17 An introduction to general relativity

Even as the evidence for the Standard Model became stronger and stronger in the 1970s and beyond, so the evidence for general relativity grew in the latter half of the twentieth century. Any discussion of the Standard Model and physics beyond it must confront Einstein's theory at two levels. First, general relativity and the Standard Model are very successful at describing the history of the universe and its present behavior on large scales. General relativity gives rise to the big bang theory of cosmology, which, coupled with our understanding of atomic and nuclear physics, explains – indeed predicted – features of the observed universe. But there are features of the observed universe which cannot be accounted for within the Standard Model and general relativity. These include dark matter and dark energy, the origin of the asymmetry between matter and antimatter, the origin of the seeds of cosmic structure (inflation) and more. Apart from these observational difficulties, there are also serious questions of principle. We cannot simply add Einstein's theory onto the Standard Model. The resulting structure is not renormalizable and cannot represent in any sense a complete theory. Black holes, when combined with quantum mechanics, raise further puzzles. In this book we will encounter both these aspects of Einstein's theory. Within extensions of the Standard Model, in the next few chapters we will attempt to explain some features of the observed universe. The second, more theoretical, level is addressed in the third part of this book. String theory, our most promising framework for a comprehensive theory of all interactions, encompasses general relativity in an essential way; some would even argue that what we mean by string theory is the quantum theory of general relativity.

The purpose of this chapter is to introduce some concepts and formulas that are essential to the applications of general relativity in this text. No previous knowledge of general relativity is assumed. We will approach the subject from the perspective of field theory, focusing on the dynamical degrees of freedom and the equations of motion. We will not give as much attention to the beautiful – and conceptually critical – geometric aspects of the subject, though we will return to some of these in the chapters on string theory. Those interested in a more in-depth treatment of general relativity will eventually want to study some of the excellent texts listed in the suggested reading at the end of the chapter.

Einstein put forward his principle of relativity in 1905. At that time, one might quip, half the known laws, those of electricity and magnetism, already satisfied this principle with no modification. The other half, Newton's laws, did not. In considering how one might reconcile gravitation and special relativity, Einstein was guided by the observed equality of gravitational and inertial mass. Inertia has to do with how objects move in space–time in response to forces. Operationally, the way we define space–time, our measurements of length, time, energy and momentum, depends crucially on this notion. The fact that gravity

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couples to precisely this mass suggests that gravity has a deep connection to the nature of space–time. Considering this equivalence, Einstein noted that an observer in a freely falling elevator (in a uniform gravitational field) would write down the same laws of nature as an observer in an inertial frame without gravity. Consider, for example, an elevator full of particles interacting through a potential $V(\vec{x}_i - \vec{x}_j)$. In the inertial frame,

$$m\frac{d^2\vec{x}_i}{dt^2} = m\vec{g} - \vec{\nabla}_i V(\vec{x}_i - \vec{x}_j). \tag{17.1}$$

The coordinates of the accelerated observer are related to those of the inertial observer by

$$\vec{x}_i = \vec{x}_i' + \frac{1}{2}\vec{g}t^2; \tag{17.2}$$

so, substituting with the equations of motion (17.1), we obtain

$$m\frac{d^2\vec{x}_i'}{dt^2} = -\vec{\nabla}_i V(\vec{x}_i' - \vec{x}_j'). \tag{17.3}$$

Einstein abstracted from this thought experiment a strong version of the *equivalence principle*: the equations of motion should have the same form in any frame, inertial or not. In other words, it should be possible to write down the laws so that in any two coordinate systems, x^μ and $x'^\mu(x)$, they take the same form. This is a strong requirement. We will see that it is similar to gauge invariance, where the requirement that the laws take the same form after gauge transformations determines the dynamics.

17.1 Tensors in general relativity

To implement the equivalence principle, we begin by thinking about the invariant element d_s of distance. In an inertial frame, in special relativity,

$$ds^2 = d\vec{x}^2 - dt^2 = \eta_{\mu\nu}dx^\mu dx^\nu. \tag{17.4}$$

Note here that we have changed the sign of the metric, as we said we would do, from that used earlier in this text. This is the convention of most workers and texts in general relativity and string theory. The above coordinate transformation for the accelerated observer alters the line element. This suggests we consider the generalization

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu. \tag{17.5}$$

The *metric tensor* $g_{\mu\nu}$ encodes the physical effects of gravitation. We will see that there is a non-trivial gravitational field when we cannot find coordinates which make $g_{\mu\nu} = \eta_{\mu\nu}$ everywhere.

