mathsf{A}_{n,+}$ acts trivially and K acts by 1.

Definition 8.24 The *n-vacuum module* of level θ is the L_∞ $\widehat{\mathfrak{g}}_{n,\theta}$ module

$$V_{n,\theta}(\mathfrak{g}) \stackrel{\text{def}}{=} U(\widehat{\mathfrak{g}}_{n,\theta}) \otimes_{U(\mathfrak{g}\otimes\mathsf{A}_{n,+}\oplus K)} \mathbb{C}_1. \tag{8.2.11}$$

We will not develop any theory behind these vacuum modules, but will raise some questions pertaining to them at the end of the next section. Also, we offer one geometric interpretation in terms of factorization algebras in §8.3.1.

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8.2.7 Higher Virasoro Algebra

So far we have not touched upon multivariable versions of the Virasoro algebra. For the Kac–Moody algebra based on $\mathfrak{g}$, one starts by simply tensoring with the dg algebra A_n to obtain the n-current algebra $\mathfrak{g} \otimes \mathsf{A}_n$.[5] When $n = 1$ the Virasoro algebra is a central extension of the Witt algebra. So, a more basic question is what the n-dimensional version of the Witt algebra should be.

Recall that A_n is a derived model for the sheaf of functions on punctured affine space. For the n-dimensional version of the Witt algebra, one should consider derivations of the dg algebra A_n. Geometrically, this corresponds to looking at a subalgebra of the derived global sections of the tangent bundle $\mathrm{T}_{\mathbb{A}^n \setminus 0}$ on $\mathbb{A}^n \setminus 0$. Recall that vector fields form a Lie algebra under the commutator; we require any model for the derived global sections to have the structure of a dg Lie algebra which extends this. Since the tangent bundle of $\mathbb{A}^n \setminus 0$ admits a framing, we obtain the following model for this dg Lie algebra.

Definition 8.25 Let witt_n denote the cochain complex

$$\mathsf{A}_n \otimes \mathbb{C}\{\partial_{z_1}, \ldots, \partial_{z_n}\} \cong \mathsf{A}_n \otimes \mathbb{C}^n,$$

where the differential is defined by $\overline{\partial}(\alpha \otimes \partial_{z_i}) = (\overline{\partial}\alpha) \otimes \partial_{z_i}$, for $\alpha \in \mathsf{A}_n$. The bracket

$$\left[\alpha \otimes \partial_{z_i}, \beta \otimes \partial_{z_j}\right] = \alpha L_{\partial_{z_i}}(\beta) \otimes \partial_{z_j} - (-1)^{|\alpha||\beta|} \beta L_{\partial_{z_j}}(\alpha) \otimes \partial_{z_i}$$

endows $\left(\mathsf{witt}_n, \overline{\partial}, [\cdot,\cdot]\right)$ with the structure of a dg Lie algebra. Furthermore $H^\bullet(\mathsf{witt}_n)$ agrees with the graded Lie structure on $H^\bullet(\mathbb{A}^n \setminus 0, \mathrm{T})$.

From the embedding of dg algebras $\mathsf{A}_n \hookrightarrow \Omega^{0,\bullet}(\mathbb{C}^n \setminus 0)$ we obtain an embedding of dg Lie algebras $\mathsf{witt}_n \hookrightarrow \Omega^{0,\bullet}(\mathbb{C}^n \setminus 0, \mathrm{T})$. Again, this is almost a quasi-isomorphism of dg Lie algebras; at the level of cohomology, this embedding is dense.

We address the problem of classifying central extensions of the n-Witt algebra. A key difference with the case $n = 1$, and the ordinary Witt algebra, is that there is no longer a one-dimensional space of central extensions.

Higher Central Charges

In [18], Hennion and Kapranov have used methods of factorization homology to partially characterize central extensions of the n-Witt algebra in terms of the Lie algebra cohomology of *formal vector fields* w_n. The Lie algebra w_n is

[5] One could think of this as derived sections of the adjoint bundle associated to some principal G-bundle on punctured affine space.

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defined as the ∞-jets of sections of the tangent bundle on (non-punctured) affine space and its cohomology has been extensively studied by Fuks [13].

Theorem 8.26 ([18]) *There is a map*

$$H^{2n+1}(\mathsf{w}_n) \to H^2(\mathsf{witt}_n). \tag{8.2.12}$$

In other words, any class $\mathsf{c} \in H^{2n+1}(\mathsf{w}_n)$ *defines a central extension of the n-Witt algebra that we denote by* $\mathrm{vir}_{n,\mathsf{c}}$.

In upcoming work [35] we offer another construction of central extensions of witt_n from cohomology classes of formal vector fields based on the method of "descent" for the de Rham cohomology of ∞-jets of the tangent bundle; for some details see the final chapter of [34]. This construction will divert from the content of this series, but we mention a particular example to get a sense of these central extensions.

