ABSTRACT DEFINITIONS FOR REFLECTION GROUPS

G. C. SHEPHARD

Introduction. This paper is an appendix to "Finite unitary reflection groups" by J. A. Todd and the author **(3)** in which the irreducible finite groups generated by reflections in unitary space **(2**, p. 364**)** are enumerated and abstract definitions are given for all those n-dimensional groups which may be generated by n reflections. (All real reflection groups are of this kind.) The purpose of this paper is to supply abstract definitions for all the remaining groups, namely those that require more than n reflections to generate them.

Some of the original definitions in our paper have been elegantly modified by Coxeter **(1)**. The definitions given here are constructed in such a way as to form natural extensions of his scheme.

1. The groups $G(m, r, 3)$**.** Let θ be a primitive mth root of unity, r a divisor of m, and $s = m/r$. Then by permuting the rows (or columns) of the diagonal $n \times n$ matrices

$$\operatorname{diag}\{\theta^{\rho_1}, \theta^{\rho_2}, \dots, \theta^{\rho_n}\}, \qquad \sum \rho_i \equiv 0 \pmod r,$$

in every possible way, we obtain a group of $sm^{n-1}n!$ matrices which is denoted by $G(m, r, n)$ **(3**, p. 277**)**. When $r = 1$ or $r = m$, this group is generated by n reflections: $G(m, 1, n)$ is the symmetry group of the polytope $\gamma_n{}^m$ **(2**, p. 374**)** and so, following the notation of **(3)**, we denote it by $[\gamma_n{}^m]$. Also, when $n > 2$, $G(m, m, n) = [\frac{1}{m}\gamma_n{}^m]$ is $[1\ 1\ (n-2)^m]^3$ in Coxeter's notation **(1**, pp. 248–249**)**.

Consider first the groups $G(m, r, 3)$ with $r \neq 1, m$. The matrices

1.1 $$\mathbf{P} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \mathbf{Q} = \begin{pmatrix} 0 & \theta & 0 \\ \theta^{-1} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{R} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathbf{T} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \theta^r \end{pmatrix}$$

clearly generate $G(m, r, 3)$, and if we denote by P, Q, R, T respectively the corresponding group elements, then the following relations are satisfied:

1.2 $$P^2 = Q^2 = R^2 = (QR)^3 = (RP)^3 = (PQ)^3 = E,$$

1.3 $$\begin{cases} (PTP)(RT^{-1}R) = (RT^{-1}R)(PTP) = (QPRP)^r, \\ TQ = QT, \quad T(PRP) = (PRP)T, \quad T^s = E. \end{cases}$$

We shall show that these relations form an abstract definition for the group.

Since $(PTP)^s = PT^sP = E$, $(RT^{-1}R)^s = RT^{-s}R = E$, and $PTP \leftrightarrows RT^{-1}R$, we have

$$(QPRP)^m = ((QPRP)^r)^s = (PTP)^s(RT^{-1}R)^s = E,$$

Received April 3, 1956.

https://doi.org/10.4153/CJM-1957-033-x Published online by Cambridge University Press

that is,

1.4 $$(QPRP)^m = E.$$

It is known that relations 1.2 and 1.4 constitute an abstract definition for the group $G(m, m, 3) = [1\ 1\ 1]^m$ of order $6m^2$ (**1**, p. 248). Denote an arbitrary element of this group by G; each occurrence of G in an expression stands for a (possibly) different element of the group.

By 1.3, T, and therefore T^u, commutes with PRP and so $PT^uPR = RPT^uP$. This implies

$$(QPRP)^{ru} = (PT^uP)(RT^{-u}R) = RPT^uPT^{-u}R,$$

from which $T^uP = GT^u$. Similarly making use of the fact that T commutes with $RPR = PRP$, we get $T^uR = GT^u$.

Let W be any word, i.e. finite product of the elements P, Q, R, T. Then W must be of the form

$$GT^{u_1}GT^{u_2}G\ldots GT^{u_p}G.$$

Let $w(W)$ be the number of factors P, Q, R occurring in all the elements G of W *except* the first. Since

$$T^{u_1}P = GT^{u_1},\ T^{u_1}Q = QT^{u_1},\ T^{u_1}R = GT^{u_1},$$

whatever letter P, Q, or R follows T^{u_1} in W it is possible to substitute some product of group elements so as to form an equivalent word W' with $w(W) - w(W') \geqslant 1$. Repeating this process and amalgamating powers of T when they become adjacent, W may be reduced, by a finite number of substitutions, to the standard form

$$W^* = GT^u, \qquad 0 \leqslant u \leqslant s - 1,$$

with $w(W^*) = 0$. Hence the given relations define a group of order $(6m^2)s$ which is $G(m, r, 3)$.