To develop a dynamical theory, we would like to write down invariant actions (which will yield covariant equations). This problem has two parts. We need to couple the fields that we already have to the metric in an invariant way. We also require an analog of the field strength for gravity, which will determine the dynamics of $g_{\mu\nu}$ in much the same way as the field strength $F_{\mu\nu}$ determines the dynamics of the gauge field A_μ. This object is the

Riemann tensor, $\mathcal{R}^{\mu}_{\nu\rho\sigma}$. We will see later that the formal analogy can be made very precise: An object, the spin connection ω_μ, constructed out of the metric tensor plays the role of A^μ. The close analogy will also be seen when we discuss Kaluza–Klein theories, where higher-dimensional general coordinate transformations become lower-dimensional gauge transformations.

We first describe how derivatives and $g_{\mu\nu}$ transform under coordinate transformations. Writing

$$x^\mu = x^\mu(x') \tag{17.6}$$

we have

$$\partial'_\mu \phi(x') = \frac{\partial x^\rho}{\partial x^{\mu\prime}} \partial_\rho \phi(x) = \Lambda_\mu^{\ \rho}(x)\partial_\rho \phi(x). \tag{17.7}$$

An object which transforms like $\partial_\rho \phi$ is said to be a covariant vector. An object which transforms like $\partial_{\rho_1}\phi\partial_{\rho_2}\phi \cdots \partial_{\rho_n}\phi$ is said to be an nth rank covariant tensor; $g_{\mu\nu}$ is an important example of such a tensor. We can obtain the transformation law for $g_{\mu\nu}$ from the invariance of the line element:

$$g'_{\mu\nu} dx^{\mu'} dx^{\nu'} = g_{\mu\nu} \frac{\partial x^\mu}{\partial x^{\rho\prime}} \frac{\partial x^\nu}{\partial x^{\sigma\prime}} dx^{\rho\prime} dx^{\sigma\prime}, \tag{17.8}$$

so

$$g'_{\mu\nu} = g_{\rho\sigma} \frac{\partial x^\rho}{\partial x^{\mu\prime}} \frac{\partial x^\sigma}{\partial x^{\nu\prime}}. \tag{17.9}$$

Now, dx^μ transforms according to the inverse of Λ:

$$dx'^\mu = \frac{\partial x'^\mu}{\partial x^\rho} dx^\rho, \tag{17.10}$$

where dx^μ is said to be a contravariant vector. Indices can be raised and lowered with $g_{\mu\nu}$; if V^ν is a contravariant vector then $g_{\mu\nu}V^\nu$ transforms as a covariant vector, for example.

Another important object is the volume element, d^4x. Under a coordinate transformation,

$$d^4x = \left| \frac{\partial x}{\partial x'} \right| d^4x'. \tag{17.11}$$

The object in between the vertical lines is the Jacobian of the coordinate transformation, $|\det \Lambda|$. The quantity $\sqrt{-\det g}$ transforms in exactly the opposite fashion. So the four-volume, is invariant.

$$\int d^4x \sqrt{-\det g}. \tag{17.12}$$

We will consider a real scalar field ϕ. The action, before the inclusion of gravity, is

$$S = \int d^4x \frac{1}{2}(-\partial_\mu \phi\, \partial_\nu \phi\, \eta^{\mu\nu} - m^2\phi^2). \tag{17.13}$$

To make this invariant we can replace $\eta^{\mu\nu}$ by $g^{\mu\nu}$ and include a factor $\sqrt{\det(-g)}$ along with d^4x. Then

$$S = \int d^4x \sqrt{\det(-g)} \frac{1}{2}(-\partial_\mu \phi\, \partial_\nu \phi\, g^{\mu\nu} - m^2\phi^2). \tag{17.14}$$

The equations of motion should be *covariant*. They must generalize the equation

$$\partial^2\phi = -V'(\phi). \tag{17.15}$$

The first derivative of ϕ, we have seen, transforms as a vector, V_μ, under coordinate transformations, but the second derivative does not transform simply:

$$\begin{aligned}\partial_\mu V_\nu &= \partial_\mu\left(\frac{\partial x^{\rho\prime}}{\partial x^\nu}V'_\rho\right)\\ &= \frac{\partial x^{\rho\prime}}{\partial x^\nu}\frac{\partial x^{\sigma\prime}}{\partial x^\mu}\partial'_\sigma V'_\rho + \frac{\partial^2 x^{\rho\prime}}{\partial x^\mu\partial x^\nu}V_\rho.\end{aligned} \tag{17.16}$$

To compensate for the extra, inhomogeneous, term we need a covariant derivative, as in gauge theories. Rather than look at the equations of motion directly, however, we can integrate the scalar field Lagrangian by parts to obtain *second* derivatives. This yields

$$\sqrt{-g}(g^{\mu\nu}\partial_\mu\partial_\nu\phi + \partial_\mu g^{\mu\nu}\partial_\nu\phi) + g^{\mu\nu}\partial_\mu\sqrt{-g}\phi\,\partial_\nu\phi. \tag{17.17}$$

