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Abstract

The local Langlands conjectures imply that to every generic supercuspidal irreducible representation of G_2 over a p-adic field, one can associate a generic supercuspidal irreducible representation of either PGSp_6 or PGL_3. We prove this conjectural dichotomy, demonstrating a precise correspondence between certain representations of G_2 and other representations of PGSp_6 and PGL_3. This correspondence arises from theta correspondences in E_6 and E_7, analysis of Shalika functionals, and spin L-functions. Our main result reduces the conjectural Langlands parameterization of generic supercuspidal irreducible representations of G_2 to a single conjecture about the parameterization for PGSp_6.

Contents

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Introduction

Let k be a finite extension of $\mathbb{Q}_p$, p a prime number; we work here with the k-points of algebraic groups. In this paper, we prove a precise correspondence between the generic supercuspidal irreducible representations (abbreviated to 'irreps') of the exceptional group G_2 and certain generic supercuspidal irreps of the classical groups PGL_3 and PGSp_6. This correspondence is phrased as a *dichotomy*, in which to every generic supercuspidal irrep τ of G_2 we associate *either* a generic supercuspidal irrep σ of PGSp_6 whose spin L-function has a pole at $s=0$ *or* a contragredient pair (or self-contragredient singleton) of generic supercuspidal irreps $\{\rho, \tilde{\rho}\}$ of PGL_3. Symbolically, we write this dichotomy as a function Δ:

$$\Delta : \mathrm{Irr}_g^\circ(G_2) \rightarrow \mathrm{Irr}_{g,\mathrm{Spin}}^\circ(\mathrm{PGSp}_6) \sqcup \frac{\mathrm{Irr}_g^\circ(\mathrm{PGL}_3)}{\mathrm{Contra}}.$$

After constructing this function Δ, we prove that it is bijective when $p \neq 2$. When $p=2$, we can prove that Δ is injective, but there is a subtlety involving self-dual supercuspidal irreps of PGL_3 which prevents a proof of bijectivity for now.

This main result is suggested by Langlands' conjectural parameterization of the generic supercuspidal irreps of these groups G_2, PGL_3, and PGSp_6. For this reason, we demonstrate the precise dichotomy at the level of Langlands parameters in the first section. The results on Langlands parameters depend essentially on the structure theory of the complex simple groups $G_2(\mathbb{C})$, $\mathrm{SL}_3(\mathbb{C})$, and $\mathrm{Spin}_7(\mathbb{C})$: embeddings of $\mathrm{SL}_3(\mathbb{C})$ into $G_2(\mathbb{C})$, embeddings of $G_2(\mathbb{C})$ into $\mathrm{Spin}_7(\mathbb{C})$, and classification of parabolic and other subgroups.

The second section is devoted to the structure theory of certain algebraic groups over the p-adic field k, including constructions and embeddings of exceptional groups and their parabolic subgroups. At different times in this paper, we require different embeddings of groups. As Jacquet modules play a crucial role, we describe in detail two kinds of parabolic subgroups: minuscule parabolics (arising from Jordan algebras) and two-step parabolic subgroups arising from the structurable algebras of Allison [All78, All79].

The third section provides the definition of the dichotomy map Δ. Specifically, the dichotomy is realized via theta correspondences using dual pairs $G_2 \times \mathrm{PGL}_3 \subset E_6$ and $G_2 \times \mathrm{PGSp}_6 \subset E_7$, and the minimal representations (see [GS05]) of E_6 and E_7. Such theta correspondences have been studied in the literature; we mention the results of Ginzburg–Jiang [GJ01], Gan–Savin [GS03, GS04], Savin [Sav99], Magaard–Savin [MS97], Loke–Savin [LS07], and Gross–Savin [GS98]. We refine results of Ginzburg–Rallis–Soudry [GRS97], who first considered the 'tower of theta correspondences' for G_2. Using extensive analysis of Jacquet modules for the minimal representations of E_6 and E_7, we are able to demonstrate that this pair of theta correspondences determines a dichotomy function Δ, taking a generic supercuspidal irrep of G_2 either to a (unique, up to isomorphism) generic supercuspidal irrep of PGSp_6 or to a (unique, up to isomorphism and contragredient) generic supercuspidal irrep of PGL_3.

