ON ω-APPROXIMATELY CONTINUOUS PERRON-STIELTJES AND DENJOY-STIELTJES INTEGRAL

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1. Introduction

The aim of the present paper is to introduce a definition of the Perron-Stieltjes integral employing the notion of approximate derivative with respect to a non-decreasing function ω and to study some of the properties of the integral. Various authors have studied the Perron integral and Perron-Stieltjes integral in different ways, most of which can be found in the references appended in the list of the bibliography. Among them Ridder [10] uses the concept of approximate ω-derivative but he assumes that the monotone function α associated with ω is continuous. Finally we consider a more general type of integral, the ω-approximately continuous Denjoy-Stieltjes integral, defined descriptively by the method of Saks [11].

Let ω be a finite non-decreasing function defined on the real line Ω. Following the definitions of Jeffery [6], p. 617, the "outer ω-measure" and the "ω-measure" of a linear set E are here denoted respectively by $\omega^*(E)$ and $|E|_\omega$.

Throughout the paper we use the following notations. S denotes the set of points of continuity of ω, $D = \Omega \setminus S$, S_0 denotes the union of the pairwise disjoint open intervals on each of which ω is constant. S_0^- and S_0^+ denote respectively the sets of the left and the right end points of the intervals of S_0; $S_1 = S_0^- \cup S_0^+$, $S_2 = S \cap S_1$ and $S_3 = S \setminus (S_0 \cup S_2)$. Then the sets D and S_1 are countable and for any two distinct points x_1, x_2, at least one of which belongs to S_3, we have $\omega(x_1) \neq \omega(x_2)$.

We require the following known results and definitions.

THEOREM 1.1. ([6], lemma 2, p. 618). *Let E be a bounded linear set. Let each point x of E be the left end-point of a sequence of closed intervals $[x, x + h_{x,i}]$ for which $h_{x,i} \to 0$ and $\mathscr{I}$ denote the family of all intervals thus associated with the set E. Then for every $\varepsilon > 0$, there exists a finite family of pairwise disjoint closed intervals $\Delta_1, \Delta_2, \cdots, \Delta_N$ in $\mathscr{I}$ such that*

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$$\sum_{i=1}^{N} \omega^*(E \cap \Delta_i) > \omega^*(E) - \varepsilon, \quad \sum_{i=1}^{N} |\Delta_i|_\omega < \omega^*(E) +$$

THEOREM 1.2. *This theorem is the analogue of Theorem 1.1. for closed intervals with 'x' as the right end-point.*

DEFINITION 1.1. ([4], §2, p. 347). Two sets A and B are said to be ω-*separated* if for every $\varepsilon > 0$ there exist open sets $G_1 \supset A$, $G_2 \supset B$ such that $|G_1 \cap G_2|_\omega < \varepsilon$.

THEOREM 1.3. ([4], §2; [7], Th. 2.20, p. 59.). *Let* $A = A_1 \cup A_2$. *If* A *is* ω-*measurable and* A_1, A_2 *are* ω-*separated, then* A_1 *and* A_2 *are also* ω-*measurable. Let the bounded sets* A_1 *and* A_2 *be not* ω-*separated. If* $E_1 \subset A_1, E_2 \subset A_2$ *with* $\omega^*(E_i)$ *sufficiently close to* $\omega^*(A_i)$, $i = 1,2$, *then* E_1 *and* E_2 *are not* ω-*separated.*

THEOREM 1.4. ([4], §2; [7], Th. 3.9, p. 77 (Lusin's Theorem)). *Let* $f(x)$ *be* ω-*measurable on the* ω-*measurable set* A *with* $|A|_\omega < +\infty$. *Then corresponding to every* $\varepsilon > 0$ *there exists a closed set* $E \subset A$ *with* $|A \setminus E|_N < \varepsilon$ *such that* $f(x)$ *is continuous on* E *(relative to* E).

2. ω-density and ω-approximate continuity

DEFINITION 2.1. (cf. [4], §3). Let A be any linear set and x be any given real number. Let $v = [x, x + h], v^0 = (x, x + h), h > 0$. We set

$$d(x,h) = \begin{cases} \omega^*(A \cap v)/|v|_\omega & \text{if } |v|_\omega \neq 0, \\ 1 & \text{if } |v|_\omega = 0 \text{ and } A \cap v^0 \neq \varnothing, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\lim_{h\to 0} \sup d(x,h)$ and $\lim_{h\to 0} \inf d(x,h)$ are called respectively the *upper and lower right* ω-*densities* of A at x. If these limits are equal, the common value is the *right* ω-*density* of A at x. *Left* ω-*densities* are defined similarly. If all the four ω-densities of A at x are equal to one another, the common value is the ω-*density* of A at x.

When the ω-density of A at x exists, x is called a point of dispersion (ω) or a point of density (ω) according as the ω-density is 0 or 1.

NOTE 2.1. If $x \in S_0$ and x is a limit point of A on both sides then clearly x is a point of density (ω) of A. If $x \in A \cap D$, then x is a point of density (ω) of A; if $x \in D \setminus A$ then x is a point of dispersion (ω) of A.

In [4] Chakrabarty has defined the ω-density of subsets of S_3 while our definition is applicable to all linear sets. We can verify that for any two linear sets A and B the theorems 3.1–3.6 of [4] are true. Whenever necessary we shall refer them for the corresponding results of ω-densities according to our definitions. From theorems 3.1, 3.2 and 3.6 [4] we obtain the theorem

THEOREM 2.1. *An arbitrary linear set has ω-density 0 or 1 a.e. (ω) on Ω.*

DEFINITION 2.2. (cf. [1], p. 267, [5], p. 793). Let $\phi(x)$ be a finite function defined on the set $A \subset \Omega$. For any real number ξ, the g.l.b. of the real numbers k for which the set

$$\{x; \; x \in A, \; x \geqq \xi \text{ and } \phi(x) > k\}$$

has ω-density 0 at ξ is denoted by $u_\omega^+(\phi, \xi)$ or simply by $u_\omega^+(\xi)$ and is called the *upper right approximate limit* (ω) of $\phi(x)$ at ξ relative to A. The l.u.b. of the real numbers k for which the set

$$\{x; \; x \in A, \; x \geqq \xi \text{ and } \phi(x) \leqq k\}$$

has ξ as a point of dispersion (ω) is denoted by $l_\omega^+(\phi, \xi)$ or simply by $l_\omega^+(\xi)$ and is called the *lower right approximate limit* (ω) of $\phi(x)$ at ξ relative to A. The *upper and lower left approximate limits* (ω), $u_\omega^-(\xi)$ and $l_\omega^-(\xi)$, are defined analogously The extended real numbers

$$u_\omega(\xi) = \max \cdot \{u_\omega^+(\xi), u_\omega^-(\xi)\}, \;\; l_\omega(\xi) = \min \cdot \{l_\omega^+(\xi), l_\omega^-(\xi)\}$$

are respectively the *upper and lower approximate limits* (ω) of $\phi(x)$ at ξ relative to A. If $u_\omega(\xi) = l_\omega(\xi)$, the common value, $\phi_\omega(\xi)$, is the *ω-approximate limit* of $\phi(x)$ at ξ. If further $\phi_\omega(\xi) = \phi(\xi)$, then $\phi(x)$ is said to be *ω-approximately continuous* at ξ relative to A.

For the remaining part of this section we suppose that A is a fixed subset of Ω and $\phi(x)$ is a finite function on A. To avoid repetition we shall drop the phrase "relative to A".

THEOREM 2.2. *$u_\omega(\xi)$ is the g.l.b. of real numbers k for which the set*

$$E^k[\phi] = \{x; \; x \in A \text{ and } \phi(x) > k\}$$

has ω-density 0 at ξ; $l_\omega(\xi)$ is the l.u.b. of real numbers k for which the set

$$E_k[\phi] = \{x; \; x \in A \text{ and } \phi(x) \leqq k\}$$

has ω-density 0 at ξ.

PROOF. Let M be the g.l.b. of real numbers k for which the set $E^k[\phi]$ has ω-density 0 at ξ. Then clearly

(1) $M \geqq u_\omega(\xi).$

If $u_\omega(\xi) = \infty$, then equality holds in (1). If $u_\omega(\xi) = -\infty$, then $u_\omega^+(\xi) = u_\omega^-(\xi) = -\infty$. This implies that for arbitrary large $k > 0$ the sets

$$\{x; \; x \in A, \; x \geqq \xi \text{ and } \phi(x) > -k\}, \; \{x; \; x \in A, \; x \leqq \xi \text{ and } \phi(x) > -k\}$$

have ω-density 0 at ξ, which gives that ξ is a point of dispersion (ω) for the set $E^{-k}[\phi]$ so that $M \leq -k$. Since $k > 0$ is arbitrary, $M = -\infty = u_\omega(\xi)$.

Now let $-\infty < u_\omega(\xi) < +\infty$. Then for any $\varepsilon > 0$, the set

$$\{x; x \in A \text{ and } \phi(x) > u_\omega(\xi) + \varepsilon\}$$

has ω-density 0 at ξ. This gives $M \leq u_\omega(\xi) + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, $M \leq u_\omega(\xi)$. So from (1) we have $M = u_\omega(\xi)$. This completes the proof of the first part. The proof of the other part is similar.

