# DUALS OF BANACH SPACES OF ENTIRE FUNCTIONS

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**1. Introduction.** Let $w$ be a strictly positive function on $\mathbb{C}$ and let $H_\infty^w$, respectively $H_0^w$, denote the Banach spaces of those entire functions $\varphi(z)$ with $|\varphi(z)| = O(w(z))$ and $|\varphi(z)| = o(w(z))$. In this generality, these spaces may contain only constants, but for many functions $w(z)$ these will be interesting Banach spaces with norm

$$\|\varphi\|_w = \sup\{|\varphi(z)|/w(z) : z \in \mathbb{C}\}.$$

We study two specific problems.

(A) For which weight functions $w$ is $H_\infty^w$ isomorphic, possibly isometrically, to $(H_0^w)^{**}$?

(B) For which weight functions $w$ can the first dual $(H_0^w)^*$ be identified with a space of functions analytic on some subset of $\mathbb{C}$?

Some answers to (A) were given about 20 years ago by J. Shapiro, A. Shields and G. D. Taylor; unpublished, but see **[2]** where (A) is studied by an abstract functional analytic approach. Problem (B) has been much discussed for functions analytic in the unit disk $\Delta$, e.g. for Bloch functions in **[1]**. Our motivation for the study of these problems, however, comes from **[4]**, where it is shown that the answer is "yes" to both problems for the following very special weight functions $w$.

Let $K$ be a compact convex subset of the closed unit disk and set

$$w(\lambda) = \max\{|\exp \lambda z| : z \in K\}.$$

Although it rarely happens that $(H_0^w)^*$ is a Banach algebra under its natural pointwise product, it is for such a function $w$.

The results obtained are satisfactory only in the case of radial weights, i.e. $w(\lambda) = w(|\lambda|)$, where $w$ grows faster than any polynomial. We give some partial results for the non-radial case and indicate where the difficulties arise.

**2. The weight functions.** Given any positive $w$ on $\mathbb{C}$ we define

$$\tilde{w}(\lambda) = \sup\{|\varphi(\lambda)| : \varphi \in H_\infty^w,\ \|\varphi\|_w \leqq 1\}.$$

Clearly $\tilde{w} \leqq w$ and $H_\infty^{\tilde{w}} = H_\infty^w$. It is clear also, from Montel's theorem, that for each $\lambda \in \mathbb{C}$ there exists $\varphi_\lambda(z) \in H_\infty^w$ with $\tilde{w}(\lambda) = |\varphi_\lambda(\lambda)|$. Also $\tilde{w}$ is lower semicontinuous since it is the supremum of a family of continuous functions; frequently $\tilde{w}$ is actually continuous.

We say that $w$ is radial if $w(\lambda) = w(|\lambda|)$. It then follows that $w$ is a strictly increasing function, unless $H_\infty^w$ contains only constants, and that $\log \tilde{w}(r)$ is a convex increasing function of $\log r$. Thus for radial functions a necessary condition that $w(\lambda) \equiv \tilde{w}(\lambda)$ is that $\log w(t)$ be a convex increasing function of $\log t$. Perhaps this condition is also sufficient; some evidence for this is provided by **[3]**. For our results we might as well replace $w$ by $\tilde{w}$; so henceforth we assume that $w = \tilde{w} > 0$.

We wish to disregard the case when $H_\infty^w$ is finite dimensional. In the case of radial weights this means discarding weights such as $w(\lambda) = 1 + |\lambda|^k$, $k \in \mathbb{N}$, where $H_\infty^w$ is the space of all polynomials of degree at most $k$ and $H_0^w$ is the space of all polynomials of degree at most $k-1$. For this reason, when $w$ is radial we assume that for $0<t<1$,

$$\lim_{r\to\infty} w(tr)/w(r)=0. \tag{2.1}$$

For non-radial transcendental weights $w(\lambda)$, the space $H_\infty^w$ may still be finite dimensional, e.g. if $w(\lambda)=(1+|\lambda|^k)\,|\exp\lambda|$.