Consider the 2-Witt algebra $\mathsf{witt}_2 = \mathsf{A}_2\{\partial_{z_1}, \partial_{z_2}\}$. The Jacobian of a vector field on $\mathbb{C}^2$ is a 2×2 matrix whose entries are functions. This definition extends to the 2-Witt algebra; the Jacobian of an element $\xi = \sum \alpha_i \partial_{z_i}$ of witt_2 is the 2×2 matrix with values in A_2 whose ij entry is

$$[\mathrm{Jac}(\xi)]^i_j = \partial_{z_i}\alpha_j \in \mathsf{A}_2.$$

Consider the following two cochains of witt_2

$$\begin{aligned}
\psi_1(\xi_0, \xi_1, \xi_2) &= \mathrm{Res}_{z=0}\mathrm{Tr}\left(\mathrm{Jac}(\xi_0)\partial\mathrm{Jac}(\xi_1)\partial\mathrm{Jac}(\xi_2)\right), \\
\psi_2(\xi_0, \xi_1, \xi_2) &= \mathrm{Res}_{z=0}\mathrm{Tr}\left(\mathrm{Jac}(\xi_0)\right)\mathrm{Tr}\left(\partial\mathrm{Jac}(\xi_1)\partial\mathrm{Jac}(\xi_2)\right).
\end{aligned}$$

It is an involved exercise to verify directly that both ψ_1, ψ_2 are cocycles of cohomological degree 2. We expect that these two cocycles are inequivalent. This problem is related to the question of the injectivity of the map (8.2.12).

We end this section with a series of questions and goals that we hope can be attacked in the near future.

- Is there a "free field" realization of the higher dimensional Virasoro algebra on the Fock module introduced in §8.2.6? See [16] for a simple case of free field realization for n-Kac–Moody algebras.
- For which invariant polynomials $\theta \in \mathrm{Sym}^{n+1}(\mathfrak{g}^*)^{\mathfrak{g}}$ (if any) does the vacuum module $V_{n,\theta}(\mathfrak{g})$ have an action by the higher dimensional Virasoro algebra? In other words, formulate a higher dimensional version of the Segal–Sugawara construction.
- In [33] we began an effort to compute characters of higher dimensional Kac–Moody and Virasoro algebras. Analogs of Weyl–Kac character formula have yet to be determined.

- These algebras describe symmetries of higher dimensional *holomorphic* field theories. Progress towards this has been developed in [16, 33, 34]. Recently, Kapranov has utilized similar techniques to characterize an infinite-dimensional conformal algebra extending the conformal algebra on $\mathbb{R}^n$ for all n [25]; it would be interesting to see what role such algebras play in non-holomorphic theories.

8.3 Further Topics and Applications

8.3.1 Factorization Algebras and a Geometric Perspective

As we've mentioned above, the Virasoro algebra describes the symmetries in conformal field theory. The so-called "observables" of a conformal field theory are described by a mathematical object called a vertex algebra [7, 12, 23]. Algebro-geometrically, vertex algebras are closely related to chiral algebras developed by Beilinson and Drinfeld [5]. More recently, the Costello–Gwilliam theory of *factorization algebras* has provided another geometric formulation of vertex algebras [9]. The rigorous definition of a vertex algebra is quite complicated at first glance. The main advantage of the perspective that factorization algebras offer is in their intrinsic link to the observables of a conformal field theory. One connection between factorization algebras and CFT is given in [9, Theorem 2.2.1], which provides a construction of a vertex algebra from a factorization algebra. We also point to work of Bruegmann who has further elucidated the connection between vertex algebras and factorization algebras [8].

(Pre)factorization Algebras

We will not give a complete account of factorization algebras. Rather, we extract some geometric intuition and sketch how it connects to the algebraic situation that we have developed in the previous sections. Additionally, we will only glance at the theory of *pre*factorization algebras; the "gluing" axiom will play no role for us. Following [9], a prefactorization algebra is an algebra over a certain *colored* operad. A colored operad is similar to an operad except that there is a set of colors which can be fed into the multi-operations. It sometimes goes by the name "(symmetric) multicategory" and is also closely related to the notion of a pseudotensor category, see [2]. For a nice review of colored operads (equivalently, multicategories) we refer to [9, §A.2].

Recall that the collection of open sets on a manifold M forms a poset. This can be enhanced to the structure of a colored operad $\mathcal{C}_M$ as follows. For a collection of open sets $\{U_i, V\}$ define the set of multi-operations

$$\mathcal{C}_M\left(\boxtimes_{i\in I} U_i, V\right)$$

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by:

- the singleton set $\{\star\}$ if the collection of open sets $\{U_i\}$ is mutually disjoint and each U_i is contained in V;
- the empty set, otherwise.

Definition 8.27 A *prefactorization algebra* $\mathcal{F}$ on a manifold M is a $\mathcal{C}_M$-algebra. Unpacking, this is an assignment

$$\mathcal{F}\colon U \subset M \mapsto A(U)$$

together with a "multiplication rule"

$$m_{\{U_i\},V} : \otimes_{i\in I}\mathcal{F}(U_i) \to \mathcal{F}(V)$$

whenever the open sets $\{U_i\}$ are mutually disjoint and contained in V. These multiplications are required to satisfy the natural associativity axiom.