THEOREM 2.3. $u_\omega^-(x) = u_\omega^+(x)$ and $l_\omega^-(x) = l_\omega^+(x)$ at almost all points (ω) of Ω.

The proof is analogous to that of the corresponding result of Chow Shu-Er ([5], p. 795, §2, lemma 1.).

NOTE 2.2. $\phi(x)$ is ω-approximately continuous at each point of the set $A \cap D$.

For, if $\xi \in A \cap D$, then each of the sets

$$\{x; x \in A \text{ and } \phi(x) \leqq \phi(\xi) - \varepsilon\}, \{x; x \in A \text{ and } \phi(x) > \phi(\xi) + \varepsilon\}$$

has ξ as a point of dispersion (ω). This implies that

$$l_\omega(\xi) \geqq \phi(\xi) - \varepsilon \text{ and } u_\omega(\xi) \leqq \phi(\xi) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have

$$u_\omega(\xi) \leqq \phi(\xi) \leqq l_\omega(\xi).$$

Again, ξ is a point of density (ω) of each of the sets

$$\{x; x \in A \text{ and } \phi(x) > \phi(\xi) - \varepsilon\}, \{x; x \in A \text{ and } \phi(x) < \phi(\xi) + \varepsilon\},$$

which gives

$$l_\omega(\xi) \leqq \phi(\xi) \leqq u_\omega(\xi);$$

whence

$$l_\omega(\xi) = \phi(\xi) = u_\omega(\xi).$$

LEMMA 2.1. If each point of a set B is a point of density (ω) of the set A, then at any point α the four ω-densities of A are greater than or equal to the corresponding four ω-densities of B.

PROOF. Let the upper right ω-density of B at α be $k > 0$. If $\alpha \in D$, then by Note 2.1 α must belong to B and therefore to A. If $\alpha \in S_0 \cup (S \cap S_0^-)$, then α must be a limiting point of B, and therefore of A, on the right. In these cases A and B both have right ω-density unity at α.

Next suppose that $\alpha \in S \setminus (S_0 \cup S_0^-)$. Choose $\varepsilon > 0$ arbitrarily. Then there exists a strictly decreasing sequence $\{\alpha_n\}$ converging to α such that

(2) $\omega^*(B \cap [\alpha, \alpha_n]) / | [\alpha, \alpha_n] |_\omega > k/(1 + \varepsilon)$, for all n.

Let $P_i = B \cap (\alpha, \alpha_i]$ and $\varepsilon_i = | [\alpha, \alpha_i] |_\omega \cdot k \cdot \varepsilon^2/(1 + \varepsilon)$, $i = 1, 2, \cdots$. Take any positive integer n. For every $\xi \in P_n$ there exists a strictly decreasing sequence of closed intervals $[\xi_i', \xi] \subset (\alpha, \xi]$ with $\xi_i' \to \xi -$ such that

(3) $\omega^*(A \cap [\xi_i', \xi]) / | [\xi_i', \xi] |_\omega > 1 - \varepsilon, \qquad i = 1, 2, \cdots$.

So by theorem 1.2 there exists a finite number of pairwise disjoint such intervals $[\eta_1, \xi_1], [\eta_2, \xi_2], \cdots, [\eta_N, \xi_N]$ for which

(4) $\sum_{i=1}^N \omega^*(P_n \cap [\eta_i, \xi_i]) > \omega^*(P_n) - \varepsilon_n$.

From (2), (3) and (4) we deduce

$$\sum_{i=1}^N \omega^*(A \cap [\eta_i, \xi_i]) > (1 - \varepsilon) \cdot \sum_{i=1}^N | [\eta_i, \xi_i] |_\omega$$

(5) $\geqq (1 - \varepsilon) \cdot \sum_{i=1}^N \omega^*(P_n \cap [\eta_i, \xi_i]) > (1 - \varepsilon) \cdot [\omega^*(P_n) - \varepsilon_n]$

$$> (1 - \varepsilon) \cdot (\varepsilon_n - \varepsilon_n \cdot \varepsilon^2)/\varepsilon^2 = (1 - \varepsilon)^2 \cdot k \cdot | [\alpha, \alpha_n] |_\omega.$$

Since the intervals $\{[\eta_i, \xi_i]\}$ are pairwise disjoint and $\alpha < \eta_i < \xi_i < \alpha_n$, we get from (5),

(6) $\omega^*(A \cap [\alpha, \alpha_n]) / | [\alpha, \alpha_n] |_\omega > (1 - \varepsilon)^2 \cdot k, \qquad n = 1, 2, \cdots$.

Relation (6) implies that the upper right ω-density of A at α is $\geqq (1 - \varepsilon)^2 \cdot k$. Since $\varepsilon > 0$ is arbitrary, it follows that the upper right ω-density of A at α is $\geqq k$.

Similar proofs hold in the other cases.

THEOREM 2.4. *The functions $u_\omega(x)$ and $l_\omega(x)$ are ω-measurable on the real line Ω.*

PROOF. The functions $u_\omega(x)$ and $l_\omega(x)$ are defined on Ω with values in the extended real number system.

Given any real number a, let us write

$$E_a = \{x; u_\omega(x) < a\} \quad \text{and} \quad E^a = \{x; u_\omega(x) \geqq a\}.$$

We define the sets $A_1, A_2, \cdots$ by

$$A_n = \{x; u_\omega(x) < a - \frac{1}{n}\}.$$

Then $E_a = \bigcup_{n=1}^\infty A_n$. Take a fixed n and write $E = \{x; x \in A$ and $\phi(x) > a - 1/n\}$. Then by Theorem 2.2 it follows that no point of E^a is a point of dispersion (ω) of E. Hence by Theorem 2.1, E has ω-density unity at almost all points (ω) of E^a, while E has ω-density 0 at each point of A_n. By Lemma 2.1 it follows easily that E^a has ω-density 0 at almost all points (ω) of A_n. Therefore E^a has ω-density 0 at almost all points (ω) of E_a. So by Theorem 3.3. [4] and Theorem 1.3 it follows that the sets E_a and E^a are ω-measurable. Hence $u_\omega(x)$ is ω-measurable on Ω.

An analogous proof holds for $l_\omega(x)$.

COROLLARY 2.4.1. *The functions* $u_\omega^-(x)$, $u_\omega^+(x)$, $l_\omega^-(x)$, $l_\omega(x)$ *are all ω-measurable on Ω.*

PROOF. By Theorem 2.3 $u_\omega(x) = u_\omega^+(x) = u_\omega^-(x)$ a.e. (ω) and $l_\omega(x) = l_\omega^+(x)$ $= l_\omega^-(x)$ a.e. (ω). These observations in conjunction with Theorem 2.4 establish the corollary.

COROLLARY 2.4.2. *Let A be ω-measurable. Then $\phi(x)$ is ω-measurable on A if, and only if, $\phi(x)$ is ω-approximately continuous a.e. (ω) on A.*

PROOF. Let $\phi(x)$ be ω-approximately continuous a.e. (ω) on A. Then $\phi(x) = u_\omega(x) = l_\omega(x)$ a.e. (ω) on A. Since A is ω-measurable, by Theorem 2.4 it follows that $\phi(x)$ is ω-measurable on A.

Conversely, let $\phi(x)$ be ω-measurable on A. Then using Theorem 1.4 it can be proved in the usual way that $\phi(x)$ is ω-approximately continuous a.e. (ω) on A, remembering that $\phi(x)$ is ω-approximately continuous at each point of $A \cap D$ (Note 2.2.).

THEOREM 2.5. $u_\omega^+(x) \geqq l_\omega^+(x) [u_\omega^-(x) \geqq l_\omega^-(x)]$, *if the upper right [left] ω-density of A at x is > 0.*

These results follow easily from the definitions.

COROLLARY 2.5.1. *If ξ be not a point of dispersion (ω) of A, then $l_\omega(\xi) \leqq u_\omega(\xi)$.*

THEOREM 2.6. $l_\omega(x) \leqq \phi(x) \leqq u_\omega(x)$ *a.e. (ω) on A.*

PROOF. Let $E = \{x; x \in A$ and $\phi(x) > u_\omega(x)\}$. For any two positive integers n, k let us write

$$A_{n,k} = \{x; x \in E, \phi(x) - u_\omega(x) > \frac{1}{n} \text{ and } \phi(x) > - k\}.$$

Suppose, if possible, $\omega^*(A_{n,k}) > 0$ for some pair (n, k). Then $A_{n,k}$ contains a component P, every point of which is a point of density (ω) of P and $\omega^*(P) = \omega^*(A_{n,k})$. Now $\phi(x) > - k$ for all $x \in P$, so that $u_\omega(x) \geqq - k$. Hence $\phi(x) > (1/n) - k$ for all $x \in P$. This implies, similarly, $\phi(x) > (2/n) - k$ for all $x \in P$. Repeating the process we get $\phi(x) > (m/n) - k$ for all $x \in P$ and for every positive integer m. This contradicts that $\phi(x)$ is a finite function. Thus $\omega^*(A_{n,k}) = 0$ for all (n, k), and hence

$$\omega^*(E) = \omega^*(\bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} A_{n,k}) = 0.$$

Similarly the ω-measure of the set

$$\{x; x \in A \text{ and } \phi(x) < l_\omega(x)\}$$

is zero. This proves the theorem.