**3. The duality.** We recall the reasoning of [4] for the case $w(\lambda)=\exp(|\lambda|)$. Here $H_0^w$ is a closed subspace of $C_0^w$ and $(C_0^w)^*\approx M^w$, the space of regular Borel measures $\mu$ on $\mathbb{C}$ for which

$$\|\mu\|=\int_{\mathbb{C}} w\,|d\mu|<\infty.$$

Thus $(H_0^w)^*\approx M^w/(H_0^w)^\perp$. This abstract identification of $(H_0^w)^*$ is similar to that for $(\text{lip}\,\alpha)^*$ obtained in [5]. The "concrete" identification of $(H_0^w)^*$ is obtained as follows. For the closed unit disk $\bar{\Delta}$ let $A^w$ denote the set of those functions $f:\bar{\Delta}\to\mathbb{C}$ of the form

$$f(z)=\int_{\mathbb{C}}\exp(z\lambda)\,d\mu(\lambda)\qquad(z\in\bar{\Delta}), \tag{3.1}$$

for some $\mu\in M^w$. The formula

$$\langle\varphi,f\rangle=\int_{\mathbb{C}}\varphi\,d\mu, \tag{3.2}$$

where $f$ is given by (3.1), gives a pairing between $H_\infty^w$ and $A^w$. This involves showing that $\mu\in(H_\infty^w)^\perp$ if and only if $\mu\perp\exp(z\lambda)$, $z\in\bar{\Delta}$. It is then routine to check that $(A^w)^*\approx H_\infty^w$ and $(H_0^w)^*\approx A^w$.

For the rest of this section we assume only that $w$ is strictly positive even though we occasionally mention the implication of $w$ being radial. Choose any $\chi(z)\in H_\infty^w$ with $\|\chi\|_w=1$ and define $K=K(\chi)$ by

$$K=\{z\in\bar{\Delta}:|\chi(z\lambda)|\leqq w(\lambda)\text{ for all }\lambda\in\mathbb{C}\}. \tag{3.3}$$

Clearly $K$ is a closed subset of $\bar{\Delta}$ and when $w$ is radial we have $K=\bar{\Delta}$. In §4 we shall further restrict our choice of $\chi$. As in (3.1) we let $A^w$ denote the set of functions $f:K\to\mathbb{C}$ of the form

$$f(z)=\int_{\mathbb{C}}\chi(z\lambda)\,d\mu(\lambda)\qquad(z\in K),$$

for some $\mu\in M^w$. We then say that $\mu$ *represents* $f$. Let

$$Z=\{\mu\in M^w:\mu\text{ represents }0\}.$$

Then $Z$ is a closed subspace of $M^w$ and $A^w\approx M^w/Z$. Thus for $\|f\|$ we take

$$\|f\|=\inf\{\|\mu\|:\mu\text{ represents }f\}.$$

Now $f$ is continuous on $K$ and analytic on $K^0$; because each function

$$f_n(z)=\int_{|\lambda|\leqq n}\chi(z\lambda)\,d\mu(\lambda)$$

is analytic on $K^0$ ($=\Delta$ when $w$ is radial) and $f_n\to f$ locally uniformly in $K^0$.

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We now show how (3.2) gives a well-defined pairing. To this end, set

$$B^w = \left\{\varphi \in H_\infty^w : \int_{\mathbb{C}} \varphi \, d\mu = 0 \text{ for all } \mu \in Z\right\},$$

so that (3.2) gives a well-defined pairing on $(B^w, A^w)$.

THEOREM 1. *$(A^w)^*$ is isometrically isomorphic to $B^w$.*

*Proof.* For $\varphi \in B^w$ set $\Gamma_\varphi(f) = \langle \varphi, f \rangle$, where $f \in A^w$. Then $|\Gamma_\varphi(f)| \leqq \|\varphi\| \, \|\mu\|$ for any $\mu$ that represents $f$. Thus $|\Gamma_\varphi(f)| \leqq \|\varphi\| \, \|f\|$ so that $\Gamma_\varphi \in (A^w)^*$ with $\|\Gamma_\varphi\| \leqq \|\varphi\|$. Evidently the map $\varphi \to \Gamma_\varphi$ is linear. To show that $\|\Gamma_\varphi\| \geqq \|\varphi\|$ we choose, for each $\lambda \in \mathbb{C}$, the mass $\mu_\lambda$ to be the point measure at $\lambda$. Let $\mu_\lambda$ represent $f_\lambda$ so that, in fact

$$f_\lambda(z) = \chi(\lambda z) \qquad (z \in K).$$

Then $\|f_\lambda\| \leqq w(\lambda)$ and hence

$$\begin{aligned} \|\Gamma_\varphi\| &\geqq \sup_\lambda |\Gamma_\varphi(w(\lambda)^{-1} f_\lambda)| \\ &= \sup_\lambda w(\lambda)^{-1} |\langle f_\lambda, \varphi \rangle| \\ &= \sup_\lambda w(\lambda)^{-1} |\varphi(\lambda)| = \|\varphi\|, \end{aligned}$$

as required.