We have not explicitly written out the associativity axiom; we refer to [9, §1.1] for a more precise definition. To get a feel for this notion of associativity, we mention a special case. A prefactorization algebra is called *locally constant* if for every embedding of open balls $B \hookrightarrow B'$ in M the induced map $A(B) \xrightarrow{\simeq} A(B')$ is an equivalence.

Theorem 8.28 (Lurie) *There is an equivalence of categories between locally constant (pre)factorization algebras on $\mathbb{R}_{>0}$ (or any homeomorphic space) and associative algebras.*

This theorem is valid only at the level of ∞-categories. The notion of an associative algebra should be taken up to homotopy which can be modeled by the category of A_∞ algebras or algebras over the operad of little 1-disks. Extending this, Lurie [30, §5.4.5] identifies locally constant factorization algebras on $\mathbb{R}^n$, $n \geqslant 1$ with algebras over the operad of little n-disks introduced by Boardman and Vogt [6].

Current Algebras

To connect with the style of infinite-dimensional algebras we have encountered in this note we will not be directly concerned with locally constant factorization algebras (though they will play some role). Instead, we are interested in the concept of a factorization algebra which depends *holomorphically* on the manifold; in particular the manifold must come equipped with a complex structure. Since the full structure of a holomorphic factorization algebra will not play a significant role in our discussion we divert the interested reader to the textbook [9, §5] or the survey [17].

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In physics, "conserved currents" refer to observables of a physical system whose integrals over cycles lead to conserved quantities. We will find that the algebras we have introduced in §8.2 arise from the value of holomorphic factorization algebras on codimension one spheres. The structure of the factorization algebras allows one to multiply currents which we will see corresponds to taking the enveloping algebra of the underlying Lie algebra of currents.

Let us begin by asking about the "factorization model" of the simplest example, the Heisenberg algebra. We have already encountered the Dolbeault complex $\Omega^{0,\bullet}(X)$ of a complex manifold X. If X is not compact, one defines the compactly supported forms as a subalgebra $\Omega^{0,\bullet}_c(X) \subset \Omega^{0,\bullet}(X)$.

Suppose $U \subset X$ is an open set in the complex manifold. Then, consider the cochain complex $\mathrm{Sym}(\Omega^{0,\bullet}_c(U)[1])$. This is the graded symmetric algebra on the shifted compactly supported Dolbeault forms. The assignment $U \mapsto \mathrm{Sym}(\Omega^{0,\bullet}_c(U)[1])$ actually extends to the structure of a prefactorization algebra on X. The multiplication map $m_{U,V;W}$ for the configuration $i : U \sqcup V \hookrightarrow W$ is defined by the following composition

$$\begin{tikzcd}
\mathrm{Sym}\left(\Omega^{0,\bullet}_c(U)[1]\right) \otimes \mathrm{Sym}\left(\Omega^{0,\bullet}_c(V)[1]\right) \arrow[r, "\cong"] \arrow[dr, "m_{U,V;W}"'] & \mathrm{Sym}\left(\Omega^{0,\bullet}_c(U)[1] \oplus \Omega^{0,\bullet}_c(V)[1]\right) \arrow[d, "i_*"] \\
& \mathrm{Sym}\left(\Omega^{0,\bullet}_c(W)[1]\right).
\end{tikzcd}$$

The shift by $[1]$ may seem arbitrary, but leads to an interesting consequence as we will now see. For the remainder of this section, let $\mathcal{F}$ stand for this prefactorization algebra.

We restrict to the complex manifold $X = \mathbb{C}^n$ and consider the following types of open sets. Choose holomorphic coordinates $(z_1, \ldots, z_n)$. The radius $\mathrm{rad} = |z_1|^2 + \cdots + |z_n|^2$ defines a map $\mathrm{rad}\colon \mathbb{C}^n \setminus 0 \to \mathbb{R}_{>0}$. For $r < R$ the open set $\mathrm{rad}^{-1}(r < R) \subset \mathbb{C}^n$ is in the neighborhood of a $(2n-1)$-sphere in $\mathbb{C}^n$ centered at zero of radius $(r+R)/2$.

Recall the dg algebra A_n modeling punctured affine space that we introduced in §8.2. We realized this as a subalgebra of (non-compactly supported) Dolbeault forms $\Omega^{0,\bullet}(\mathbb{C}^n \setminus 0)$. We relate compactly supported Dolbeault forms to A_n using a version of Serre duality. Indeed, let ρ be a smooth function on $\mathbb{C}^n \setminus 0$ which takes value 1 near the outer boundary of $\mathrm{rad}^{-1}(r < R)$ and value zero near the inner boundary. We further normalize ρ so that

$$\mathsf{int}_{\mathbb{C}^n} \overline{\partial} \rho \, \mathrm{d}^n z = 1.$$

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The assignment $\alpha \mapsto \alpha \wedge \overline{\partial}(\rho)$ defines a cochain map

$$i_\rho : \mathsf{A}_n \hookrightarrow \Omega_c^{0,\bullet}\big(\mathrm{rad}^{-1}(r<R)\big)[1].$$

Notice that the map is defined by wedging with a particular $(0,1)$-form, hence the cohomological shift. One can use Stokes' theorem to see that this map intertwines the $\overline{\partial}$ operators. This map has a very special property, which is proved using Serre duality: at the level of cohomology the map is *dense*. Applying the symmetric algebra functor $\mathrm{Sym}(-)$, we see that i_ρ gives a map of cochain complexes

$$i_\rho : \mathrm{Sym}(\mathsf{A}_n) \to \mathcal{F}\left(\mathrm{rad}^{-1}(r<R)\right).$$

Instead of considering fixed values $r<R$ we can ask about the *pushforward* of the factorization algebra $\mathrm{rad}_*\mathcal{F}$ along the radius map. This one-dimensional factorization algebra on $\mathbb{R}_{>0}$ is *almost* locally constant. Consider the following construction.