3. Approximate ω-derivatives of functions in the class $\mathscr{F}$

We denote by $\mathscr{F}$ the class of all finite functions $f(x)$ defined on Ω such that if $x_0 \in D$, then $f(x)$ tends to finite limits as x tends to $x_0 +$ and $x_0 -$ over the set S; these limits are denoted by $f(x_0 +)$ and $f(x_0 -)$ respectively. If $f \in \mathscr{F}$, we define the function $\hat{f}$ as follows:

$$\hat{f}(x) = f(x) \qquad\qquad \text{if } x \in S,$$
$$= \tfrac{1}{2}[f(x +) + f(x -)] \text{ if } x \in D.$$

In this section we consider only the functions of the class $\mathscr{F}$.

DEFINITION 3.1. For any given real number ξ we define the function $\psi(f; \xi, x)$ by

$$\psi(f; \xi, x) = \begin{cases} [\hat{f}(x) - \hat{f}(\xi)]/[\bar{\omega}(x) - \bar{\omega}(\xi)] & \text{if } \bar{\omega}(x) \neq \bar{\omega}(\xi), \\ [f(\xi +) - f(\xi -)]/[\omega(\xi +) - \omega(\xi -)] & \text{if } x = \xi \in D, \\ 0 & \text{otherwise.} \end{cases}$$

The *upper and lower approximate limits* (ω) of $\psi(f; \xi, x)$ at ξ are defined to be the upper and lower approximate ω-derivatives of f at ξ, and are denoted by $\overline{AD}f_\omega(\xi)$ and $\underline{AD}f_\omega(\xi)$ respectively. The *right (left) upper and lower approximate* ω-derivatives are defined in a corresponding way. They are denoted by $AD^+f_\omega(\xi)$, $AD_+f_\omega(\xi)$, $AD^-f_\omega(\xi)$ and $AD_-f_\omega(\xi)$ respectively. If $\overline{AD}f_\omega(\xi) = \underline{AD}f_\omega(\xi)$, the common value is denoted by $(ap)f'_\omega(\xi)$ and it is called the *approximate* ω-*derivative* of $f(x)$ at ξ. If $AD^+f_\omega(\xi) = AD_+f_\omega(\xi)$, the common value $(ap)f'_{+\omega}(\xi)$ is the *right approximate* ω-*derivative*. We define $(ap)f'_{-\omega}(\xi)$ analogously.

The ordinary upper and lower right [left] ω-derivatives of $f(x)$ at ξ are respectively the upper and lower limits of ψ $(f; \xi, x)$ as $x \to \xi + [x \to \xi -]$. These are denoted by $D^+f_\omega(\xi)$ and $D_+f_\omega(\xi)[D^-f_\omega(\xi)$ and $D_-f_\omega(\xi)]$. If $D^+f_\omega(\xi) = D_+f_\omega(\xi)[D^-f_\omega(\xi) = D_-f_\omega(\xi)]$ the common value is denoted by $f'_{+\omega}(\xi)[f'_{-\omega}(\xi)]$. If all the four ω-derivatives at ξ are equal to one another, the common value is denoted by $f'_\omega(\xi)$ and it is called the ω-*derivative* of $f(x)$ at ξ.

It is clear that $f'_\omega(x) = 0$ for every $x \in S_0$ and $f'_\omega(x) = [f(x +) - f(x -)]/[\omega(x +) - \omega(x -)]$ for every $x \in D$.

THEOREM 3.1.

(i) $AD_-f_\omega(x) \leqq AD^-f_\omega(x)$

(ii) $AD_+f_\omega(x) \leqq AD^+f_\omega(x)$

(iii) $\underline{AD}f_\omega(x) \leqq \overline{AD}f_\omega(x)$.

THEOREM 3.2. *If $f'_\omega(\xi)$ exists, then $(ap)f'_\omega(\xi)$ also exists and $(ap)f'_\omega(\xi) = f_\omega(\xi)$.*

THEOREM 3.3.

(i) $AD_\pm(-f)_\omega(x) \;\; = \; -AD^\pm f_\omega(x)$

(ii) $AD^\pm(cf)_\omega(x) \;\; = \; c \cdot AD^\pm f_\omega(x)$ if $c > 0$

(iii) $AD^\pm(f+g)_\omega(x) \;\leqq\; AD^\pm f_\omega(x) + AD^\pm g_\omega(x)$

(iv) $AD_\pm(f+g)_\omega(x) \;\geqq\; AD_\pm f_\omega(x) + AD_\pm g_\omega(x),$

equality holding in (iii) and (iv) if for at least one of the functions $f(x)$ and $g(x)$ the upper and lower approximate ω-derivatives on the appropriate side are equal, it being assumed that the expressions on the right in (iii) and (iv) are meaningful.

The proofs are straightforward and simple.

THEOREM 3.4. *If $f(x)$ be ω-measurable on $[a,b]$, then the approximate ω-derivatives of $f(x)$ are also so on $[a,b]$.*

PROOF. We prove the theorem for $AD^+ f_\omega(x)$. The proofs in other cases are analogous. Since $\left| S_0 \cup S_2 \right|_\omega = 0$ and every subset of D is ω-measurable, $AD^+ f_\omega(x)$ is ω-measurable on $(S_0 \cup S_2 \cup D) \cap [a,b]$. It is, therefore, sufficient to prove that $AD^+ f_\omega(x)$ is ω-measurable on $S'_3 = [a,b] \cap S_3$.

Suppose, if possible, that $AD^+ f_\omega(x)$ is not ω-measurable on S'_3. Then from theorem 1.3 it follows that there is a real number r such that the sets

$$E_r = \{x; \, x \in S'_3 \text{ and } AD^+ f_\omega(x) < r\}, \; E^r = \{x; \, x \in S'_3 \text{ and } AD^+ f_\omega(x) \geqq r\}$$

are not ω-separated. So by the same theorem there exists $\varepsilon > 0$ such that any set $E_1 \subset E_r$ with $\omega^*(E_1) > \omega^*(E_r) - 2\varepsilon$ is not ω-separated from any set $E_2 \subset E^r$ with $\omega^*(E_2) > \omega^*(E^r) - 2\varepsilon$. Let

$$A_n = \{x; \, x \in E_r \text{ and } AD^+ f_\omega(x) < r - \frac{1}{n}\}, \qquad n = 1, 2, \cdots.$$

Then $A_n \subset A_{n+1}$ and $E_r = \bigcup_{n=1}^\infty A_n$. So $\omega^*(A_n) \to \omega^*(E_r)$ as $n \to \infty$. Therefore we can choose a positive integer N such that $\omega^*(A_N) > \omega^*(E_r) - \varepsilon$. For every n we denote by B_n the set of points $x \in E^r$ such that for each $x \in B_n$ the set

$$\{x'; \, x' > x \text{ and } [\bar{f}(x') - \bar{f}(x)]/[\bar\omega(x') - \bar\omega(x)] > r - \frac{1}{2N}\}$$

has upper right ω-density $> (1/n)$ at x. Then $B_n \subset B_{n+1}$ and $E^r = \bigcup_{n=1}^\infty B_n$. So there exists a positive integer m with $\omega^*(B_m) > \omega^*(E^r) - \varepsilon$. For each n, denote by C_n the set of points $x \in A_N$ for which

$$[\tilde{f}(x') - \tilde{f}(x)]/[\overline{\omega}(x') - \overline{\omega}(x)] < r - \frac{1}{N},$$

except possibly for a set $\{x'\}$ with

(7) $$\frac{\omega^*(\{x'\} \cap [x, x + h])}{|[x, x + h]|_\omega} < \frac{1}{4m} \text{ whenever } 0 < h \leqq \frac{1}{n}.$$

Clearly $C_n \subset C_{n+1}$ and $A_N = \bigcup_{n=1}^\infty C_n$. So for some k,

$$\omega^*(C_k) > \omega^*(A_N) - \varepsilon > \omega^*(E_r) - 2\varepsilon.$$

By the assumption made about the number ε, it now follows that the sets B_m and C_k are not ω-separated. Hence by theorem 3.6 [4] the set of points of B_m at each of which the ω-density of C_k is 1 is of outer ω-measure > 0. Also $f(x)$ is ω-approximately continuous a.e. (ω) on $[a, b]$. So we can find a point $\xi \in B_m$ which is a point of density (ω) of C_k and at which $f(x)$ is ω-approximately continuous. Then from the definition of B_m we see that the set

$$B = \left\{x; x > \xi \text{ and } [\tilde{f}(x) - \tilde{f}(\xi)]/[\overline{\omega}(x) - \overline{\omega}(\xi)] > r - \frac{1}{2N}\right\}$$

has upper right ω-density $> (1/m)$ at ξ. So we can find h_0 with $0 < h_0 < (1/k)$ such that

(8) $$\omega^*(B \cap [\xi, \xi + h_0]) > \frac{1}{m} \cdot |[\xi, \xi + h_0]|_\omega.$$

Since $\xi \in S'_3$ we can find δ with $0 < \delta < h_0$, $\xi + \delta \in S$ such that