To prove that $\Gamma$ is onto we fix $T \in (A^w)^*$ and let $\varphi(\lambda) = T(f_\lambda)$, where $\lambda \in \mathbb{C}$. Suppose that the entire function $f_\lambda(z) = \chi(\lambda z)$ has expansion

$$f_\lambda(z) = \sum_{n=0}^{\infty} a_n \lambda^n z^n \qquad (z \in K).$$

Let $u(z)$ be the identity function, $u(z) = z$ for $z \in K$, and note that the function

$$a_n u^n = (1/2\pi i) \int_{|\zeta|=R} \zeta^{-n-1} \chi(z\zeta) \, d\zeta$$

belongs to $A^w$. For $|\lambda| < R$ the series $\sum a_n \lambda^n z^n$ converges to $f_\lambda(z)$ in $A^w$ and hence

$$\varphi(\lambda) = \sum_{n=0}^{\infty} a_n T(u^n) \lambda^n.$$

Thus $\varphi$ is entire and since $|\varphi(\lambda)| \leqq \|T\| \, \|f_\lambda\| \leqq \|T\| \, w(\lambda)$ we have that $\varphi \in H_\infty^w$. Suppose now that

$$\int_{\mathbb{C}} \chi(z\lambda) \, d\mu(\lambda) = 0 \qquad (z \in K).$$

Then

$$\int_{\mathbb{C}} \varphi \, d\mu = \int_{\mathbb{C}} T(f_\lambda) \, d\mu(\lambda) = T\left(\int_{\mathbb{C}} f_\lambda \, d\mu(\lambda)\right) = 0,$$

since $\int_{\mathbb{C}} f_\lambda(z) \, d\mu(\lambda)$ is zero for $z \in K$, as may be seen by applying point-evaluation

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functionals. Hence $\varphi \in B^w$. Finally, if $f \in A^w$ is represented by $\mu$, we have

$$\Gamma_\varphi(f) = \int_{\mathbb{C}} \varphi \, d\mu = \int_{\mathbb{C}} T(f_\lambda) \, d\mu(\lambda) = T\left(\int_{\mathbb{C}} f_\lambda \, d\mu(\lambda)\right) = Tf$$

so that $\Gamma$ is onto, as required.

**4. Radial weights.** The reasoning above includes the case of non-radial weights. For some $w$ and $\chi$, the set $K$ need not be all of $\bar{\Delta}$. It can have empty interior and we may even have $K = \{1\}$, for example, with $w(\lambda) = (1 + |\lambda|^k) \, |\exp \lambda|$ and $\chi(\lambda) = \exp \lambda$. The set $K$ need not be connected; with the $w$ above and with $\chi(\lambda) = \lambda^k \exp \lambda$ we obtain $K = \{0, 1\}$. We do not know whether it is possible to choose $\chi$ so that $K$ is convex. We consider now the case of radial weights; non-radial weights are discussed in §5.

THEOREM 2. *Let $w$ be radial and satisfy* (2.1). *For suitable choice of $\chi$ we have* $H_\infty^w = B^w = (A^w)^*$.

*Proof.* Since, by assumption, $\log w(r) = \log \tilde{w}(r)$ is a positive unbounded convex function of $\log r$, it is possible to construct a function $\chi(z) = \sum_{n=0}^{\infty} a_n z^n$ in $H_\infty^w$ with $a_n \neq 0$, $n \geqq 0$ (see [3, Theorem 2] where a more precise result is proved). Suppose that $\varphi \in H_\infty^w$ and let $\mu$ represent 0, so that

$$\int_{\mathbb{C}} \chi(z\lambda) \, d\mu(\lambda) = 0 \qquad (z \in \bar{\Delta}).$$

For $z \in \bar{\Delta}$ set

$$g(z) = \int_{\mathbb{C}} \varphi(z\lambda) \, d\mu(\lambda),$$

so that $g$ is analytic for $z \in \Delta$. It is easy to justify repeated differentiation at $z = 0$; indeed since $\varphi \in H_\infty^w$ we have a bound

$$|\lambda^n \varphi^{(n)}(\lambda)| \leqq M_n w(2\lambda) \qquad (\lambda \in \mathbb{C}).$$