- There is a natural action of the torus T^n on $\mathbb{C}^n \setminus 0$ given by rotations $(e^{2\pi i\theta_1}z_1,\ldots,e^{2\pi i\theta_n}z_n)$. This action extends to the factorization algebra $\mathrm{rad}_*\mathcal{F}$. Let $\mathcal{F}(\mathrm{rad}^{-1}(r<R))^{(\vec{N})}$ denote the subspace of $\mathrm{rad}_*(\mathcal{F})(r<R)$ where T^n acts by weight $\vec{N} \in \mathbb{Z}^n$. Then, the assignment

$$(r<R) \subset \mathbb{R}_{>0} \mapsto \oplus_{N\in\mathbb{Z}}\mathcal{F}(\mathrm{rad}^{-1}(r<R))^{(N)}$$

  defines a *locally constant* factorization algebra on $\mathbb{R}_{>0}$.
- By Theorem 8.28 this construction defines an associative dg algebra that we denote by $\oint\mathcal{F}$. The map i_ρ defines an isomorphism of associative dg algebras

$$i_\rho : \mathrm{Sym}(\mathsf{A}_n) \xrightarrow{\cong} \oint\mathcal{F}.$$

This case was somewhat boring as we used the data of the factorization algebra $\mathcal{F}$ to extract a *commutative* dg algebra $\mathrm{Sym}(\mathsf{A}_n)$. Recall, the Heisenberg algebra is a central extension of the (abelian) Lie algebra of algebraic functions on the punctured disk. So, a natural question to ask is where the central extension is hiding at the level of the factorization algebra construction. The key idea is that at the level of factorization algebras, this extension appears as a *deformation* of the differential $\overline{\partial}$.

Consider the (familiar looking) multi-linear functional ψ on the compactly supported forms $\Omega_c^{0,\bullet}(\mathbb{C}^n \setminus 0)$:

$$\psi(\omega_0,\ldots,\omega_n) = \mathsf{int}_{\mathbb{C}^n\setminus 0}\omega_0\partial\omega_1\cdots\partial\omega_n.$$

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Notice that instead of taking the residue, as in Definition 8.17 we are integrating along all of $\mathbb{C}^n \setminus 0$. Because of this, we see that ψ is actually of *cohomological degree* $+1$ in the sense that for the value $\psi(\omega_0, \dots, \omega_n)$ to be nonzero one must have $|\omega_0| + \cdots + |\omega_n| = n$.

Using ψ, we can perform the following deformation of the factorization algebra $\mathcal{F}$. To an open set $U \subset \mathbb{C}^n$ consider the cochain complex

$$\widetilde{\mathcal{F}}(U) = \left(\mathrm{Sym}\left(\Omega_c^{0,\bullet}(U)[1]\right)\,,\, \overline{\partial} + \psi\right).$$

Notice that the term in the differential is the one used in the definition of $\mathcal{F}$, the functional ψ is the deformation.

Proceeding just as we did above by looking at codimension one spheres we can build a locally constant one-dimensional factorization algebra out of $\widetilde{\mathcal{F}}$ and hence an A_∞ algebra $\oint \widetilde{\mathcal{F}}$.

Proposition 8.15 *Let* h_n *be the n-Heisenberg algebra. Then, there is a quasi-isomorphism of* A_∞ *algebras* $i_\rho : U(\mathsf{h}_n) \to \oint \widetilde{\mathcal{F}}$.

The key to the proof of this proposition is the observation that the functional ψ is compatible with the cocycle φ_n of Lemma 8.11 under the map i_ρ defined above. We refer to [16] for more details and for the following generalization. Recall that $\mathfrak{g} \otimes \Omega^{0,\bullet}(X)$ has the structure of a dg Lie algebra; the same is true if we replace forms by compactly supported forms.