(9) $$|[\xi, x]|_\omega < \frac{1}{4m} \cdot |[\xi, \xi + h_0]|_\omega$$

for any x in $[\xi, \xi + \delta]$. Take an arbitrary but fixed point x of $C_k \cap (\xi, \xi + \delta)$ and set

$$C = \left\{x'; [\tilde{f}(x') - \tilde{f}(x)]/[\overline{\omega}(x') - \overline{\omega}(x)] \geqq r - \frac{1}{N}\right\}.$$

Then by (7) we have

(10) $$\frac{\omega^*(C \cap [x, x + h])}{|[x, x + h]|_\omega} < \frac{1}{4m} \text{ when } 0 < h \leqq \frac{1}{k}.$$

Noting that $\xi + h_0 - x < (1/k)$ we get, using (9) and (10),

(11) $$\omega^*(C \cap [\xi, \xi + h_0]) = \omega^*(C \cap [\xi, x]) + \omega^*(C \cap [x, \xi + h_0])$$

$$< |[\xi, x]|_\omega + \frac{1}{4m} \cdot |[x, \xi + h_0]|_\omega$$

$$< \frac{1}{4m} \cdot \left| [\xi, \xi + h_0] \right|_\omega + \frac{1}{4m} \cdot \left| [\xi, \xi + h_0] \right|_\omega = \frac{1}{2m} \cdot \left| [\xi, \xi + h_0] \right|_\omega.$$

Write $P = B \setminus C$. Then from (8) and (11) we deduce

(12) $$\frac{1}{2m} \cdot \left| [\xi, \xi + h_0] \right|_\omega < \omega^*(P \cap [\xi, \xi + h_0]).$$

Also, using (9), we get

(13) $$\omega^*(P \cap [\xi, \xi + \delta]) \leqq \left| [\xi, \xi + \delta] \right|_\omega < \frac{1}{4m} \cdot \left| [\xi, \xi + h_0] \right|_\omega.$$

From (12) and (13) we conclude that there are points $\xi' \in P$ with $\xi + \delta < \xi' < \xi + h_0$. For such a ξ' we have simultaneously

(14) $$[\tilde{f}(\xi') - \tilde{f}(\xi)] / [\overline{\omega}(\xi') - \overline{\omega}(\xi)] > r - \frac{1}{2N}$$

and

(15) $$[\tilde{f}(\xi') - \tilde{f}(x)] / [\overline{\omega}(\xi') - \overline{\omega}(x)] < r - \frac{1}{N}.$$

From (14) and (15) we get

$$\tilde{f}(x - \tilde{f}(\xi) > r \cdot [\overline{\omega}(x) - \overline{\omega}(\xi)] + \frac{1}{2N} \cdot [\overline{\omega}(\xi') + \overline{\omega}(\xi) - 2\overline{\omega}(x)].$$

So we have, as $x, \xi \in S$,

(16) $$f(x) - f(\xi) > r \cdot [\omega(x) - \omega(\xi)] + \frac{1}{2N} \cdot [\omega(\xi + \delta) + \omega(\xi) - 2\omega(x)].$$

The right hand side of (16) tends to $(1/2N) \cdot [\omega(\xi + \delta) - \omega(\xi)]$ as $x \to \xi +$. So there is a δ_0 $(0 < \delta_0 < \delta)$ such that

(17) $$f(x) - f(\xi) > \frac{1}{4N} \cdot [\omega(\xi + \delta) - \omega(\xi)] > 0$$

for all $x \in C_k \cap (\xi, \xi + \delta_0)$. Since C_k has ω-density 1 at ξ, the relation (17) contradicts that $f(x)$ is ω-approximately continuous at ξ. This proves the theorem.

THEOREM 3.5. *If $f(x)$ be ω-measurable on $[a, b]$, then the ω-derivatives of $f(x)$ are also so on $[a, b]$.*

PROOF. We prove the theorem for $D^+ f_\omega(x)$. The proofs in other cases are analogous.

As remarked in Theorem 3.4 it is sufficient to show that $D^+ f_\omega(x)$ is ω-measurable on S_3'. Suppose, if possible, that $D^+ f_\omega(x)$ is not ω-measurable on S_3'. Then there is a real number r such that the sets

$$E_r = \{x; x \in S_3' \text{ and } D^+ f_\omega(x) < r\}, \ E^r = \{x; x \in S_3' \text{ and } D^+ f_\omega(x) \geqq r\}$$

are not ω-separated.

Let $\{c_i\}$ be a strictly increasing sequence of real numbers tending to r. Let E_{ik} denote the set of points $\xi \in E_r$ for which

$$(18) \qquad \frac{\bar{f}(\xi + h) - \bar{f}(\xi)}{\bar{\omega}(\xi + h) - \bar{\omega}(\xi)} < c_i \text{ whenever } 0 < h < \frac{1}{k}.$$

Then for sufficiently large i and k (henceforth fixed) the sets E_{ik} and E^r are not ω-separated. This in conjunction with a.e. (ω) ω-approximate continuity of $f(x)$ and the Theorem 3.6 [4] gives that there is a subset $E \subset E^r$ with $\omega^*(E) > 0$, at each point of which $f(x)$ is ω-approximately continuous and E_{ik} has ω-density 1. Let $\alpha \in E$ and c be a real number with $c_i < c < r$. Since $D^+f_\omega(\alpha) > c$, there exists h' with $0 < h' < (1/k)$ such that

$$(19) \qquad \bar{f}[(\alpha + h') - \bar{f}(\alpha)]/[\bar{\omega}(\alpha + h') - \bar{\omega}(\alpha)] > c.$$

Since $\alpha \in S$ is a point of density (ω) of E_{ik}, α is a limit point of E_{ik} on both sides. We take any point $\xi \in (\alpha, \alpha + h') \cap E_{ik}$ and write $h = \alpha + h' - \xi$. Then $0 < h < (1/k)$. The relations (18) and (19) give

$$\bar{f}(\alpha + h') - \bar{f}(\alpha) > c \cdot [\bar{\omega}(\alpha + h') - \bar{\omega}(\alpha)]$$

and

$$\bar{f}(\xi + h) - \bar{f}(\xi) < c_i \cdot [\bar{\omega}(\xi + h) - \bar{\omega}(\xi)].$$

So we get, as $\xi + h = \alpha + h'$ and $\alpha, \xi \in S$,

$$f(\xi) - f(\alpha) > [\bar{\omega}(\alpha + h') - \omega(\alpha)]$$
$$\cdot \{c - c_i \cdot [\bar{\omega}(\alpha + h') - \omega(\xi)]/[\bar{\omega}(\alpha + h') - \omega(\alpha)]\},$$

the right hand side of which tends to $[\bar{\omega}(\alpha + h') - \omega(\alpha)] \cdot (c - c_i) > 0$, as $\xi \to \alpha +$ over E_{ik}. This contradicts the fact that $f(x)$ is ω-approximately continuous at α. This proves the theorem.

THEOREM 3.6. If $f(x)$ is monotone on $[a, b]$, then $f'_\omega(x)$ exists finitely a.e. (ω) on $[a, b]$.

This can be proved in the usual way.

THEOREM 3.7. Let $f(x)$ be non-decreasing on each interval of $(S_0 \cup S_2) \cap [a, b]$. If $AD_+f_\omega(x) \geqq 0$ on $[a, b), AD_-f_\omega(x) \geqq 0$ on $(a, b]$, then $f(x)$ is non-decreasing on $[a, b]$.

PROOF. Let $\alpha, \beta (> \alpha)$ be any two points in $[a, b]$. Choose $\varepsilon > 0$ arbitrarily. Let

$$E = \{x; x \geqq \alpha \text{ and } \psi(f; \alpha, x) \geqq -\varepsilon\}.$$

From the hypothesis it is clear that $x \in E$ implies

$$(20) \qquad \tilde{f}(x) - \tilde{f}(\alpha) \geqq -\varepsilon \cdot [\overline{\omega}(x) - \overline{\omega}(\alpha)].$$

Since $AD_+ f_\omega(\alpha) \geqq 0$, the set E has right ω-density 1 at α. So there is a point $\xi_1 > \alpha$ of E such that

$$\omega^*(E \cap [\alpha, \xi_1]) \geqq \tfrac{1}{2} \cdot |[\alpha, \xi_1]|_\omega.$$

Now let

$$E_1 = \{x; x \geqq \xi_1 \text{ and } \psi(f; \xi_1, x) \geqq -\varepsilon\}.$$

Then $x \in E_1$ implies

$$(21) \qquad \tilde{f}(x) - \tilde{f}(\xi_1) \geqq -\varepsilon \cdot [\overline{\omega}(x) - \overline{\omega}(\xi_1)].$$

Taking $x = \xi_1$ in (20) and then adding with (21) we get $\tilde{f}(x) - \tilde{f}(\alpha) \geqq -\varepsilon$ $\cdot [\overline{\omega}(x) - \overline{\omega}(\alpha)]$, which shews that $E_1 \subset E$. Since $\underline{AD} f_\omega(\xi_1) \geqq 0$, the set E has right ω-density 1 at ξ_1. So there is a point $\xi_2 > \xi_1$ of E such that

$$\omega^*(E \cap [\xi_1, \xi_2]) \geqq \tfrac{1}{2} |[\xi_1, \xi_2]|_\omega.$$

Proceeding thus we obtain a strictly increasing sequence $\{\xi_n\}$ from E such that $\xi_n < \beta$ and

$$(22) \qquad \omega^*(E \cap [\xi_{n-1}, \xi_n]) \geqq \tfrac{1}{2} |[\xi_{n-1}, \xi_n]|_\omega.$$

