the sphere in two real points, an infinite number meeting it in two imaginary points, and four generators of each system touching it. This result is obtained analytically from equation (1), which shows that the generators through $(a\cos\alpha, b\sin\alpha, 0)$ touch the sphere if $\sin^2\alpha = (c^4 - b^4)/(a^4 - b^4)$ and intersect the sphere in real or imaginary points according as $\sin^2\alpha \gtreqless (c^4 - b^4)/(a^4 - b^4)$. If P is the point of contact of one of the generators which touch the sphere, that generator is the shortest distance between two coincident generators of the opposite system, and thus P lies on a line of striction.

Lastly, if $b^2 > r^2$, then $c^2 > a^2 > b^2$. In this case the H_1 and sphere have no real common points, and hence no two of the generators of the H_1 intersect at right angles. Since the generators of the H_1 and its asymptotic cone $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$ are parallel, the greatest angle between a pair of generators is $2\tan^{-1}b/c$.

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Note on the Principal Axes of a Normal Section of a Cylinder which Envelopes an Ellipsoid.—This note deals with methods of determining the lengths and direction-cosines of the axes. If the lengths are 2α and 2β, then α^{-2} and β^{-2} are the two non-zero roots of the discriminating cubic for the cylinder, and thus are easily found. The following methods apply geometrical considerations and determine both the lengths and the direction-cosines.

Let the ellipsoid be $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, and let the generators of the cylinder be parallel to $x/l = y/m = z/n$. Consider the normal section of the cylinder through the centre of the ellipsoid. The plane of the section is $lx + my + nz = 0$. The perpendiculars from the centre to the common tangent planes of the ellipsoid and cylinder lie in the plane $lx + my + nz = 0$, and are the perpendiculars to the tangents to the section of the cylinder by this plane. Let a common tangent plane be $\lambda x + \mu y + \nu z = \sqrt{a^2\lambda^2 + b^2\mu^2 + c^2\nu^2}$, and let the perpendicular from the centre to this plane be of length r.

Then
$$r^2 = \frac{a^2\lambda^2 + b^2\mu^2 + c^2\nu^2}{\lambda^2 + \mu^2 + \nu^2},$$

and hence the perpendicular lies on the cone

$$(a^2 - r^2)x^2 + (b^2 - r^2)y^2 + (c^2 - r^2)z^2 = 0. \quad \text{...............(1)}$$

It also lies in the plane $lx + my + nz = 0$. ....(2)

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The plane and cone intersect in two lines through the centre which are the perpendiculars to the four tangents to the normal section that are at a distance r from the centre. If these perpendiculars coincide, they coincide along an axis. Hence if the plane touches the cone, $2r$ is the length of an axis. The condition of tangency is

$$\frac{l^2}{a^2-r^2}+\frac{m^2}{b^2-r^2}+\frac{n^2}{c^2-r^2}=0,$$

a quadratic in r^2 which gives the lengths of the semi-axes. If λ, μ, ν are the direction-cosines of the axis of length $2r$, the plane (2) touches the cone (1) along the line $x/\lambda=y/\mu=z/\nu$, and therefore

$$\frac{\lambda(a^2-r^2)}{l}=\frac{\mu(b^2-r^2)}{m}=\frac{\nu(c^2-r^2)}{n}.$$

These results can also be obtained by considering the intersection of the cylinder and a sphere whose centre is on the axis of the cylinder, and whose radius is equal to a principal semi-axis of a normal section. If the reduced equation of the cylinder is $x^2/\alpha^2+y^2/\beta^2=1$, the equation $\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}=\frac{x^2+y^2+z^2}{r^2}$, where $r^2=\alpha^2$ or β^2 shows that the intersection consists of a pair of planes (real or imaginary), whose line of intersection is real, and is one of the principal axes. (The planes give circular sections of the cylinder). The equation of the cylinder is

$$\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}\right)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)-\left(\frac{lx}{a^2}+\frac{my}{b^2}+\frac{nz}{c^2}\right)^2=0,$$

i.e. $$k(\mathrm{E}-1)-\mathrm{P}^2=0, \text{ say.}$$

Hence $k\left(\mathrm{E}-\frac{x^2+y^2+z^2}{r^2}\right)-\mathrm{P}^2=0$, if r is a principal semi-axis, represents a pair of planes, and their line of intersection is the axis. Differentiating with respect to x, y, z, we obtain three planes through the line of intersection,

$$kx\left(\frac{1}{a^2}-\frac{1}{r^2}\right)-\frac{l}{a^2}\mathrm{P}=0,$$

$$ky\left(\frac{1}{b^2}-\frac{1}{r^2}\right)-\frac{m}{b^2}\mathrm{P}=0,$$

$$kz\left(\frac{1}{c^2}-\frac{1}{r^2}\right)-\frac{n}{c^2}\mathrm{P}=0.$$

Hence the line of intersection is

$$\frac{x(a^2-r^2)}{l}=\frac{y(b^2-r^2)}{m}=\frac{z(c^2-r^2)}{n},$$

and since this lies in the plane $lx+my+nz=0$,

$$\frac{l^2}{a^2-r^2}+\frac{m^2}{b^2-r^2}+\frac{n^2}{c^2-r^2}=0. \quad \ldots\ldots\ldots\ldots\ldots(3)$$

