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A NOTE ON ISOMORPHISMS OF MULTIPLIER ALGEBRAS

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1. **Introduction.** Let A_1, A_2 be commutative semi-simple Banach algebras and $M(A_1)$, $M(A_2)$ their multiplier algebras. Birtel in [2] has proved that every isomorphism of A_1 onto A_2 induces an isomorphism of $M(A_1)$ onto $M(A_2)$. In this note, we extend this result to the noncommutative case. We also show that if A is a dual A^*-algebra which is a dense two-sided ideal of a B^*-algebra B, then $M(A)$ is isomorphic to $M(B)$. Thus the converse of the previous result cannot hold. All algebras under consideration are over the complex field.

Let A be a semi-simple Banach algebra. A mapping f on A into itself is called a multiplier if $(fx)y = x(fy)$ for all x, y in A. It has been shown that f is linear and continuous and $f(xy) = (fx)y$ for all x, y in A. Let $M(A)$ be the set of all multipliers of A. Then $M(A)$ is a semi-simple commutative Banach algebra with identity; $M(A)$ is called the multiplier algebra of A.

2. **Multiplier algebras of A^*-algebras.** In this section, unless otherwise stated, A will be an A^*-algebra with norm $\|\cdot\|$ which is a dense two-sided ideal of a B^*-algebra B with norm $|\cdot|$.

LEMMA 2.1. *If A is commutative, then for each $f \in M(A)$, f is a multiplier of B.*

Proof. Let M be a maximal modular ideal of A and let u be an identity for A modular M. Let N be the closure of M in B. If follows easily from [4; p. 18, Lemma 4] that $u \notin N$ and so N is a modular ideal of B. Now it is easy to see that A and B have the same carrier space X. Therefore $B = C_0(X)$, the algebra of all continuous complex-valued functions on X vanishing at infinity. Let $f \in M(A)$. By [6; p. 1135, Theorem 3.1], f can be considered as a bounded continuous function on X. Since $M(B)$ is the algebra of all bounded continuous functions on X (see [6; p. 1131]), $f \in M(B)$.

LEMMA 2.2. *Let A be a semi-simple Banach algebra and E a maximal commutative subalgebra of A. If $f \in M(A)$, then $f_E \in M(E)$, where F_E denotes the restriction of f to E.*

Proof. Let $f \in M(A)$ and x, $y \in E$. Since $(fx)y = y(fx)$, fx commutes with E. Hence by the maximality of E, $fx \in E$. Therefore $f_E \in M(E)$. This completes the proof.

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For any set S in a Banach algebra A, let $L(S)$ and $R(S)$ denote the left and right annihilators of S in A, respectively. Then A is called a dual algebra if, for every closed left ideal I and for every closed right ideal J, we have $I=L(R(I))$ and $J=R(L(J))$.

LEMMA 2.3. *Let A be a dual algebra. If $f\in M(B)$, then $f_A\in M(A)$.*

Proof. Let $\{e_\alpha\}$ be a maximal orthogonal family of self-adjoint minimal idempotents in A and let $x\in A$. By [4; p. 30, Theorem 16], $x=\sum_\alpha e_\alpha x$ in the norm $\|\cdot\|$. Hence there is only a countable number of e_α for which $e_\alpha x\neq 0$; say $e_{\alpha_1}, e_{\alpha_2}, \ldots$. Let $f\in M(B)$. For any two positive integers m, $n(m\leq n)$, [4; p. 18, Lemma 4] shows that

$$\left\|\sum_{i=m}^{n} e_{\alpha_i}(fx)\right\|\leq k\left|f\left(\sum_{i=m}^{n} e_{\alpha_i}\right)\right|\left\|\sum_{i=m}^{n} e_{\alpha_i}x\right\|$$
$$\leq k\,|f|\left\|\sum_{i=m}^{n} e_{\alpha_i}x\right\|,$$

where k is a constant and $|f|$ is the operator bound of f in B. Therefore $\{\sum_{i=1}^{n} e_{\alpha_i}fx\}$ is a Cauchy sequence in A. It follows easily that $fx\in A$. Hence $f_A\in M(A)$.

THEOREM 2.4. *Let A be a dual A^*-algebra which is a dense two-sided ideal of a B^*-algebra B. Then there exists an isomorphism of $M(A)$ onto $M(B)$.*

Proof. Let $f\in M(A)$. We shall show that f can be uniquely extended to a mapping $f'\in M(B)$. Let $x\in B$ be a hermitian element and let E be a maximal commutative *-subalgebra of B containing x. Let $F=E\cap A$. It is easy to show that F is a maximal commutative *-subalgebra of A which is a dense two-sided ideal of E. By [4; p. 31, Theorem 19], F is dual. By Lemmas 2.1 and 2.2, $f_F\in M(E)$. Let $\{x_n\}\subset F$ be a sequence converging to x in $|\cdot|$. Since $f_F\in M(E)$, $f_Fx=\lim_n fx_n$ in $|\cdot|$.

We claim that f_Fx is independent of the choice of E. In fact, let E' be a maximal commutative *-subalgebra of B containing x and let $F'=E'\cap A$. Let $\{y_n\}$ be a sequence in F' converging to x in $|.|$. Then $f_{F'}x=\lim_n fy_n$ in $|.|$. Let $z\in A$. By [4; p. 18, Lemma 4], we have

$$|(f_{F'}x-f_Fx)z|\leq|f_{F'}x-fy_n|\,|z|+k\,|y_n-x_n|\,\|fz\|+|fx_n-f_Fx|\,|z|,$$

where k is a constant. Hence $(f_{F'}x-f_Fx)A=(0)$ and so $f_{F'}x=f_Fx$. Therefore f_Fx is independent of the choice of E. We write $f'x=f_Fx$. Let $y\in b$ and write $y=y_1+iy_2$, where y_1, y_2 are hermitian. Define

$$f'y=f'y_1+if'y_2.$$

It is straightforward to show that f' is a multiplier of B such that $f'_A=f$; clearly f' is unique. This together with Lemma 2.3 shows that $f\to f'$ is a one-one mapping of $M(A)$ onto $M(B)$. It is now easy to see that $f \to f'$ is an isomorphism of $M(A)$ onto $M(B)$. This completes the proof.

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