By (20) we have

$$(23) \qquad \tilde{f}(\xi_n) - \tilde{f}(\alpha) \geqq -\varepsilon \cdot [\overline{\omega}(\xi_n) - \overline{\omega}(\alpha)].$$

Let $\xi = \lim_{n \to \infty} \xi_n$. If $\xi \in D$, making $n \to \infty$ in (23) we get

$$f(\xi -) - \tilde{f}(\alpha) \geqq -\varepsilon \cdot [\omega(\xi -) - \overline{\omega}(\alpha)].$$

Also as $f'_\omega(\xi) \geqq 0$, we have $f(\xi +) \geqq f(\xi -)$. We deduce that $\xi \in E$. Next suppose that $\xi \in S$. Then using (22) we easily deduce that

$$\omega^*(E \cap [\xi_n, \xi]) \geqq \tfrac{1}{2} \cdot |[\xi_n, \xi]|_\omega, \qquad n = 1, 2, \cdots.$$

This implies that the upper left ω-density of E at ξ is $\geqq \tfrac{1}{2}$. If $\xi \notin E$ we must have

$$\tilde{f}(\xi) - \tilde{f}(\alpha) < -\varepsilon \cdot [\overline{\omega}(\xi) - \overline{\omega}(\alpha)].$$

Hence for all $x \in (\alpha, \xi) \cap E$ we have

$$(24) \qquad \tilde{f}(\xi) - \tilde{f}(x) < -\varepsilon \cdot [\overline{\omega}(\xi) - \overline{\omega}(x)].$$

Since E has upper left ω-density $\geqq \tfrac{1}{2}$ at ξ, the relation (24) contradicts that $AD_- f_\omega(\xi) \geqq 0$. Hence $\xi \in E$.

If $\xi < \beta$, we cover the interval $[\alpha, \beta]$ by a Lebesgue chain and thus obtain

$$(25) \qquad \tilde{f}(\beta) - \tilde{f}(\alpha) \geqq -\varepsilon \cdot [\overline{\omega}(\beta) - \overline{\omega}(\alpha)].$$

Since $\varepsilon > 0$ is arbitrary the relation (25) gives $\bar{f}(\beta) \geqq \bar{f}(\alpha)$, which proves the theorem.

THEOREM 3.8. *Let* $E \subset [a, b] \setminus S_0$ *with* $|E|_\omega = 0$ *and* k *be any given positive number. Then there exists a non-decreasing function* $\sigma(x)$ *on* Ω *such that* $D_+\sigma_\omega(x) = +\infty$ *for every* $x \in E \setminus S_0^-$ *and* $D_-\sigma_\omega(x) = +\infty$ *for every* $x \in E \setminus S_0^+$ *and* $\bar{\sigma}(b) = k$, $\bar{\sigma}(a) = 0$.

PROOF. Clearly $E \subset S$. Let $\{G_n\}$ be a sequence of open sets containing E with $G_{n+1} \subset G_n$ and $|G_n|_\omega < (1/4^n)$. We define the functions $\mu_n(x)$ on Ω by

$$\mu_n(x) = 2^n \cdot |G_n \cap (-\infty, x]|_\omega,$$

and set

$$\rho(x) = \Sigma_{n=1}^\infty \mu_n(x).$$

Then $\rho(x)$ is finite and non-decreasing on Ω. Let $x_0 \in E \setminus S_0^-$. For every n there is an open interval $I_n \subset G_n$ with $x_0 \in I_n$. Let $N(h)$ denote the number of open sets G_n which contain the interval $[x_0, x_0 + h]$. Then it is obvious that $N(h) \to \infty$ as $h \to 0 +$. So for an arbitrary $h > 0$ we have

$$\rho(x_0 + h) - \rho(x_0) = \Sigma_{n=1}^\infty 2^n \cdot |G_n \cap [x_0, x_0 + h]|_\omega$$

$$\geqq \Sigma_{n=1}^{N(h)} 2^n \cdot |G_n \cap [x_0, x_0 + h]|_\omega$$

$$\geqq 2^{N(h)} \cdot |[x_0, x_0 + h]|_\omega.$$

Hence it follows that for all $x > x_0$,

$$\bar{\rho}(x) - \bar{\rho}(x_0) \geqq [\bar{\omega}(x) - \bar{\omega}(x_0)] \cdot 2^{N(x-x_0)},$$

which implies

$$D_+\rho_\omega(x_0) = +\infty.$$

Similarly, if $x \in E \setminus S_0^+$, we can show that $D_-\rho_\omega(x) = +\infty$. Then the function

$$\sigma(x) = k \cdot [\rho(x) - \bar{\rho}(a)]/[\bar{\rho}(b) - \bar{\rho}(a)]$$

fulfills all the conditions of the theorem.

4. The ω-approximately continuous Perron-Stieltjes Integral

DEFINITION 4.1. A function $M(x)$ in $\mathcal{F}$ is called a *major function* of an arbitrary function $f(x)$ on $[a, b]$ if

(i) $\bar{M}(a) = 0$,

(ii) $\underline{ADM}_\omega(x) > -\infty$ for all $x \in [a, b]$,

(iii) $AD_-M_\omega(x) \geqq f(x)$ for every $x \in (a, b] \setminus [S_0 \cup (S \cap S_0^+)]$

and $AD_+ M_\omega(x) \geqq f(x)$ for every $x \in [a, b) \setminus [S_0 \cup (S \cap S_0^-)]$,

(iv) $M(x)$ is non-decreasing on each interval of $[a, b] \cap (S_0 \cup S_2)$.

A function $m(x)$ in $\mathscr{F}$ is a minor function of $f(x)$ on $[a, b]$ if

(i') $\bar{m}(a) = 0$,

(ii') $\overline{ADm_\omega}(x) < +\infty$ for all $x \in [a, b]$,

(iii') $AD^- m_\omega(x) \leqq f(x)$ for all $x \in (a, b] \setminus [S_0 \cup (S \cap S_0^+)]$

and $AD^+ m_\omega(x) \leqq f(x)$ for all $x \in [a, b) \setminus [S_0 \cup (S \cap S_0^-)]$,

(iv') $m(x)$ is non-increasing on each interval of $[a, b] \cap (S_0 \cup S_2)$.

NOTE. The existence of at least one major function and at least one minor function of $f(x)$ on $[a, b]$ guarantees that $f(x)$ is finite on $D \cap [a, b]$.

THEOREM 4.1. *If $M(x)$ be a major function and $m(x)$ be a minor function of $f(x)$ on $[a, b]$, then $\bar{M} - \bar{m}$ is non-decreasing on $[a, b]$, and in particular $\bar{M}(b) \geqq \bar{m}(b)$.*

PROOF. Let $g(x) = M(x) - m(x)$. Then $g \in \mathscr{F}$. By definitions of major and minor functions, $\underline{ADM_\omega}(x) - \overline{ADm_\omega}(x)$ is well defined on $[a, b]$. So, using Theorem 3.3 we deduce that $AD_+ g_\omega(x) \geqq 0$ on $[a, b)$ and $AD_- g_\omega(x) \geqq 0$ on $(a, b]$. Further $g(x)$ is non-decreasing on each interval of $[a, b] \cap (S_0 \cup S_2)$. Therefore by Theorem 3.7 $\bar{g}(x) = \bar{M}(x) - \bar{m}(x)$ is non-decreasing on $[a, b]$. In particular $\bar{M}(b) - \bar{m}(b) \geqq \bar{M}(a) - \bar{m}(a) = 0$. This completes the proof of the theorem.

DEFINITION 4.2. Given any function $f(x)$ on $[a, b]$, we define the function $\omega_f(x)$ on Ω as follows:

$$\omega_f(x) = f(x) \cdot [\omega(x +) - \omega(x -)] \text{ if } x \in D \cap [a, b],$$
$$= 0 \qquad\qquad\qquad\qquad \text{elsewhere.}$$

DEFINITION 4.3. Let $f(x)$ be any function defined on $[a, b]$ possessing at least one major function $M(x)$ and at least one minor function $m(x)$ on $[a, b]$. Then $f(x)$ is said to be ω-approximately Perron-Stieltjes integrable, in short (APS)-integrable, on $[a, b]$ if $\sup_m \bar{m}(b) = \inf_M \bar{M}(b)$. The common value is then denoted by $(APS) - \int_a^b f d\omega$ and is called the (APS)-integral of $f(x)$ on $[a, b]$.

Since there is no chance of confusion with other types of integrals we delete the prefix (APS) from our notation of integrals that follow. It is obvious from the definition that

THEOREM 4.2. *A necessary and sufficient condition that $\int_a^b f d\omega$ may exist*

is that for every $\varepsilon > 0$ *there is a major function* $M(x)$ *and a minor function* $m(x)$ *of* $f(x)$ *on* $[a, b]$ *such that* $\bar{M}(b) - \bar{m}(b) < \varepsilon$.