To bring this into a convenient form, we need a formula for the derivative of a determinant. We can work this out using a trick we have used repeatedly in the case of the path integral. Write

$$\det M = \exp(\mathrm{Tr}\ln M) \tag{17.18}$$

so that

$$\begin{aligned}\det(M+\delta M) &\approx \exp[\mathrm{Tr}\ln M + \ln(1+M^{-1}\delta M)]\\ &= (\det M)(1+M^{-1}\delta M).\end{aligned} \tag{17.19}$$

Thus, for example,

$$\frac{d\det M}{dM_{ij}} = M^{-1}_{ij}\det M. \tag{17.20}$$

Putting all this together, we have the quadratic term in the action for a scalar field:

$$\phi\left(g^{\mu\nu}\partial_\mu\partial_\nu\phi + \partial_\mu g^{\mu\nu}\partial_\nu\phi + g^{\mu\nu}\frac{1}{2}g^{\rho\sigma}\partial_\mu g_{\rho\sigma}\partial_\nu\phi\right). \tag{17.21}$$

Writing this as

$$\phi g^{\mu\nu}D_\mu\partial_\nu\phi, \tag{17.22}$$

we have for the covariant derivative

$$D_\mu V_\nu = \partial_\mu V_\nu - \Gamma^\lambda_{\mu\nu}V_\lambda. \tag{17.23}$$

Here

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\rho}(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}). \tag{17.24}$$

Note that $\Gamma^\lambda_{\mu\nu}$ is symmetric in μ, ν. The covariant derivative is often denoted by a semicolon and a Greek letter in the subscript or superscript:

$$D_\mu V_\nu \equiv V_{\mu;\nu}. \tag{17.25}$$

The reader can check that

$$\Gamma^\lambda_{\mu\nu} = \Gamma^{\lambda\prime}_{\mu\nu} - \frac{\partial^2 x^\lambda}{\partial x^\mu \partial x^\nu}, \tag{17.26}$$

which just compensates the extra term in the transformation law (17.16). Here Γ is known as the *affine connection* (the components of Γ are also sometimes referred to as the Christoffel symbols and Γ itself as the Christoffel connection; it is sometimes written as $\left\{ {}^{\;\mu}_{\nu\;\rho} \right\}$). With this definition,

$$D_\mu V_\nu = \partial_\mu V_\nu - \Gamma^\lambda_{\mu\nu} V_\lambda \tag{17.27}$$

transforms like a tensor with two indices, $V_{\mu\nu}$. Similarly, acting on contravariant vectors:

$$D_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda \tag{17.28}$$

transforms correctly. You can also check that $V_{\mu;\nu;\rho}$ transforms as a third-rank covariant tensor, and so on.

To get some practice, and to see how the metric tensor can encode gravity, let us use the covariant derivative to describe the motion of a free particle. In an inertial frame, without gravity,

$$\frac{d^2 x^\mu}{d\tau^2} = 0, \tag{17.29}$$

where $\tau = g_{\mu\nu} dx^\mu dx^\nu$ is the proper time is made covariant by first rewriting it as

$$\frac{dx^\rho}{d\tau} \frac{\partial}{\partial x^\rho} \left(\frac{dx^\mu}{d\tau} \right) = 0. \tag{17.30}$$

We need to replace the derivative $\partial/\partial x^\rho$ by a covariant derivative. The covariant version of the left-hand side of Eq. (17.29) is then

$$\frac{dx^\rho}{d\tau} D_\rho \left(\frac{\partial x^\mu}{\partial \tau} \right). \tag{17.31}$$

This becomes, using Eq. (17.28),

$$\frac{\partial x^\rho}{\partial \tau} \frac{\partial^2 x^\mu}{\partial x^\rho \partial \tau} + \Gamma^\mu_{\rho\sigma} \frac{\partial x^\sigma}{\partial \tau} \frac{\partial x^\rho}{\partial \tau}. \tag{17.32}$$

So the equation of motion is

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma} \frac{\partial x^\sigma}{\partial \tau} \frac{\partial x^\rho}{\partial \tau} = 0. \tag{17.33}$$

This is known as the geodesic equation. Viewed as Euclidean equations, the solutions are geodesics. For a sphere embedded in flat three-dimensional space, for example, the solutions of this equation are easily seen to be great circles. We should be able to recover

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Newton's equation for a weak gravitational field. For a weak static gravitational field we might expect that