Theorem 8.29 ([16]) *Let* $\mathfrak{g}$ *be a Lie algebra and* $\theta \in \mathrm{Sym}^{n+1}(\mathfrak{g}^*)^{\mathfrak{g}}$ *an invariant polynomial. Let* $\mathcal{F}_\theta(\mathfrak{g})$ *be the factorization algebra which assigns to an open set* $U \subset \mathbb{C}^n$ *the cochain complex*

$$\widetilde{C}_\bullet^\theta\left(\mathfrak{g} \otimes \Omega_c^{0,\bullet}(U)\right) = \left(\mathrm{Sym}(\mathfrak{g} \otimes \Omega_c^{0,\bullet}(U)[1]\,,\, \overline{\partial} + \mathrm{d}_{\mathrm{CE}} + \psi_\theta\right).$$

Then, there is an equivalence of A_∞ *algebras*

$$U(\widehat{\mathfrak{g}}_{n,\theta}) \to \oint \mathcal{F}_\theta(\mathfrak{g}).$$

In this proposition, ψ_θ is the multi-linear functional on $\mathfrak{g} \otimes \Omega_c^{0,\bullet}$ defined by $\mathrm{int}\,\theta(\omega_0 \partial \omega_1 \cdots \partial \omega_n)$. The complex $\widetilde{\mathrm{C}}_\bullet$ is a deformation of the Chevalley–Eilenberg complex computing the Lie algebra *homology* of $\mathfrak{g} \otimes \Omega_c^{0,\bullet}$. There is a completely analogous result for the Virasoro algebra, and its higher dimensional versions. We refer to [34, Chapter 3] for more details.

To summarize, we have used the "radial" part of the structure of factorization algebras on $\mathbb{C}^n$ to extract the current algebras, and central extensions thereof. Of course, there is much more data present in the factorization algebra than just its radial part. In complex dimension one, the factorization product of disks, for instance, is the geometric underpinning of the operator product expansion in the context of vertex algebras. We will not directly involve ourselves with the product of disks, but we discuss another configuration which will lead us to a familiar theory of modules for infinite-dimensional Lie algebras.

Vacuum Modules

We've already mentioned the passage from associative algebras to locally constant factorization algebras on $\mathbb{R}_{>0}$. This equivalence is best thought of as an equivalence between locally constant factorization algebras and the operad of little intervals $\{(a,b) \mid a<b\}$ in $\mathbb{R}_{>0}$. The underlying vector space (or cochain complex) of the associative algebra is the value of the prefactorization algebra on such an interval $A=\mathcal{F}((a,b))$.

What happens when we look at a factorization algebra $\mathcal{F}$ on $\mathbb{R}_{\geqslant 0}$ instead? The condition of being locally constant is replaced by the notion of a "constructible" factorization algebra, we refer to [1] for a full development. At the operadic level, this has the effect of introducing a new type of interval of the form $[0,a)$ together with operations corresponding to configurations of intervals

$$(b,c)\sqcup(d,e)\hookrightarrow(a,f),$$
$$[0,c)\sqcup(d,e)\hookrightarrow[0,f),$$

where $0<a<b<c<d<e<f$. The first type of configuration gives rise to an algebra structure on $A=\mathcal{F}((b,c))$, just as it did in the case of $\mathbb{R}_{>0}$. The second type of configuration endows $M=\mathcal{F}([0,c))$ with the structure of an A-module. The following result is a very special case of the general formalism developed in [1]. See also [9, Chapter 8].

Theorem 8.30 *A constructible factorization algebra on $\mathbb{R}_{\geqslant 0}$ is equivalent to the data of an associative algebra A together with an A-module M.*

We have shown how factorization algebras on $\mathbb{C}^n$ give rise to associative algebras by restricting to the "radial" part $\oint\mathcal{F}$. Extending this, we consider the radial projection map as $\mathrm{rad}\colon \mathbb{C}^n\to\mathbb{R}_{\geqslant 0}$. Restricting to $\mathbb{R}_{>0}$ we know how to recover $\oint\mathcal{F}$ by looking at S^1-eigenspaces. Now, we look at the S^1-eigenspaces of $\mathcal{F}([0,r))$, this is simply the value of $\mathcal{F}$ on the n-disk $D_0(r)$ of radius r centered at zero. Denote the direct sum of all of the eigenspaces by $V_{\mathcal{F}}$.

Proposition 8.16 *Suppose that $\mathcal{F}$ is a holomorphic factorization algebra (such as $\mathcal{F}=\mathrm{Sym}(\Omega_c^{0,\bullet}[1])$). Then the subspace $V_{\mathcal{F}}\subset\mathcal{F}(D_r^n(0))$ is a module for the A_∞ algebra $\oint\mathcal{F}$.*

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Let's work out an example of this in the case of the n-Heisenberg algebra. We have already seen that the radial part of the factorization algebra $\mathcal{F} = \mathrm{Sym}(\Omega_c^{0,\bullet}[1])$, where the differential $\overline{\partial} + \psi$ recovers the enveloping algebra of the n-Heisenberg algebra h_n. We will argue that $V_{\mathcal{F}}$ is the vacuum module $\mathsf{F}_n(0,1)$ for h_n constructed in §8.2.6.

To see that the cohomology of $V_{\mathcal{F}}$ is the right thing is not difficult. Indeed, by Serre duality $H^\bullet(\Omega_c^{0,\bullet}(D_0(r)))$ can be identified with the continuous linear dual of the space of holomorphic n-forms $\Omega^{n,hol}(D_0(r))$ concentrated in degree $+n$. The residue defines a linear embedding

$$\mathsf{A}_{n,-} \hookrightarrow \Omega^{n,hol}(D_0(r))^\vee$$

sending α to the functional $\omega \mapsto \mathrm{Res}_{z=0}(\alpha \wedge \omega)$, which one can show agrees precisely with the subspace of S^1-eigenspaces. Applying the symmetric algebra one then finds that $H^\bullet V_{\mathcal{F}} \cong \mathrm{Sym}(\mathsf{A}_{n,-}[1-n])$, which we compare to Proposition 8.14.