THEOREM 4.3. *If* $\int_a^b f \, d\omega$ *exists and if* $a < c < b$, *then each of the integrals* $\int_a^c f \, d\omega$ *and* $\int_c^b f \, d\omega$ *exists and further*

$$\int_a^c f d\omega + \int_c^b f d\omega = \int_a^b f d\omega.$$

PROOF. Let $\varepsilon > 0$ be given. Then there exists a major function $M(x)$ and a minor function $m(x)$ of $f(x)$ on $[a, b]$ such that

(26) $\bar{M}(b) - \bar{m}(b) < \varepsilon.$

Now $M(x)$ and $m(x)$ qualify as major and minor functions of $f(x)$ on $[a, c]$, while $M(x) - \bar{M}(c)$ and $m(x) - \bar{m}(c)$ serve for those on $[c, b]$. Now by Theorem 4.1 and relation (26) we get

$$\bar{M}(c) - \bar{m}(c) \leqq \bar{M}(b) - \bar{m}(b) < \varepsilon,$$

and

$$[\bar{M}(b) - \bar{M}(c)] - [\bar{m}(b) - \bar{m}(c)] < \varepsilon,$$

which imply the existence of $\int_a^c f \, d\omega$ and $\int_c^b f \, d\omega$ respectively. Again, by definition we have

$$\bar{m}(c) \leqq \int_a^c f d\omega \leqq \bar{M}(c)$$

and

$$\bar{m}(b) - \bar{m}(c) \leqq \int_c^b f d\omega \leqq \bar{M}(b) - \bar{M}(c),$$

which give on adding

(27) $\bar{m}(b) \leqq \int_a^c f d\omega + \int_c^b f d\omega \leqq \bar{M}(b).$

Also we have

(28) $\bar{m}(b) \leqq \int_a^b f d\omega \leqq \bar{M}(b).$

Relations (26), (27) and (28) give that

$$\left| \int_a^c f d\omega + \int_c^b f d\omega - \int_a^b f d\omega \right| < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$\int_a^c f d\omega + \int_c^b f d\omega = \int_a^b f d\omega.$$

This completes the proof of the theorem.

COROLLARY 4.3.1. *If $\int_a^b f d\omega$ exists, then for $a \leqq \alpha < \beta \leqq b$, $\int_\alpha^\beta f d\omega$ exists.*

THEOREM 4.4. *If $\int_a^c f d\omega$ and $\int_c^b f d\omega$ both exist, where $a < c < b$, then $\int_a^b f d\omega$ exists.*

PROOF. Let $\varepsilon > 0$ be given. Then there is a major function $M_1(x)$ and a minor function $m_1(x)$ of $f(x)$ on $[a, c]$ such that

(29) $$\bar{M}_1(c) - \bar{m}_1(c) < \varepsilon;$$

there are major and minor functions $M_2(x)$ and $m_2(x)$ on $[c, b]$ such that

(30) $$\bar{M}_2(b) - \bar{m}_2(b) < \varepsilon.$$

Let us define

$$M(x) = M_1(x) \quad \text{for } x \leqq c,$$
$$= M_2(x) + \bar{M}_1(c) \quad \text{for } x > c;$$

and

$$m(x) = m_1(x) \quad \text{for } x \leqq c,$$
$$= m_2(x) + \bar{m}_1(c) \quad \text{for } x > c.$$

Then clearly M, $m \in \mathscr{F}$. We can easily verify that $M(x)$ and $m(x)$ qualify as major and minor functions respectively of $f(x)$ on $[a, b]$. Now we have, using (29) and (30),

$$\bar{M}(b) - \bar{m}(b) = [\bar{M}_2(b) + \bar{M}_1(c)] - [\bar{m}_2(b) + \bar{m}_1(c)]$$
$$= [\bar{M}_2(b) - \bar{m}_2(b)] + [\bar{M}_1(c) - \bar{m}_1(c)] < 2\varepsilon.$$

Hence we conclude that $\int_a^b f d\omega$ exists.

DEFINITION 4.4. If $\int_a^b f d\omega$ exists, we define the function $F(x)$ by

$$F(x) = \begin{cases} \displaystyle\int_a^x f d\omega & \text{if } a < x \leqq b, \\ F(b) + \tfrac{1}{2}\omega_f(b) & \text{if } \quad x > b, \\ -\tfrac{1}{2}\omega_f(a) & \text{if } \quad x < a, \\ 0 & \text{if } \quad x = a. \end{cases}$$

Then $F(x)$ is called the *indefinite (APS)-integral* of $f(x)$ on $[a, b]$.

THEOREM 4.5. *If $\int_a^b f d\omega$ exists, then the corresponding indefinite integral $F \in \mathscr{F}$. Specifically, $F(x+) = F(x) + \frac{1}{2}\omega_f(x)$ and $F(x-) = F(x) - \frac{1}{2}\omega_f(x)$ if $x \in D$.*

PROOF. It is sufficient to restrict our consideration to $[a, b]$ only. We have $F(a-) = -\frac{1}{2}\omega_f(a) = F(a) - \frac{1}{2}\omega_f(a)$ and $F(b+) = F(b) + \frac{1}{2}\omega_f(b)$. Now choose $\varepsilon > 0$ arbitrarily. Then there is a major function $M(x)$ and a minor function $m(x)$ of $f(x)$ on $[a, b]$ such that

(31) $$\bar{M}(b) - \frac{1}{2}\varepsilon < F(b) < \bar{m}(b) + \frac{1}{2}\varepsilon.$$

Suppose that $a \in D$. Then by definition 4.1, (iii, iii′) we have $m(a+) - m(a-) \leqq \omega_f(a) \leqq M(a+) - M(a-)$. But $\bar{M}(a) = \bar{m}(a) = 0$. Hence we deduce,

(32) $$m(a+) \leqq \frac{1}{2}\omega_f(a) \leqq M(a+).$$

If $\delta > 0$ be sufficiently small and $x \in S \cap (a, a + \delta)$, then we have

(33) $$m(a+) - \varepsilon < m(x) \leqq M(x) < M(a+) + \varepsilon$$

and

(34) $$m(x) \leqq F(x) \leqq M(x).$$

From (31), (32), (33) and (34) we deduce that

$$\left| F(x) - \frac{1}{2}\omega_f(a) \right| < 3\varepsilon \text{ for all } x \in S \cap (a, a + \delta).$$

Hence

$$F(a+) = \frac{1}{2}\omega_f(a) = F(a) + \frac{1}{2}\omega_f(a).$$

Next, suppose that $b \in D$. Then there is a sufficiently small $h > 0$ such that for $x \in S \cap (b - h, b)$,

(35) $$m(b-) - \frac{1}{2}\varepsilon < m(x) \leqq F(x) \leqq M(x) < M(b-) + \frac{1}{2}\varepsilon.$$

Also we have

(36) $$m(b+) - m(b-) \leqq \omega_f(b) \leqq M(b+) - M(b-).$$

From (31), (35) and (36) we deduce that

$$-\frac{1}{2}\varepsilon < F(x) - [F(b) - \frac{1}{2}\omega_f(b)] < \frac{1}{2}\varepsilon \text{ for all } x \in S \cap (b - h, b).$$

Hence $F(b-) = F(b) - \frac{1}{2}\omega_f(b)$.

Now let $c \in D \cap (a, b)$. Let $F_1(x)$ and $F_2(x)$ be respectively the indefinite integrals of $f(x)$ on $[a, c]$ and $[c, b]$. Then $F(x) = F_1(x)$ for $x \leqq c$ and $F(x) = F_2(x) + F(c)$ for $x \geqq c$. Hence by what has been proved above, we get

$$F(c-) = F_1(c-) = F_1(c) - \frac{1}{2}\omega_f(c) = F(c) - \frac{1}{2}\omega_f(c),$$

and

$$F(c+) = F_2(c+) + F(c) = \tfrac{1}{2}\omega_f(c) + F(c).$$

This completes the proof of the Theorem.

COROLLARY 4.5.1. $F'_\omega(c) = f(c)$ for every $c \in D \cap [a, b]$.

THEOREM 4.6. *Let $F(x)$ be the indefinite integral of $f(x)$ on $[a, b]$. Then for any major function $M(x)$ and any minor function $m(x)$ of $f(x)$ on $[a, b]$, the functions $\bar{M}(x) - F(x)$ and $F(x) - \bar{m}(x)$ are non-decreasing on $[a, b]$.*

PROOF. Let $a \leqq x_1 < x_2 \leqq b$. Then $M(x) - \bar{M}(x_1)$ is a major function and $m(x) - \bar{m}(x_1)$ is a minor function of $f(x)$ on $[x_1, x_2]$. Now the theorem follows from the inequality

$$\bar{m}(x_2) - \bar{m}(x_1) \leqq F(x_2) - F(x_1) = \int_{x_1}^{x_2} f d\omega \leqq \bar{M}(x_2) - \bar{M}(x_1).$$

THEOREM 4.7. *The (APS)-integral of an arbitrary function $f(x)$ is zero on each closed interval on $S_0 \cup S_2$.*

PROOF. Let $[a, b]$ be an interval on $S_0 \cup S_2$. Then $M(x) \equiv 0$ qualifies both as a major function and a minor function of $f(x)$ on $[a, b]$. So $\int_a^b f d\omega = 0$.