Thus, for $|z| \leqq 1/2$, for example, we have

$$g^{(n)}(z) = \int_{\mathbb{C}} \lambda^n \varphi^{(n)}(z\lambda) \, d\mu(\lambda)$$

and so, in particular,

$$g^{(n)}(0) = \varphi^{(n)}(0) \int_{\mathbb{C}} \lambda^n \, d\mu(\lambda).$$

This holds for all $\varphi \in H_\infty^w$. We take $\varphi(z) = \chi(z)$ to obtain, for all $n \geqq 0$,

$$g^{(n)}(0) = n! \, a_n \int_{\mathbb{C}} \lambda^n \, d\mu(\lambda) = 0.$$

Hence $g(z)\equiv 0$ for $|z|<1$ and, by continuity,

$$0=g(1)=\int_{\mathbb{C}}\varphi(\lambda)\,d\mu(\lambda),$$

completing the proof.

We now consider $(H_0^w)^*$.

THEOREM 3. *Let $w$ be radial and satisfy* (2.1). *Then* $(H_0^w)^*=A^w$.

*Proof.* Let $B_0^w=B^w\cap H_0^w$. When $w$ is radial $B^w=H_\infty^w$ and so $B_0^w=H_0^w$. Since $B_0^w$ is a subspace of $C_0^w$ we have the usual isometric isomorphism (using the notation of §3)

$$(B_0^w)^*=(C_0^w)^*/(B_0^w)^\perp=M^w/(B_0^w)^\perp.$$

Since $A^w=M^w/Z$ and, obviously $Z\subset(B_0^w)^\perp$, the theorem is proved once we show that $(B_0^w)^\perp\subset Z$. Suppose, then, that $\mu\in M^w$ with $\int_{\mathbb{C}}\varphi\,d\mu=0$ for all $\varphi\in B_0^w$. We need to show that

$$\int_{\mathbb{C}}\chi_z(\lambda)\,d\mu(\lambda)=0 \tag{4.1}$$

for each $z\in K=\bar{\Delta}$, where $\chi_z(\lambda)=\chi(z\lambda)$. But $\chi_z\in B_0^w$ and so (4.1) holds for all $z\in\Delta$ and the result follows by continuity.

In the non-radial case we may still get that $(H_0^w)^*=A^w$ and $(A^w)^*=H_\infty^w$. For example, it is clear that the arguments in Theorems 2 and 3 hold for any weight $w$ with a radial minorant (i.e. $w(\lambda)\geqq w_1(\lambda)$ for $\lambda\in\mathbb{C}$ with $w_1$ radial and positive). For other non-radial cases special conditions or special methods involving the Borel transform or Paley–Wiener theorem are required; see [**4**] for details.

## 5. Concluding remarks.

In the case of radial weights, the approximation condition of [**2**] yields immediately that $(H_0^w)^{**}=H_\infty^w$. The methods of the present paper give the isometric result. In the radial case the method is independent of the choice of $\chi$ provided that $\chi^n(0)\neq 0$, $n\geqq 0$. Of course, the set $K$ can vary with $\chi$.

There are non-radial weight functions for which $(H_0^w)^{**}\neq H_\infty^w$, though the problem becomes finite-dimensional. Take $w(\lambda)=(1+|\lambda|^k)\,|\exp\lambda|$ and $\chi(\lambda)=\exp\lambda$. Then $K=\{1\}$ and $A^w$, $B^w$ are both 1-dimensional with $B_0^w=\{0\}$. Clearly $\dim H_0^w=k$ and $\dim H_\infty^w=k+1$.

The results are also very sensitive to a change in weight function. Suppose that $w_n(\lambda)\to w(\lambda)$ uniformly on compact subsets of $\mathbb{C}$. We may have $(H_0^{w_n})^{**}=H_\infty^{w_n}$ but $(H_0^w)^{**}\neq H_\infty^w$, or we may have that $(H_0^{w_n})^{**}\neq H_\infty^{w_n}$ but $(H_0^w)^{**}=H_\infty^w$.

The problems and ideas of the present work have counterparts in $\mathbb{C}^n$. However the essential difficulties remain and so we do not consider the matter further.

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