We consider the module structure in the case $n = 1$. The characterizing property of the vacuum module involved the existence of a vector $|0\rangle \in V_{\mathcal{F}}$ such that $b[n]|0\rangle = 0$ for all $n \geqslant 0$. The vacuum vector is simply $|0\rangle = 1 \in \mathrm{Sym}^0$. To represent $b[n] = z^n \in \mathrm{Sym}(\mathbb{C}[z,z^{-1}]) \cong \oint \mathcal{F}$ in the value of the factorization algebra on an annulus we use the bump function $\overline{\partial}\rho$ of the previous section. The element $\widetilde{b}[n] = z^n \overline{\partial}\rho \in \mathcal{F}(\mathbb{A})$ represents $b[n]$ in cohomology.

We want to compute the value of $\widetilde{b}[n] \otimes 1$ under the factorization product

$$m : \mathcal{F}(D_0(r)) \otimes \mathcal{F}(\mathrm{rad}^{-1}(R < R')) \to \mathcal{F}(D_0(r')),$$

where $r < R < R' < r'$. It suffices to show that when viewed as an element of $\mathcal{F}(D_0(r'))$, the class $\widetilde{b}[n]$ is cohomologically trivial for $n \geqslant 0$. Indeed, when $n \geqslant 0$, the element ρz^n defines an element of $\mathcal{F}(D_0(r'))$ which satisfies $\overline{\partial}(\rho z^n) = \widetilde{b}[n]$.

Proposition 8.17 *Consider the factorization algebra* $\mathcal{F} = \left(\mathrm{Sym}(\Omega_c^{0,\bullet}[1]), \overline{\partial} + \psi\right)$ *on* $\mathbb{C}^n$. *The module* $V_{\mathcal{F}}$ *for the* A_∞ *algebra* $\oint \mathcal{F} \simeq U(h_n)$ *is equivalent to the vacuum Fock module* $F_n(0,1)$.

A very similar result holds for n-Kac–Moody algebras $\widehat{\mathfrak{g}}_{n,\theta}$ and their associated vacuum modules that we introduced in §8.2.6. Associated to any Lie algebra $\mathfrak{g}$ and degree $n+1$ invariant polynomial θ there is the factorization algebra $\mathcal{F}_{\mathfrak{g},\theta}$ on $\mathbb{C}^n$ described in Theorem 8.29. Its radial part $\oint \mathcal{F}_{\mathfrak{g},\theta}$ returns the enveloping algebra of $\widehat{\mathfrak{g}}_{n,\theta}$. The module $V_{\mathcal{F}_{\mathfrak{g},\theta}}$ is equivalent to the level θ vacuum module $V_{n,\theta}(\mathfrak{g})$. We refer to [16] for more details.

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8.3.2 Super Enhancements

Super Lie algebras play a role in symmetries of physical systems which have objects of different parity. Most common examples of such systems arise from *supersymmetry* which involve fields or particles that have both "bosonic" and "fermionic" statistics.

The blend of supersymmetry with infinite-dimensional Lie algebras is rich from both a mathematical and physical perspective. Physically, super enhancements of the (ordinary) Virasoro algebra, for instance, play a significant role in superstring theory in an analogous way that the Virasoro algebra appears in conformal field theory or in the bosonic string. We will see an example of one such enhancement below. Mathematically, Kac has classified all finite-dimensional super Lie algebras which naturally leads to the theory of affine super Lie algebras [22]. The connection between the character theory of affine super Lie algebras and number theory is a deep and fascinating subject on its own [24].

In this section we will follow a similar trend and consider super enhancements of higher dimensional Kac–Moody and Virasoro algebras. These algebras appear as symmetries in higher dimensional supersymmetric field theories. We do not stress that realization here, but refer to [16, 32, 33] for accounts of this.

These enhancements are interesting in their own right and most certainly lead to a rich theory of representations extending what we have dipped our toes into throughout this note. One aspect of the theory that we focus on is how certain super enhancements of the multivariable algebras that we have introduced interpolate between ordinary (single-variable) infinite-dimensional Lie algebras, like affine algebras.

Example: the "Topological" Virasoro Algebra

We begin with an example of a super enhancement of a familiar infinite-dimensional Lie algebra. Recall that the Virasoro algebra is linearly generated by elements $\{L_n\}_{n\in\mathbb{Z}}$. The super Lie algebra $\mathsf{vir}_{1|1}$ spanned by even elements $\{L_n, J_n\}_{n\in\mathbb{Z}}$ and odd elements $\{G_n, Q_n\}_{n\in\mathbb{Z}}$ satisfies the commutation relations

$$
\begin{aligned}
[L_n, L_m] &= (n-m)L_{n+m} - \frac{c}{2}n(n^2-1)\delta_{n+m,0},\\
[L_n, G_m] &= -mG_{n+m}, \quad [L_n, Q_m] = (n-m)Q_{n+m},\\
[G_n, Q_m] &= L_{n+m} + nJ_{n+m} + \frac{c}{2}(n^2-n)\delta_{n+m,0},\\
[J_n, J_m] &= cn\delta_{n+m,0},\\
[J_n, G_m] &= G_{n+m}, \quad [J_n, Q_m] = -Q_{n+m}.
\end{aligned}
$$