COROLLARY 4.7.1. *If $\int_a^a f d\omega$ exists, then the indefinite integral is constant on each interval of $(S_0 \cup S_2) \cap [a, b]$.*

This follows from Theorems 4.3 and 4.7.

THEOREM 4.8. *Let $f \in \mathscr{F}$. If $(ap)f'_\omega(x)$ exists finitely on $[a, b]$ and $f(x)$ is constant on the intervals of $(S_0 \cup S_2) \cap [a, b]$, then $\int_a^b (ap)f'_\omega(x)d\omega$ exists and has the value $\bar{f}(b) - \bar{f}(a)$.*

PROOF. The function $f(x) - \bar{f}(a)$ qualifies both as a major function and a minor function of $(ap)f'_\omega(x)$ on $[a, b]$. Hence the theorem follows.

THEOREM 4.9. *Let $\int_a^b f d\omega$ exist and $F(x)$ be the corresponding indefinite integral. Then $(ap)F'_\omega(x)$ exists finitely a.e. (ω) and $(ap)F'_\omega(x) = f(x)$ a.e. (ω) on $[a, b]$.*

PROOF. By Theorem 4.5 F belongs to the class $\mathscr{F}$ and by Corollary 4.5.1 $F'_\omega(\xi) = f(\xi)$ at every point $\xi \in D \cap [a, b]$. Now let

$$A = \{x; x \in (a, b) \cap S_3 \text{ and } \underline{ADF}_\omega(x) < f(x)\}.$$

Choose $\varepsilon > 0$ arbitrarily. For each positive integer n, let

$$A_n = \{x; x \in A \text{ and } \underline{ADF}_\omega(x) < f(x) - \varepsilon/2^n\}.$$

There is a major function $M(x)$ of $f(x)$ on $[a, b]$ such that

(37) $$\bar{M}(b) - F(b) < \varepsilon^2/2^{2n+1}.$$

Write $R(x) = \bar{M}(x) - F(x)$. Then by Theorem 4.6 $R(x)$ is non-decreasing on $[a, b]$ and hence by Theorem 3.6 $R'_\omega(x)$ exists finitely a.e. (ω) on $[a, b]$. If $x \in A_n$, $\overline{ADM_\omega(x) - ADF_\omega(x)} = R'_\omega(x) > (\varepsilon/2^n)$ provided $R'_\omega(x)$ exists. Let E denote the set of points of $(a, b) \cap S_3$ where $R'_\omega(x)$ exists finitely. Then

$$E \cap A_n \subset B_n = \{x; x \in E \text{ and } R'_\omega(x) > \varepsilon/2^n\}.$$

To each point $\xi \in B_n$ we can associate a sequence of closed intervals $\{[\xi, \xi_i]\}$ with $\xi_i \in (a, b) \cap S$ and $\xi_i \to \xi +$ such that

(38) $\qquad [R(\xi_i) - R(\xi)]/[\omega(\xi_i) - \omega(\xi)] > \dfrac{\varepsilon}{2^n}, \qquad i = 1, 2, \cdots.$

By Theorem 1.1 we can choose a finite number of disjoint such intervals $\Delta_1, \Delta_2, \cdots, \Delta_r$ for which

(39) $\qquad \sum_{i=1}^{r} \omega^*(B_n \cap \Delta_i) > \omega^*(B_n) - \varepsilon/2^{n+1}.$

From (37), (38) and (39) we deduce that $\omega^*(B_n) < \varepsilon/2^n$, and hence $\omega^*(A_n) < \varepsilon/2^n$. But $A = \bigcup_{n=1}^{\infty} A_n$. Therefore $\omega^*(A) < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, it follows that $\omega^*(A) = 0$.

If $B = \{x; x \in (a, b) \cap S_3 \text{ and } \overline{ADF_\omega}(x) > f(x)\}$, we can show, by introducing a minor function, that $\omega^*(B) = 0$. Hence if $x \in (a, b) \cap S_3 \setminus (A \cup B)$ then

$$\overline{ADF_\omega}(x) \leqq f(x) \leqq \underline{ADF_\omega}(x),$$

which gives that $(ap)F'_\omega(x)$ exists and equals $f(x)$. Since $|S_0 \cup S_2 \cup A \cup B|_\omega = 0$, we conclude that $(ap)F'_\omega(x)$ exists and equals $f(x)$ a.e. (ω) on $[a, b]$.

Now let $R(x) = \bar{M}(x) - F(x)$ and $r(x) = F(x) - \bar{m}(x)$, where $M(x)$ and $m(x)$ are major and minor functions of $f(x)$ on $[a, b]$. Let C denote the set of points in $[a, b]$ at which $(ap)F'_\omega(x)$ exists and $R'_\omega(x)$, $r'_\omega(x)$ both exist finitely. Then $|C|_\omega = |[a, b]|_\omega$. If $x \in C$, we have

$$(ap)F'_\omega(x) = \underline{AD\bar{M}_\omega(x)} - R'_\omega(x) = \underline{ADM_\omega(x)} - R'_\omega(x) > -\infty$$

and

$$(ap)F'_\omega(x) = \overline{AD\bar{m}_\omega(x)} + r'_\omega(x) = \overline{ADm_\omega(x)} + r'_\omega(x) < +\infty.$$

So $(ap)F'_\omega(x)$ is finite on C. Hence the theorem follows.

THEOREM 4.10. *If $\int_a^b f \, d\omega$ exists then the corresponding indefinite integral $F(x)$ is ω-approximately continuous on $[a, b]$.*

PROOF. Let $\varepsilon > 0$ be given. Then there is a major function $M(x)$ and a minor function $m(x)$ of $f(x)$ on $[a, b]$ such that for all $x \in [a, b]$,

(40) $\qquad 0 \leqq \bar{M}(x) - F(x) < \varepsilon \text{ and } 0 \leqq F(x) - \bar{m}(x) < \varepsilon.$

By Note 2.2 $F(x)$ is ω-approximately continuous at every point of D. Now choose $c \in [a, b] \cap S$ arbitrarily. Since $ADM_\omega(c) > -\infty$ and $\overline{ADm}_\omega(c) < +\infty$, we can find $K > 0$ sufficiently large so that except for a set $\{x\}$ with ω-density 0 at c, we have

$$\bar{M}(x) - M(c) \leqq -K \cdot [\bar{\omega}(x) - \omega(c)], \qquad x < c$$

(41)
$$\bar{M}(x) - M(c) \geqq -K \cdot [\bar{\omega}(x) - \omega(c)], \qquad x > c$$

$$\bar{m}(x) - m(c) \geqq K \cdot [\bar{\omega}(x) - \omega(c)], \qquad x < c$$

$$\bar{m}(x) - m(c) \leqq K \cdot [\bar{\omega}(x) - \omega(c)], \qquad x > c.$$

From (40) and (41) we deduce that

$$\bigl|F(x) - F(c)\bigr| < \varepsilon + K \cdot \bigl|\bar{\omega}(x) - \omega(c)\bigr|,$$

except for a set $\{x\}$ with ω-density 0 at c. Since $\bar{\omega}(x) \to \omega(c)$ as $x \to c$, it follows that $F(x)$ is ω-approximately continuous at c. Hence the theorem follows.

COROLLARY 4.10.1. *Let* $f(x)$ *be* (APS)-*integrable on* $[a, b]$. *Then* $f(x)$ *is necessarily* ω-*measurable and finite a.e.* (ω) *on* $[a, b]$.

This follows from Theorems 4.9, 4.10, 3.4 and Corollary 2.4.2.

COROLLARY 4.10.2. *If* $f(x)$ *be* (APS)-*integrable on* $[a, b]$ *then the major and minor functions of* $f(x)$ *possess finite approximate* ω-*derivatives a.e.* (ω) *on* $[a, b]$.

THEOREM 4.11. *If* $\int_a^b f d\omega$ *exists and* $g(x) = f(x)$ *a.e.*(ω) *on* $[a, b]$, *then* $\int_a^b g d\omega$ *exists and* $\int_a^b g d\omega = \int_a^b f d\omega$.

PROOF. Let E denote the set of points of $[a, b] \setminus S_0$ for which $g(x) \neq fx)$. Then $|E|_\omega = 0$. So given $\varepsilon > 0$ there exists, by Theorem 3.8, a non-decreasing function $\sigma(x)$ with $\bar{\sigma}(b) = \varepsilon$ and $\bar{\sigma}(a) = 0$. There are major and minor functions of $f(x)$ on $[a, b]$ such that $\bar{M}(b) - \bar{m}(b) < \varepsilon$. Then $M(x) + \sigma(x)$ and $m(x) - \sigma(x)$ are major and minor functions of $g(x)$ on $[a, b]$ such that $[\bar{M}(b) + \bar{\sigma}(b)] - [\bar{m}(b) - \bar{\sigma}(b)] < 3\varepsilon$. Since $\varepsilon > 0$ is arbitrary, it follows that $\int_a^b g d\omega$ exists. Finally, the relation $\bar{m}(b) - \varepsilon = \bar{m}(b) - \bar{\sigma}(b) \leqq \int_a^b g d\omega \leqq \bar{M}(b) + \bar{\sigma}(b) = \bar{M}(b) + \varepsilon$ gives that $\int_a^b g d\omega = \int_a^b f d\omega$.