The fourth section is devoted to proving the injectivity of the dichotomy map Δ, through a study of Whittaker and Shalika functionals. When considering a generic supercuspidal irrep ρ of PGL_3, the fibre $\Delta^{-1}(\{\rho, \tilde{\rho}\})$ has cardinality at most equal to the dimension of a space of Whittaker functionals on ρ. The uniqueness of Whittaker functionals immediately yields injectivity of the dichotomy map in this case. However, when considering a generic supercuspidal irrep σ of PGSp_6, the fibre $\Delta^{-1}(\sigma)$ has cardinality equal to the dimension of a space of Shalika functionals on σ. Here, the 'Shalika subgroup' is nearly isomorphic to $\mathrm{GL}_2(k[\epsilon]/\epsilon^3)$, embedded appropriately in GSp_6. This subgroup is a cubic analogue of the Shalika subgroup $\mathrm{GL}_n(k[\epsilon]/\epsilon^2)$ studied by Jacquet–Rallis [JR96] and others (see [JNQ08] for a recent example). In this fourth section, we prove a result of some independent interest; the uniqueness of such Shalika functionals for arbitrary supercuspidal irreps of GSp_6. It almost immediately follows that the dichotomy map is injective.

The fifth section is devoted to characterizing the image of the dichotomy map Δ, finishing the proof of a bijection when $p \neq 2$. The dichotomy map surjects onto the set of generic non-self-contragredient (an automatic condition when $p \neq 2$) supercuspidal irreps of PGL_3. When $p = 2$, we cannot yet exclude the possibility that a generic self-contragredient supercuspidal irrep of PGL_3 occurs in the theta correspondence with a generic supercuspidal irrep τ of G_2, and also a generic supercuspidal irrep of PGSp_6 occurs in the theta correspondence with the same τ. In other words, we cannot yet prove that a 'second occurrence' in a tower of theta lifts is not supercuspidal in residue characteristic two. From the way that we define our dichotomy map Δ, we cannot therefore prove that the image of Δ includes all self-contragredient supercuspidal irreps of PGL_3, though all such irreps of PGL_3 occur in a theta correspondence with a generic supercuspidal irrep of G_2.

The fifth section focuses on the set of generic supercuspidal irreps of PGSp_6 in the image of Δ. Precisely those generic supercuspidal irreps of PGSp_6 with non-vanishing Shalika functional occur in this image. However, Langlands' conjectures predict another characterization of the image of dichotomy: a generic supercuspidal irrep σ of PGSp_6 should occur in the image of dichotomy if and only if its degree-eight spin L-function has a pole at $s = 0$. Thus, to characterize the image of dichotomy, we prove that σ has a non-vanishing Shalika functional if and only if $L(\sigma, \mathrm{Spin}, s)$ has a pole at $s = 0$. This is a local version of the main result of Ginzburg–Jiang [GJ01]. One direction, that a non-vanishing Shalika functional implies that the L-function has a pole, requires an analysis of the minimal representation of E_8(!), the construction of Shahidi [Sha90] of the spin L-function, and connections to reducibility points for representations of F_4 parabolically induced from GSp_6. The other direction, that if $L(\sigma, \mathrm{Spin}, s)$ has a pole at $s = 0$, then σ has a non-vanishing Shalika functional, requires the Bump–Ginzburg [BG92] integral representation of the spin L-function, results of Vo [Vo97] on this L-function, and global methods to demonstrate that the Bump–Ginzburg construction agrees (in its poles) with Shahidi's for the spin L-function.

The dichotomy proven in this paper comes close to proving Langlands' conjectural parameterization of generic supercuspidal irreps of G_2 by parameters (representations of the Weil group) with values in $G_2(\mathbb{C})$. Indeed, the dichotomy reduces this parameterization (when $p \neq 2$) to a conjecture related to Langlands' parameterization for PGSp_6. While Langlands parameters for generic irreps of PGSp_6 are now known (by functoriality for classical groups, due to Cogdell *et al.* [CKPS04] and the local Langlands correspondence for GL_7 by Henniart [Hen84b, Hen00], Kutzko–Moy [KM85], and for GL_8 as well by Harris–Taylor [HT01]), it remains to be proven that the currently understood parameterization for PGSp_6 is compatible with the degree-eight spin L-functions. Thus, the local Langlands' parameterization of generic supercuspidal irreps of G_2 is reduced to a single question about the classical group PGSp_6 when $p \neq 2$.

Of course, a complete parameterization of supercuspidal irreps of G_2 satisfying Langlands' conjectures would require also an analysis of the non-generic supercuspidal irreps, and the partition of all supercuspidal irreps into L-packets. For example, many non-generic representations arise from inner forms $PD^\times$ of PGL_3 (see [SG99]), but we do not address such phenomena in this paper.