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{17.34}$$

with $h_{\mu\nu}$ small. Since the gravitational potential in Newton's theory is a scalar, we might further guess that

$$g_{00} = -(1 + 2\phi), \quad g_{ij} = \delta_{ij}. \tag{17.35}$$

Then the non-vanishing components of the affine connection are

$$\begin{aligned} \Gamma^i_{00} &= \frac{1}{2} g^{ij} \left(\partial_0 g_{i0} + \partial_0 g_{0i} - \partial_i g_{00} \right) \\ &= \partial_i \phi \end{aligned} \tag{17.36}$$

and, similarly,

$$\Gamma^0_{0i} = -\partial_i \phi. \tag{17.37}$$

In the non-relativistic limit we can replace τ by t, and we have the equation of motion

$$\frac{d^2 x^i}{dt^2} = -\partial_i \phi. \tag{17.38}$$

17.2 Curvature

Using the covariant derivative we can construct actions for scalars and gauge fields. Fermions require some additional machinery; we will discuss this towards the end of the chapter. Instead, we turn to the problem of finding an action for the gravitational field itself. In the case of gauge fields the crucial object was the field strength, $F_{\mu\nu} = [D_\mu, D_\nu]$. For the gravitational field we will also work with the commutator of covariant derivatives operators. We write

$$[D_\mu, D_\nu] V_\rho = \mathcal{R}^\sigma_{\rho\mu\nu} V_\sigma, \tag{17.39}$$

where $\mathcal{R}$ is known as the *Riemann tensor* or *curvature tensor*. For a Euclidean space it measures what we would naturally call the curvature of the space. It is straightforward to work out an expression for $\mathcal{R}$ in terms of the affine connection:

$$\mathcal{R}^\lambda_{\mu\nu\kappa} = \partial_\kappa \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\mu\kappa} + \Gamma^\eta_{\mu\nu} \Gamma^\lambda_{\kappa\eta} - \Gamma^\eta_{\mu\kappa} \Gamma^\lambda_{\nu\eta}. \tag{17.40}$$

Unlike F, which is first order in derivatives of A, the Riemann tensor $\mathcal{R}$ is second order in derivatives of g. As a result the gravitational action will be first order in $\mathcal{R}$.

Note that $\mathcal{R}$ transforms as a tensor under coordinate transformations. It has important symmetry and cyclicity properties. These are most conveniently described by lowering the first index on $\mathcal{R}$:

$$\mathcal{R}_{\lambda\mu\nu\kappa} = \mathcal{R}_{\nu\kappa\lambda\mu}, \tag{17.41}$$

$$\mathcal{R}_{\lambda\mu\nu\kappa} = -\mathcal{R}_{\mu\lambda\nu\kappa} = -\mathcal{R}_{\lambda\mu\kappa\nu} = \mathcal{R}_{\mu\lambda\kappa\nu}, \tag{17.42}$$

$$\mathcal{R}_{\lambda\mu\nu\kappa} + \mathcal{R}_{\lambda\kappa\mu\nu} + \mathcal{R}_{\lambda\nu\kappa\mu} = 0. \tag{17.43}$$

Starting with $\mathcal{R}$ we can define other tensors. The most important is the *Ricci tensor*. This has only two indices:

$$\mathcal{R}_{\mu\kappa} = g^{\lambda\nu}\mathcal{R}_{\lambda\mu\nu\kappa}. \tag{17.44}$$

The Ricci tensor is symmetric:

$$\mathcal{R}_{\mu\kappa} = \mathcal{R}_{\kappa\mu}. \tag{17.45}$$

Also very important is the *Ricci scalar*:

$$\mathcal{R}_{\mathrm{s}} = g^{\mu\kappa}\mathcal{R}_{\mu\kappa}. \tag{17.46}$$

Note that the Riemann tensor $\mathcal{R}$ also satisfies an important identity, similar to the Bianchi identity for $F^{\mu\nu}$ (which gives the homogeneous Maxwell equations):

$$\mathcal{R}_{\lambda\mu\nu\kappa;\eta} + \mathcal{R}_{\lambda\mu\eta\nu;\kappa} + \mathcal{R}_{\lambda\mu\kappa\eta;\nu} = 0. \tag{17.47}$$

17.3 The gravitational action

Having introduced, through the Riemann tensor $\mathcal{R}$, a description of curvature, we are in a position to write down a generally covariant action for the gravitational field. Terms linear in $\mathcal{R}$, as we noted, will be second order in the derivatives of the metric, so they can provide a suitable action. The action must be a scalar, so we take

$$S_{\mathrm{grav}} = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\,\mathcal{R}. \tag{17.48}$$

To obtain the equations of motion we need to vary the complete action, including the parts involving matter fields, with respect to $g_{\mu\nu}$. We first consider the variation of the terms involving matter fields. The variation of the matter action with respect to $g_{\mu\nu}$ turns out to be nothing other than the stress–energy tensor, $T^{\mu\nu}$. Once one knows this fact, this gives what is often the easiest way to find the stress–energy tensor for a system. To see that this identification is correct, we first show that $T_{\mu\nu}$ is covariantly conserved, i.e.