THEOREM 4.12. *Let* $f(x)$ *be Lebesgue-Stieltjes summable with respect to* $\omega(x)$ *on* $[a, b]$. *Then* (APS) $- \int_a^b f d\omega$ *exists and*

$$(APS) - \int_a^b f d\omega = (LS) \int_a^b f d\omega - \tfrac{1}{2}[\omega_f(a) + \omega_f(b)].$$

THEOREM 4.13. *If* $f(x)$ *be non-negative and* (APS)-*integrable on* $[a, b]$ *then it is* (LS)-*summable on* $[a, b]$.

Proofs of these two theorems are analogous to those of the corresponding theorems 1 and 2 in [9] (p. 168–169).

5. $AC - \omega$ and $ACG - \omega$ functions belonging to the class $\mathscr{F}$

DEFINITION 5.1. A function $F \in \mathscr{F}$ is said to be $AC - \omega$ below, [$AC - \omega$ above] on a set E if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every finite set of non-overlapping intervals $\{(x_i, x_i')\}$ with end-points on E and with $\sum |[x_i, x_i']|_\omega < \delta$, we have

$$\sum \{ F(x_i') - F(x_i) \} > -\varepsilon, \ [\sum \{ F(x_i') - F(x_i) \} < \varepsilon].$$

$F(x)$ is said to be $ACG - \omega$ below [$ACG - \omega$ above] on E if E is the union of a countable number of sets on each of which $F(x)$ is $AC - \omega$ below [$AC - \omega$ above]. $F(x)$ is $ACG - \omega$ on E if it icboth $ACG - \omega$ below and $ACG - \omega$ above on E.

THEOREM 5.1. (cf. [8], lemma 4.1, p. 261). *If $\overline{ADF_\omega}(x) > -\infty$ on E and $F(x)$ be non-decreasing on $E \cap I$ for every interval I on which $\omega(x)$ is constant, then $F(x)$ is $ACG - \omega$ below on E.*

THEOREM 5.2. (cf. [8], p. 263, lines 2, sqq.). *The indefinite integral of an (APS)-integrable function on $[a, b]$ is $ACG - \omega$ on $[a, b]$.*

THEOREM 5.3. (cf. [8], lemma 2.1, p. 254). *A function $F(x)$ which is $ACG - \omega$ on $[a, b]$ necessarily fulfills the condition $(N\omega)$, that is $|F[H]| = 0$ for every set $H \subset [a, b]$ with $|H|_\omega = 0$.*

The definitions adopted in this article are analoguos to those in Yôto Kubota's paper [8]. The proofs of the Theorems 5.1–5.3 are also analogous to those of the corresponding results in [8]. Kubota, with his notion of ACG, has defined the AD-integral ([8], §2) by the method of Saks [11]. We show that the same method leads us to the "ω-approximately continuous Denjoy-Stieltjes Integral" or "the (ADS)-integral".

THEOREM 5.4. *Let $F \in \mathscr{F}$ and be such that*

(i) $\bar{F}(x) = F(x)$ *and $F(x)$ is ω-approximately continuous on $[a, b]$,*

(ii) $F(x-) \leqq F(x+)$ *for all $x \in D \cap [a, b]$,*

(iii) $F(x)$ *is non-decreasing on the intervals of $(S_0 \cup S_2) \cap [a, b]$,*

(iv) $F[E]$ *contains no interval, where*

$$E = \{x; \ x \in [a, b] \setminus (S_0 \cup S_2) \text{ and } AD_-F_\omega(x) \leqq 0\}$$

Then $F(x)$ is non-decreasing on $[a, b]$.

PROOF. Suppose, if possible, that $a \leqq c_0 < d_0 \leqq b$, while $F(d_0) < F(c_0)$. As $F[E]$ contains no interval there is y_0 such that $F(d_0) < y_0 < F(c_0)$, $y_0 \notin F[E]$. Let $A = \{x; x \in [c_0, d_0]$ and $F(x) > y_0\}$. Then $c_0 \in A$. As $F(x)$ is non-decreasing on the intervals of $(S_0 \cup S_2) \cap [a, b]$ and $F(x)$ is ω-approximately continuous at every point of $[a, b]$, so A has right ω-density 1 at every point of A. We can therefore find $c_n \in A \cap S$ such that

$$\omega^*(A \cap [c_{n-1}, c_n]) \geqq \tfrac{1}{2} |[c_{n-1}, c_n]|_\omega,$$

$c_0 < c_1 < c_2 < \cdots < d_0$. Let $c_n \to \alpha$. If $\alpha \in D$, then as $F(c_n) > y_0$ for all n, so $F(\alpha -) \geqq y_0 \cdot SoF(\alpha) \geqq y_0$. If $F(\alpha) = y_0$, then clearly $F(\alpha -) = F(\alpha +) = y_0$. This means $F'_\omega(\alpha) = 0$, which contradicts that $y_0 \notin F[E]$. Thus $F(\alpha) > y_0$. If $\alpha \in S$, then we have

$$\omega^*(A \cap [c_n, \alpha]) \geqq \tfrac{1}{2} \cdot \Sigma_{r=n}^\infty |[c_r, c_{r+1}]|_\omega \geqq \tfrac{1}{2} \cdot |[c_n, \alpha]|_\omega$$

for all $n = 1, 2, 3, \cdots$. So the upper left ω-density of A at α is > 0. We assert again that $F(\alpha) > y_0$. Suppose that $F(\alpha) = y_0$. Then for all $x \in A \cap [c_0, \alpha]$ we have

$$[F(\alpha) - F(x)]/[\overline{\omega}(\alpha) - \overline{\omega}(x)] < 0,$$

so that $AD_- F_\omega(\alpha) \leqq 0$, which contradicts that $y_0 \notin F[E]$. If $F(\alpha) < y_0$, then as $F(x)$ is ω-approximately continuous at α, the set A should have ω-density 0 at α, which is not the case. Thus in all cases $\alpha \in A$. If $\alpha < d_0$, covering $[c_0, d_0]$ by a Lebesgue chain of intervals we get $F(d_0) > y_0$, leading to a contradiction. This final contradiction establishes the theorem.

This theorem enables us to enunciate the following.

DEFINITION 5.2. Let $f(x)$ be any function defined on $[a, b]$ and suppose that there exists a function $F \in \mathscr{F}$ such that
  (i) $F(x)$ is ω-approximately continuous on $[a, b]$, $F(x) = F(x)$ on $D \cap [a, b]$,
  (ii) $F(x)$ is $ACG - \omega$ on $[a, b]$ and $F(x)$ is constant on the intervals of $(S_0 \cup S_2) \cap [a, b]$,

and

  (iii) $(ap)F'_\omega(x)$ exists finitely a.e. (ω) on $[a, b]$ and $(ap)F'_\omega(x) = f(x)$ a.e. (ω) on $[a, b]$.

Then $f(x)$ is said to be *integrable in the ω-approximately continuous Denjoy-Stieltjes sense* or *(ADS)-integrable* and we write

$$(ADS) - \int_a^b f d\omega = F(b) - F(a).$$

The function $F(x)$ is said to be an *indefinite (ADS)-integral* of $f(x)$ on $[a, b]$.

Definition 5.2 requires a uniqueness theorem which is furnished below.

THEOREM 5.5. *If $F_1(x)$ and $F_2(x)$ are any two indefinite (ADS)-integrals of $f(x)$ on $[a, b]$, then $F_2(b) - F_2(a) = F_1(b) - F_1(a)$.*

PROOF. Let $\varepsilon > 0$ be given. Consider the function $F(x) = F_1(x) - F_2(x) + \varepsilon \cdot \overline{\omega}(x)$. Then $F(x) \equiv F(x)$ is ω-approximately continuous on $[a, b]$ and $(ap)F'_\omega(x) = \varepsilon > 0$ a.e. (ω) on $[a, b]$. So the set

$$E = \{x; x \in [a, b] \text{ and } AD\_F_\omega(x) \leqq 0\}$$

is of ω-measure zero. But $F(x)$ is $ACG - \omega$ on $[a, b]$. So by Theorem 5.3 $|F[E]| = 0$. Hence $F[E]$ contains no interval. At the points of $D \cap [a, b]$, we have $F'_\omega(x) = \varepsilon > 0 \cdot$ So $F(x +) > F(x -)$. Also $F(x)$ is non-decreasing on the intervals of $(S_0 \cup S_2) \cap [a, b]$. Hence by Theorem 5.4 we have

$$F_1(b) - F_1(a) \geqq F_2(b) - F_2(a) - \varepsilon \cdot [\overline{\omega}(b) - \overline{\omega}(a)].$$

Since $\varepsilon > 0$ is arbitrary, it follows that

(42) $$F_1(b) - F_1(a) \geqq F_2(b) - F_2(a).$$

Interchanging F_1 and F_2 we get,

(43) $$F_2(b) - F_2(a) \geqq F_1(b) - F_1(a).$$

Combining (42) and (43) we get the required result.

We note that an (APS)-integrable function is necessarily (ADS)-intergrable and the two integrals have equal value. It is easy to verify that the ADS-integral possesses all the usual fundamental properties of an integral.

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