For $G(4, 2, 3)$ a simpler definition than that obtained by substituting $m = 4$ and $r = 2$ in 1.3 is given by 1.2 together with the relations

1.5 $$(QPRP)^4 = E,$$

1.6 $$T^2 = (TQ)^2 = (TPRP)^2 = (TR)^4 = P(TRQR)P(TRQR)^{-1} = E.$$

This can be checked by the Todd-Coxeter method (**1a**, Chap. 2).

2. The other groups. Now consider $G(m, r, n)$ $(r|m;\ r \neq 1, m;\ n > 3)$ of order $rm^{n-1}n!$. If we introduce $n - 3$ further generators $S_1, S_2, \ldots, S_{n-3}$ which satisfy the relations

2.1 $$\begin{cases} S_i^2 = (PS_i)^2 = (RQRS_i)^2 = (S_iS_j)^2 = (RS_1)^3 = E \\ \qquad\qquad (i, j = 1, 2, \ldots, n-3;\ |i-j| \geqslant 2), \\ (S_iS_{i-1})^3 = (RS_i)^2 = E \qquad (i = 2, 3, \ldots, n-3), \end{cases}$$

then these, together with 1.2, 1.4, form a definition for $G(m, m, n)$ of order

$m^{n-1}n!$. This is the same definition as that of (**2**, pp. 374–5) with the notation changed thus:

$$P = P_1, \quad Q = R_1Q_1R_1, \quad R = R_1, \quad S_i = R_{i+1} \qquad (i = 1, 2, \ldots, n-3).$$

The use of these new generators is due to Coxeter (**1**, p. 248) who has shown that they simplify the definitions of some of the reflection groups discussed in (**3**).

Adding now the generator T satisfying 1.3 and $TS_i = S_iT$ ($i = 1, 2, \ldots, n-3$) we get an abstract definition for $G(m, r, n)$. The proof is very similar to that of §1, remarking that every word can be reduced to the standard form $G'T^u$, where G' is an arbitrary element of $G(m, m, n)$.

This completes the discussion of $G(m, r, n)$ except for the groups where $n = 2$. Here we take as generators P, Q and T corresponding to the matrices

$$\mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \mathbf{Q} = \begin{pmatrix} 0 & \theta \\ \theta^{-1} & 0 \end{pmatrix}, \qquad \mathbf{T} = \begin{pmatrix} 1 & 0 \\ 0 & \theta^r \end{pmatrix}$$

and obtain the relations

$$P^2 = Q^2 = T^s = E, \quad T(PQ) = (PQ)T, \quad PT^{-1}PT = TPT^{-1}P = (PQ)^r,$$

which can be shown to be an abstract definition in the same way as for the the above groups.

The only remaining groups generated by more than n reflections are seven two-dimensional groups for which abstract definitions have already been given (**3**, pp. 281–282; **1**, Table XIII) and the group $[(\frac{1}{2}\gamma_3{}^4)^{+1}]$ (no. 31 in the table of (**3**, p. 301)).

This group is generated by the matrices (cf. 1.1):

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \mathbf{Q} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{R} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$\mathbf{S} = \tfrac{1}{2}\begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix}, \quad \mathbf{T} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

The corresponding group elements satisfy the relations 1.2, 1.5, 1.6 and

2.2 $\qquad S^2 = (SP)^2 = (SR)^2 = (SQ)^3 = E,$

2.3 $\qquad (ST)^3 = E.$

By §1, the relations 1.2, 1.5, 1.6 form an abstract definition for $G(4, 2, 3)$, while 1.2, 1.5, 2.2 form an abstract definition for $[1\ 1\ 2]^4 = [(\frac{1}{4}\gamma_3{}^4)^{+1}]$ (**2**, p. 374). We assert that 1.2, 1.5, 1.6, 2.2, 2.3 together form an abstract definition for the group $[(\frac{1}{2}\gamma_3{}^4)^{+1}]$. This can readily be verified by the Todd-Coxeter method by enumerating the six cosets of the subgroup $[1\ 1\ 2]^4$.

References

1. H. S. M. Coxeter, *Groups generated by unitary reflections of period two*, Can. J. Math. *9* (1957), 243–272.

1a. H. S. M. Coxeter and W. O. J. Moser, *Generators and relations*, Ergebnisse der Math., Neue Folge, *14* (1957).

2. G. C. Shephard, *Unitary groups generated by reflections*, Can. J. Math. *5* (1953), 364–383.

3. G. C. Shephard and J. A. Todd, *Finite unitary reflection groups*, Can. J. Math. *6* (1954), 274–304.

University of Birmingham

https://doi.org/10.4153/CJM-1957-033-x Published online by Cambridge University Press