$$D_\nu T^{\nu\mu} = T^{\mu\nu}{}_{;\nu} = 0. \tag{17.49}$$

By assumption the fields solve the equations of motion in the gravitational background, so the variation of the action, for any variation of the fields, is zero. Consider, then, a space–time translation:

$$x^{\mu\prime} = x^\mu + \epsilon^\mu. \tag{17.50}$$

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Starting with

$$g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x^{\mu'}} g_{\rho\sigma} \frac{\partial x^\sigma}{\partial x^{\nu'}}, \tag{17.51}$$

we have

$$g'_{\mu\nu}(x+\epsilon) = g_{\mu\nu}(x) - \partial_\mu \epsilon^\rho g_{\rho\nu} - \partial_\nu \epsilon^\sigma g_{\sigma\mu}. \tag{17.52}$$

Thus

$$\delta g_{\mu\nu}(x) = -g_{\mu\lambda}\partial_\nu \epsilon^\lambda - g_{\lambda\nu}\partial_\mu \epsilon^\lambda - \partial_\lambda g_{\mu\nu}\epsilon^\lambda. \tag{17.53}$$

Defining

$$\frac{\delta S_{\text{matt}}}{\delta g_{\mu\nu}} = T^{\mu\nu}, \tag{17.54}$$

under this particular variation of the metric we have

$$\delta S_{\text{matt}} = -\int d^4x\sqrt{-g}T^{\mu\nu}\left(g_{\mu\lambda}\partial_\nu\epsilon^\lambda + g_{\lambda\nu}\partial_\mu\epsilon^\lambda + \partial_\lambda g_{\mu\nu}\epsilon^\lambda\right). \tag{17.55}$$

Integrating the first two terms by parts and using the symmetry of the metric (and consequently the symmetry of $T^{\mu\nu}$), we obtain

$$\delta S_{\text{matt}} = \int d^4x\left[\partial_\mu(T^{\mu\lambda}\sqrt{-g}) - \frac{1}{2}\partial_\lambda g_{\mu\nu}T^{\mu\nu}\sqrt{-g}\right]\epsilon^\lambda. \tag{17.56}$$

The coefficient of ϵ^λ vanishes for fields which obey the equations of motion; this object is $T^{\mu\nu}{}_{;\mu}$. The reader can verify this last identification painstakingly or by noting that

$$\Gamma^\mu_{\mu\lambda} = \frac{1}{\sqrt{-g}}\partial_\lambda\sqrt{g}; \tag{17.57}$$

so, for a general vector, for example, we have

$$V^\mu{}_{;\mu} = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\,V^\mu\right) \tag{17.58}$$

and similarly for higher-rank tensors.

As a check, consider the stress tensor for a free massive scalar field. Once more, the action is

$$S = \int d^4x\sqrt{-g}\left(-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - \frac{1}{2}m^2\phi^2\right). \tag{17.59}$$

So, recalling our formula for the variation of the determinant,

$$T_{\mu\nu} = \frac{1}{2}\partial_\mu\phi\,\partial_\nu\phi - \frac{1}{4}g_{\mu\nu}(g^{\rho\sigma}\partial_\rho\phi\,\partial_\sigma\phi - m^2\phi^2). \tag{17.60}$$

To find the full gravitational equation – *Einstein's equation* – we need to vary also the gravitational term in the action. This is best done by explicitly constructing the variation of the curvature tensor under a small variation of the field. We leave the details for the exercises, and merely quote the final result:

$$\mathcal{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R}_s = \kappa^2 T_{\mu\nu}. \tag{17.61}$$

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We will consider many features of this equation, but it is instructive to see how we obtain Newton's expression for the gravitational field, in the limit where gravity is not too strong. We have already argued that in this case we can write

$$g_{00} = -(1 + 2\phi), \quad g^{ij} = \delta_{ij}. \tag{17.62}$$

As we have seen, the non-vanishing components of the connection are

$$\Gamma^i_{00} = \partial_i \phi, \quad \Gamma^0_{i0} = -\partial_i \phi. \tag{17.63}$$

Correspondingly, the non-zero components of the Riemann curvature tensor are

$$\mathcal{R}^i_{00j} = \partial_i \partial_j \phi = -\mathcal{R}^i_{0j0} = \mathcal{R}^0_{ij0}, \tag{17.64}$$

where the relations between the various components follow from the symmetries of the curvature tensor. From these we can construct the Ricci tensor and the Ricci scalar:

$$\mathcal{R}_{00} = \nabla^2 \phi, \quad \mathcal{R}_s = -\nabla^2 \phi. \tag{17.65}$$

So, we obtain

$$-\nabla^2 \phi = \kappa^2 T_{00}. \tag{17.66}$$

Note that from this we can identify Newton's gravitational constant in terms of κ,

$$G_N = \frac{\kappa^2}{8\pi}. \tag{17.67}$$

17.4 The Schwarzschild solution

Not long after Einstein wrote down his equations for general relativity, Schwarzschild constructed the solution of the equations for a static isotropic metric. Such a metric can be taken to have the form

$$ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\phi^2). \tag{17.68}$$

Actually, rotational invariance would allow other terms. In terms of vectors $d\vec{x}$ the most general metric has the form

$$-B(r)dt^2 + D(r)\vec{x} \cdot d\vec{x}dt + C(r)d\vec{x} \cdot d\vec{x} + D(r)(\vec{x} \cdot d\vec{x})^2. \tag{17.69}$$

By a sequence of coordinate transformations, however, one can bring the metric to the form (17.68).