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This super Lie algebra admits a lift to a $\mathbb{Z}$-graded Lie algebra by declaring that the elements $\{Q_n\}$ have degree $+1$ and the elements $\{G_n\}$ have degree -1. We will assume this grading in what follows. Notice that the ordinary Virasoro algebra appears as a subalgebra. This algebra is referred to as the "topological Virasoro" algebra [15] due to its connection with $\mathcal{N} = (2,2)$ supersymmetry.

With this example we can detail a particular construction that will be useful in the next section. Recall, a Maurer–Cartan element of a dg Lie algebra $\mathfrak{g}$ is an element $\alpha \in \mathfrak{g}$ satisfying the Maurer–Cartan equation

$$\mathrm{d}\alpha + \frac{1}{2}[\alpha, \alpha] = 0.$$

Such elements connect dg Lie algebras with the rich subject of formal deformation theory (see [19, 29], for instance). For us, it will be important to note that a Maurer–Cartan element allows one to *deform* the dg Lie algebra via

$$(\mathfrak{g}, \mathrm{d}) \rightsquigarrow (\mathfrak{g}, \mathrm{d} + [\alpha, -])$$

while keeping the bracket the same. More generally, if D is a Maurer–Cartan element in the dg Lie algebra of derivations of $\mathfrak{g}$ one can consider the deformation $\mathrm{d} + D$; in this case α determines the inner derivation $D = [\alpha, -]$.

The element $Q_0 \in \mathsf{vir}_{1|1}$ is a Maurer–Cartan element; indeed the differential is trivial and $[Q_0, Q_0] = 0$. In particular we obtain a deformed algebra $(\mathsf{vir}_{1|1}, [Q, -])$. It is actually not difficult to show that this deformation completely collapses on to the trivial Lie algebra.

Higher Dimensional Enhancements

We focus on super enhancements of the multivariable current algebra. Any Lie algebra $\mathfrak{g}$ acts on itself by the adjoint representation. Using this, one can define a super Lie algebra structure on $\mathfrak{g} \oplus \Pi\mathfrak{g}$ where $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is the usual bracket, $\mathfrak{g} \times \Pi\mathfrak{g} \to \Pi\mathfrak{g}$ is the adjoint action and $\Pi\mathfrak{g} \times \Pi\mathfrak{g} \to \mathfrak{g}$ is the zero map. If $\mathbb{C}[\varepsilon]$ denotes the "odd dual numbers," the free graded algebra generated by a single *odd* element ε, then this super Lie algebra is identical to $\mathfrak{g}[\varepsilon] = \mathfrak{g} \otimes \mathbb{C}[\varepsilon]$. In the case $\mathfrak{g} = \mathfrak{gl}(N)$, note that $\mathfrak{gl}(N)[\varepsilon]$ is a subalgebra of the matrix super Lie algebra $\mathfrak{gl}(N|N)$.

The definition of the n-current algebra makes sense for a super or graded Lie algebra. For the example at hand, we consider the n-current algebra associated to $\mathfrak{g}[\varepsilon]$ which is $\mathfrak{g}[\varepsilon] \otimes \mathsf{A}_n$. Concretely, elements are of the form $\alpha + \varepsilon\alpha'$ where $\alpha, \alpha' \in \mathsf{A}_n$. There are two gradings at hand: there is a $\mathbb{Z}$-grading arising from the $\mathbb{Z}$-grading on A_n and there is a $\mathbb{Z}/2$-grading arising from the parity of ε. In fact, this $\mathbb{Z}/2$ grading actually lifts to a $\mathbb{Z}$-grading by declaring that ε has degree -1. In this way we obtain a dg Lie algebra structure on $\mathfrak{g}[\varepsilon] \otimes \mathsf{A}_n$ whereby A_n carries

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its usual $\mathbb{Z}$-grading and ε has degree -1. We will use this totalized $\mathbb{Z}$-grading in what follows.

In the last section we saw how one can deform a dg Lie algebra; the same is true for the current algebra $\mathfrak{g}[\varepsilon] \otimes \mathsf{A}_n$. Given any derivation $D \in \mathrm{Der}(\mathfrak{g}[\varepsilon] \otimes \mathsf{A}_n)$ of cohomological degree $+1$ which satisfies

$$[\overline{\partial}, D] + \frac{1}{2}[D, D] = 0, \tag{8.3.1}$$

we can consider the deformed current algebra

$$\left(\mathfrak{g}[\varepsilon] \otimes \mathsf{A}_n\,,\ \overline{\partial}\right) \rightsquigarrow \left(\mathfrak{g}[\varepsilon] \otimes \mathsf{A}_n\,,\ \overline{\partial} + D\right).$$

Recall that A_n is a model for punctured affine space whose coordinates are $(z_1, \ldots, z_n)$. We define the following derivation of $\mathfrak{g}[\varepsilon] \otimes \mathsf{A}_n$

$$D \stackrel{\text{def}}{=} z_n \frac{\partial}{\partial \varepsilon}.$$

It is immediate to see that D satisfies (8.3.1) and so defines a deformed algebra $\left(\mathfrak{g}[\varepsilon] \otimes \mathsf{A}_n\,,\ \overline{\partial} + z_n \frac{\partial}{\partial \varepsilon}\right)$.