The form of equation (3) shows that the principal axes of the normal section of the cylinder are equal to the principal axes of the section of the ellipsoid by the plane $lx/a+my/b+nz/c=0$, and thus that these sections are equal ellipses. This equality can be established otherwise and equation (3) deduced. If P and Q are corresponding points on an ellipse and its auxiliary circle

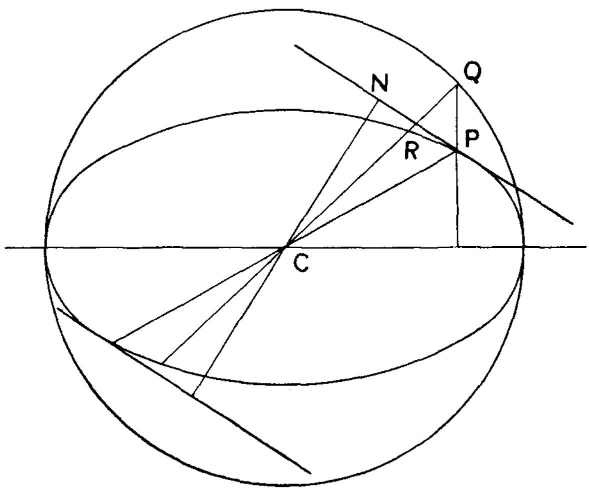

Fig. 1.

(Fig. 1), and CQ meets the ellipse in R, CR is equal to CN, the perpendicular from C to the tangent at P. ($CN^{-2}=\cos^2\theta/a^2+\sin^2\theta/b^2=CR^{-2}$, where θ is the eccentric angle of P). Thus any radius vector of an ellipse is equal to the perpendicular from the centre to a tangent. But the perpendiculars from the centre to the tangents to a normal section of the cylinder are the perpendiculars to the common tangent planes of the

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ellipsoid and cylinder. If $x/\lambda = y/\mu = z/\nu$ is a perpendicular, the tangent plane is $\lambda x + \mu y + \nu z = \sqrt{a^2\lambda^2 + b^2\mu^2 + c^2\nu^2}$, and r, the length of the perpendicular, is given by

$$r^2 = \frac{a^2\lambda^2 + b^2\mu^2 + c^2z^2}{\lambda^2 + \mu^2 + \nu^2.}.$$

We have also $l\lambda + m\mu + n\nu = 0$, ... (4)

since the perpendicular lies in the plane $lx + my + nz = 0$. Consider now the semi-diameter of the ellipsoid when equations are

$$\frac{x}{a\lambda} = \frac{y}{b\mu} = \frac{z}{c\nu} \left(= \frac{r_1}{\sqrt{a^2\lambda^2 + b^2\mu^2 + c^2\nu^2}} \right).$$

Its length is given by $r_1{}^2 = \dfrac{a^2\lambda^2 + b^2\mu^2 + c^2\nu^2}{\lambda^2 + \mu^2 + \nu^2}$, and by (4) it lies in the plane $\dfrac{lx}{a} + \dfrac{my}{b} + \dfrac{nz}{c} = 0$. ....(5)

Therefore the semi-diameters of the normal section of the cylinder are equal to the semi-diameters of the section of the ellipsoid by the plane (5), and the sections are equal ellipses.

If the tangent at P to the ellipse (Fig. 1) is parallel to $x/l = y/m$, the equations of CN and CR are $lx + my = 0$, $\dfrac{lx}{a} + \dfrac{my}{b} = 0$, and thus the results for the ellipse and ellipsoid are exactly analogous.

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A Construction by Ruler and Dividers.

1. The following method of cutting off an n^{th} part of a given straight line requires besides a ruler only a pair of dividers or other means of laying off on a given straight line a segment equal to the distance between two given points. In other words, we assume that, besides using a ruler in the usual way, we can mark off from a given straight line AX a segment AB, which shall be equal to the distance between two given points C and D.

2. The construction is as follows:—

Let AB be the given straight line.

Draw any other straight line AX from A, and cut off from AX a succession of equal parts, the $n-1^{\text{th}}$ ending at D, and the $n+1^{\text{th}}$ at C.

Join CB, and produce it to E, making BE = CB.