We will solve Einstein's equations with $T_{\mu\nu} = 0$. Corresponding to ds^2, we have the non-vanishing metric components

$$g_{rr} = A(r), \quad g_{\phi\phi} = r^2 \sin^2\theta, \quad g_{tt} = -B(r), \quad g_{\theta\theta} = r^2. \tag{17.70}$$

Our goal is to determine A and B. The equations for them follow from Einstein's equations. We first need to evaluate the non-vanishing Christoffel symbols. This is done in the exercises. While straightforward, the calculation of the connection and the curvature

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is slightly tedious, and this is an opportunity to practise using the computer packages described in the exercises. The non-vanishing components of the affine connection are

$$\Gamma^r_{rr} = \frac{1}{2A(r)}A'(r), \quad \Gamma^r_{\theta\theta} = -\frac{r}{A(r)}, \quad \Gamma^r_{\phi\phi} = -\frac{r\sin^2\theta}{A(r)},$$
$$\Gamma^r_{\phi\phi} = \frac{r\sin^2\theta}{A(r)}, \quad \Gamma^r_{tt} = \frac{1}{2A(r)}B'(r), \tag{17.71}$$

where the primes denote derivatives with respect to r. Similarly,

$$\Gamma^\theta_{r\phi} = \Gamma^\theta_{\theta r} = \frac{1}{r}, \quad \Gamma^\theta_{\phi\phi} = -\sin\theta\,\cos\theta,$$
$$\Gamma^\phi_{\phi r} = \Gamma^\phi_{r\phi} = \frac{1}{r}, \quad \Gamma^\phi_{\phi\theta} = \Gamma^\phi_{\theta\phi} = \cos\theta,$$
$$\Gamma^t_{tr} = \Gamma^t_{rt} = \frac{B'}{2B}. \tag{17.72}$$

The non-vanishing components of the Ricci tensor are

$$\mathcal{R}_{rr} = \frac{B''}{2B} - \frac{1}{4}\frac{B''}{B}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{1}{r}\frac{A'}{A}, \tag{17.73}$$

$$\mathcal{R}_{\theta\theta} = -1 + \frac{r}{2A}\left(-\frac{A'}{A} + \frac{B'}{B}\right) + \frac{1}{A}, \tag{17.74}$$

$$\mathcal{R}_{\phi\phi} = \sin^2\theta\,\mathcal{R}_{\theta\theta}, \quad \mathcal{R}_{tt} = -\frac{B''}{2A} + \frac{1}{4}\frac{B'}{A}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{1}{r}\frac{B'}{A}. \tag{17.75}$$

For empty space, Einstein's equation reduces to

$$\mathcal{R}_{\mu\nu} = 0. \tag{17.76}$$

We will require that, asymptotically, the space–time is just flat Minkowski space, so we will solve these equations with the requirement that

$$A_{r\to\infty} = B_{r\to\infty} = 1. \tag{17.77}$$

Examining the components of the Ricci tensor we see that it is enough to set $\mathcal{R}_{rr} = \mathcal{R}_{\theta\theta} = \mathcal{R}_{tt} = 0$. We can simplify the equations with a little cleverness:

$$\frac{\mathcal{R}_{rr}}{A} + \frac{\mathcal{R}_{tt}}{B} = -\frac{1}{rA}\left(\frac{A'}{A} + \frac{B'}{B}\right). \tag{17.78}$$

From this it follows that $A = 1/B$. Then, from $\mathcal{R}_{\theta\theta} = 0$, we have

$$\frac{d}{dr}(rB) - 1 = 0. \tag{17.79}$$

Thus it follows that

$$rB = r + \text{ const.} \tag{17.80}$$

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Now $B = -g_{tt}$, so, at a distance far from the origin, where the space–time is nearly flat, $B = 1 + 2\phi$, where ϕ is the gravitational potential. Hence:

$$B(r) = 1 - \frac{2MG}{r}, \quad A(r) = \left(1 - \frac{2MG}{r}\right)^{-1}. \tag{17.81}$$

17.5 Features of the Schwarzschild metric

Finally, then, we have the Schwarzschild metric:

$$ds^2 = -\left(1 - \frac{2MG_N}{r}\right) dt^2 + \left(1 - \frac{2MG_N}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2. \tag{17.82}$$

Far from the origin, this clearly describes an object of mass M. While so far we have discussed the energy–momentum tensor for matter, we have not yet discussed the energy of gravitation. The situation is similar to the problem of defining charge in a gauge theory. There, the most straightforward definition involves using the asymptotic behavior of the fields to determine the total charge. In gravity, the energy is similar. There is no invariant local definition of the energy density. But in spaces that are asymptotically flat, one can give a global notion of the energy, known as the Arnowitt, Deser and Misner (ADM) energy. Only the $1/r$ behavior of the fields enters. We will not review this here but, not surprisingly, in the present case this energy P^0 is equal to M.