Proposition 8.18 *Let* $\iota\colon \mathbb{A}^{n-1} \to \mathbb{A}^n$ *be the map* $\iota(z_1, \ldots, z_{n-1}) \mapsto (z_1, \ldots, z_{n-1}, 0)$. *The map*

$$\left(\mathfrak{g}[\varepsilon] \otimes A_n\,,\ \overline{\partial} + z_n \frac{\partial}{\partial \varepsilon}\right) \xrightarrow{\simeq} \mathfrak{g} \otimes \mathsf{A}_{n-1},$$

which sends $X \otimes \alpha$ *to* $X \otimes \iota^* \alpha$*, is a quasi-isomorphism of dg Lie algebras.*

Proof This statement follows from showing that $\iota^*\colon \mathsf{A}_n[\varepsilon] \to \mathsf{A}_{n-1}$ is a quasi-isomorphism of dg algebras. At the level of coordinates ι^* sends $\varepsilon \mapsto 0$, $z_i \mapsto z_i$ for $i \neq n$ and $z_n \mapsto 0$. It is not difficult to see that the map intertwines the differentials. To see that ι^* is a quasi-isomorphism one can write down an explicit retraction, see [32]. We remark that the key to describing the cohomology is to observe that there is a degree -1 operator on $\mathsf{A}_2[\varepsilon]$ defined by $\varepsilon \partial_{z_2}$ which satisfies $[\overline{\partial} + z_n \partial_\varepsilon, \varepsilon \partial_{z_2}] = z_n \partial_{z_n} + \varepsilon \partial_\varepsilon$. Thus, $\varepsilon \partial_{z_n}$ can be used to define a homotopy between $\mathsf{A}_n[\varepsilon]$ and the complex obtained by setting ε and z_n to zero.

The explicit formula for the retraction in the case $n = 2$ is

$$s(z^{-n}) = \frac{\overline{z}_1^n}{(z_1\overline{z}_1 + z_2\overline{z}_2)^n} - \varepsilon n \frac{\overline{z}_1^{n-1}}{(z_1\overline{z}_1 + z_2\overline{z}_2)^{n-1}} \omega,$$

where $\omega \in \mathsf{A}_2^1$ is the Bochner–Martinelli kernel as defined in (8.2.5). A direct calculation shows

$$\overline{\partial} s(z^{-n}) = z_2 n \frac{\overline{z}_1^{n-1}}{(z_1\overline{z}_1 + z_2\overline{z}_2)^{n-1}} \omega,$$

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which implies $\left(\overline{\partial}+z_2\frac{\partial}{\partial\varepsilon}\right)s(z_1^{-n})=0$. Thus showing that s is a cochain map. $\square$

Applied to the case $n=2$, this proposition implies that there is a deformation of the two-dimensional current algebra $\mathfrak{g}\otimes \mathsf{A}_2[\varepsilon]$ which is equivalent to the ordinary current algebra $\mathfrak{g}[z,z^{-1}]$. To see the affine algebra $\widehat{\mathfrak{g}}_\kappa$ as a deformation one must introduce a central extension of this 2-current algebra. For an invariant quadratic polynomial κ on $\mathfrak{g}$ consider the bilinear map

$$\widetilde{\varphi}_\kappa(\alpha+\varepsilon\alpha',\beta+\varepsilon\beta')=\mathrm{Res}^{(2)}_{z=0}\big(\kappa(\alpha,\partial\beta')\mathrm{d}z_2+\kappa(\alpha',\partial\beta)\mathrm{d}z_2\big).$$

Here, we emphasize that the righthand side denotes the two-dimensional residue. It is an exercise to see that $\widetilde{\varphi}_\kappa$ is a two-cocycle of the dg Lie algebra $(\mathfrak{g}\otimes\mathsf{A}_2[\varepsilon],\overline{\partial}+z_2\partial_\varepsilon)$ and hence defines a central extension that we denote by $\widehat{\mathfrak{g}}_{2|1,\kappa}$.

Proposition 8.19 ([32]) *The quasi-isomorphism of the above proposition extends to a quasi-isomorphism between $\widehat{\mathfrak{g}}_{2|1,\kappa}$ with differential $\overline{\partial}+z_2\partial_\varepsilon$ and the affine algebra $\widehat{\mathfrak{g}}_\kappa$.*

We remark that there are similar results at the level of vacuum, and Verma, modules for the higher dimensional Kac–Moody algebra. The above algebraic results are closely related to the work of [4] where they give a construction of vertex algebras from the observables of four-dimensional supersymmetric field theory. We refer to [32] for a further discussion of this and an extension of these results to factorization algebras.

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