The curvature of space–time near a star yields observable effects. Einstein, when he first published his theory, proposed two tests of the theory: the bending of light by the Sun and the precession of Mercury's perihelion. In the latter case the theory accounted for a known anomaly in the motion of the planet; the prediction of the bending of light was soon confirmed.

A striking feature of this metric is that it becomes singular at a particular value of r, known as the Schwarzschild radius (the horizon), given by

$$r_{\mathrm{h}} = 2MG_N. \tag{17.83}$$

At this point the coefficient of dr^2 diverges, and that of dt^2 vanishes. Both change sign, in some sense reversing the roles of r and t. This singularity is a bit of a fake. No component of the curvature becomes singular. One can exhibit this by choosing coordinates in which the metric is completely non-singular (see the exercises at the end of the chapter).

For most realistic objects, such as planets and typical stars, the r_{h} value is well within the star, where surely it is important to use a more realistic model of $T_{\mu\nu}$. But there are systems in nature where the "material" lies well within the Schwarzschild radius. These systems are known as *black holes.* The known black holes arise from the collapse of very massive stars. It is conceivable that smaller black holes arise from more microscopic processes. These systems have striking properties. Classically, light cannot escape from the region within the horizon; the curvature singularity at the origin is real. Black holes are nearly featureless. Classically, an external observer can only determine the mass, charge and angular momentum of the black hole, however complex the system which may have preceded it.

Bekenstein pointed out that the horizon area has peculiar properties and behaves much like a thermal system. Most importantly, it obeys a relation analogous to the second law of thermodynamics,

$$dA > 0. \tag{17.84}$$

Identifying the area with an entropy suggests that one can associate a temperature T_λ with the black hole, known as the *Hawking temperature*. The black hole horizon is a sphere of area $4\pi r_{\mathrm{h}}^2$. So one might guess, on dimensional grounds, that

$$T_{\mathrm{h}} = \frac{1}{8\pi G_N M}. \tag{17.85}$$

The precise constant does not follow from this argument. The reader is invited to work through an heuristic path-integral derivation in the exercises.

Quantum mechanically, Hawking showed that this temperature has a microscopic significance. When one studies quantum fields in the gravitational background, one finds that particles do escape from the black hole. These particles have a thermal spectrum with characteristic temperature T_{h}. This phenomenon is known as *Hawking radiation*.

These features of black holes raise a number of conceptual questions. For the black hole at the center of the galaxy, for example, with mass millions of times greater than the Sun, the Hawking temperature is ludicrously small. Correspondingly, the Hawking radiation is totally irrelevant. But one can imagine microscopic black holes which would evaporate in much more modest periods of time. This raises a puzzle. The Hawking radiation is strictly thermal. So one could form a black hole, say, in the collapse of a small star. The initial star is a complex system, with many features. The black hole is nearly featureless. Classically, however, one might imagine that some memory of the initial state of the system is hidden behind the horizon; this information would simply be inaccessible to the external observer. But owing to the evaporation, the black hole and its horizon eventually disappear. One is left with just a thermal bath of radiation, with features seemingly determined by the temperature (and therefore the mass). Hawking suggested that this information paradox posed a fundamental challenge for quantum mechanics: it would seem that pure states could evolve into mixed states, through the formation of a black hole. For many years this question was the subject of serious debate. One might respond to Hawking's suggestion by saying that the information is hidden in subtle correlations in the radiation, as would be the case for the burning of, say, a lump of coal initially in a pure state. But more careful consideration indicates that things cannot be quite so simple. Only in relatively recent years has string theory provided at least a partial resolution of this paradox. We will touch on this subject briefly in the chapters on string theory. In the suggested reading the reader will be referred to more thorough treatments.

17.6 Coupling spinors to gravity

In any theory ultimately intended to describe nature, both spinors and general relativity will be present. Coupling spinors to gravity requires some concepts beyond those we have utilized up to now. The usual covariant derivative is constructed for tensors under changes of coordinates. In flat space, spinors are defined by their properties under rotations or more generally, Lorentz transformations. To do the same in general relativity it is necessary, first, to introduce a local Lorentz frame at each point. The basis vectors in this frame are denoted e^a_μ. Here μ is the Lorentz index; we can think of a as labeling the different vectors. The e_μs, in four dimensions are referred to as a tetrad or *vierbein*. In other dimensions they are called *vielbein*.

Requiring that the basis vectors be orthonormal in the Lorentzian sense gives

$$e^a_\mu(x) e_{a\nu}(x) = g_{\mu\nu}(x) \tag{17.86}$$

or, equivalently,

$$e^a_\mu(x) e^{b\mu}(x) = \eta^{ab}. \tag{17.87}$$

The choice of vielbein is not unique. We can multiply e by a Lorentz matrix, $\Lambda^a_b(x)$. Using e we can change indices from space–time (sometimes called "world") indices to tangent space indices:

$$V^a = e^a_\mu V^\mu. \tag{17.88}$$

Using this we can work out the form of the connection which maintains the gauge symmetry. We require that

$$D_\mu V^a = e^{a\nu} D_\mu V_\nu. \tag{17.89}$$

The derivative on the left-hand side is equal to

$$\partial_\mu V^a + (\omega_\mu)^a_b V^b. \tag{17.90}$$

With a bit of work, one can find explicitly the connection between the spin connection and the vielbein:

$$\omega^{ab}_\mu = \frac{1}{2} e^{\nu a}\left(\partial_\mu e^b_\nu - \partial_\nu e^b_\mu\right) - \frac{1}{2} e^{\nu b}\left(\partial_\mu e^a_\nu - \partial_\nu e^a_\mu\right) - \frac{1}{2} e^{\rho a} e^{\sigma b}(\partial_\rho e_{\sigma c} - \partial_\sigma e_{\rho c}) e^c_\mu. \tag{17.91}$$

Now we put this together. First, the curvature has a simple expression in terms of the spin connection, which formally is identical to that of a Yang–Mills connection:

$$(\mathcal{R}_{\mu\nu})^a_b = \partial_\mu(\omega_\nu)^a_b - \partial_\nu(\omega_\mu)^a_b + [\omega_\mu, \omega_\nu]^a_b. \tag{17.92}$$

This is connected simply to the Riemann tensor by the basic vectors e^a_σ:

$$(\mathcal{R}_{\mu\nu})^a_b = e^a_\sigma e^\tau_b (\mathcal{R}_{\mu\nu})^\sigma_\tau. \tag{17.93}$$

We can now construct, also, a generally covariant action for spinors:

$$\int d^D x \sqrt{g} i \bar{\psi} \Gamma^a e^\mu_a \left(\partial_\mu + \frac{1}{2}\omega^{bc}_\mu \Sigma_{bc}\right)\psi. \tag{17.94}$$

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Suggested reading

There are a number of excellent textbooks on general relativity, for example those of Weinberg (1972), Wald (1984), Carroll (2004) and Hartle (2003). Many aspects of general relativity that are important for string theory are discussed in the text of Green *et al.* (1987). A review of black holes in string theory was provided by Peet (2000).

Exercises

(1) Show that $g^{\mu\nu}\partial_\nu$ transforms like dx^μ. Verify explicitly that the covariant derivative of a vector transforms correctly.

(2) Derive Eq. (17.38) by considering the following action for a particle:

$$S = -\int ds = -\int \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}. \tag{17.95}$$

(3) Verify the formula (17.40) for the Riemann tensor $\mathcal{R}$, its symmetry properties and the Bianchi identities.

(4) Repeat the derivation of the conservation of the stress tensor, being careful with each step. Derive the stress tensor for the Maxwell field of electrodynamics, $F_{\mu\nu}$. Derive Einstein's equations from the action. You will need to show first that

$$\delta R_{\mu\nu} = \left(\delta\Gamma^\lambda_{\mu\lambda}\right)_{;\nu} - \left(\delta\Gamma^\lambda_{\mu\nu}\right)_{;\lambda}.$$

(5) Download a package of programs for doing calculations in general relativity in MAPLE, MATHEMATICA or any other program you prefer. A Google search will yield several choices. Practise by computing the components of the affine connection and the curvature for the Schwarzschild solution.

(6) Here is an heuristic derivation of the Hawking temperature. Near the horizon one can choose coordinates such that the metric is almost flat. Check this using

$$\eta = 2\sqrt{r_\mathrm{h}(r - r_\mathrm{h})}, \tag{17.96}$$

$$ds^2 = -4r_\mathrm{h}^2\eta^2 dt^2 + d\eta^2 + r_\mathrm{h}^2 d\Omega_2^2. \tag{17.97}$$

Now take the time to be Euclidean, $t \to i\phi/(2r_\mathrm{h})$. Check that now this is the metric of the plane times that of a two-sphere, provided that ϕ is an angle, $0 < \phi < 2\pi$ (otherwise, the space is said to have a conical singularity). Argue that field theory on this sphere is equivalent to field theory at finite temperature T_h (you may need to read Appendix C, particularly the discussion of finite-temperature field theory).

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