0.1 Conventions

The letter k will always denote a finite extension of $\mathbb{Q}_p$, where p is a prime number. A k-algebra will always mean a *unital* (except for Lie algebras, of course), *finite-dimensional* k-algebra. An involution on a k-algebra will always mean an anti-automorphism of order two, which *fixes every element* of k. We do not assume k-algebras to be commutative or associative; in fact, non-associative algebras play a central role. For a k-vector space A, we write $\mathfrak{End}_k(A)$ for the Lie algebra of k-linear endomorphisms of A.

We fix a split Cayley algebra $\mathbb{O}$ over k, in what comes later. We also fix a smooth, non-trivial, additive character ψ_k of k. From ψ_k, we may define a smooth additive character $\psi_{\mathbb{O}}$ by

$$\psi_{\mathbb{O}}(\omega) = \psi_k(\mathrm{Tr}(\omega)) \quad \text{for all } \omega \in \mathbb{O}.$$

We use a boldface letter, such as $\mathbf{G}$, to denote an algebraic group over k. We use an ordinary letter, such as G, to denote the k-points of $\mathbf{G}$, viewed naturally as a topological group. All representations of such groups G will be assumed to be smooth representations on complex vector spaces. An *irrep* of G will mean a smooth irreducible representation of G on a complex vector space. If $G \to G'$ is a surjective group homomorphism, and π is a representation of G', we often also write π for the representation of G arising by pullback.

If $H \subset G$ is a closed subgroup, and π is a representation of H on a complex vector space V, then we write Ind_H^G for the representation of G obtained by smooth (unnormalized) induction:

$$\mathrm{Ind}_H^G \pi = \{f \in C^\infty(G, V) : f(hg) = \pi(h) f(g) \text{ for all } h \in H\}.$$

Here, $C^\infty(G, V)$ denotes the space of uniformly locally constant functions from G to V. Induction is adjoint to restriction, by the appropriate version of Frobenius reciprocity:

$$\mathrm{Hom}_G(\tau, \mathrm{Ind}_H^G \pi) \cong \mathrm{Hom}_H(\tau, \pi)$$

for every smooth representation τ of G and every smooth representation π of H.

When $H \backslash G$ is non-compact, it is often more useful to consider the smooth compact induction:

$$\text{c-}\mathrm{Ind}_H^G \pi = \{f \in \mathrm{Ind}_H^G \pi : \mathrm{Supp}(f) \subset H \cdot K \text{ for some compact subset } K \subset G\}.$$

Then $\text{c-}\mathrm{Ind}_H^G \pi$ is again a smooth representation of G, and is a subrepresentation of $\mathrm{Ind}_H^G \pi$.

If π is a representation of G, and ρ is an irrep of G, then we say that ρ is a *constituent* of π if ρ is isomorphic to a quotient of a subrepresentation of π. However, we almost exclusively work with *supercuspidal* constituents in this paper; the injectivity and projectivity of supercuspidal irreps, in the category of smooth representations, imply that when supercuspidal irreps occur as constituents, they also occur as subrepresentations and as quotients.

1. Dichotomy of parameters

1.1 The local Langlands conjectures

Recall that k is a finite extension of $\mathbb{Q}_p$, fix an algebraic closure $\bar{k}$ of k, and let $\Gamma = \mathrm{Gal}(\bar{k}/k)$. Let k^{unr} denote the maximal unramified extension of k in $\bar{k}$. There is a unique continuous isomorphism from $\mathrm{Gal}(k^{\mathrm{unr}}/k)$ to the profinite group $\hat{\mathbb{Z}}$ which sends the geometric Frobenius to 1. This isomorphism yields a surjective homomorphism from Γ to $\hat{\mathbb{Z}}$. The preimage of $\mathbb{Z}$ is the subgroup $W_k \subset \Gamma$, called the Weil group of k.

The Weil group contains $\mathrm{Gal}(\bar{k}/k^{\mathrm{unr}})$, and W_k is given the coarsest topology for which $\mathrm{Gal}(\bar{k}/k^{\mathrm{unr}})$ is an open subgroup endowed with the subspace topology from $\mathrm{Gal}(\bar{k}/k)$. Thus, there is a short exact sequence of topological groups and continuous homomorphisms:

$$1 \to \mathrm{Gal}(\bar{k}/k^{\mathrm{unr}}) \to W_k \to \mathbb{Z} \to 1.$$

Let $\mathbf{G}$ be a semisimple, split, *adjoint* algebraic group over k, and let $G = \mathbf{G}(k)$. Let $\mathrm{Irr}(G)$ denote the set of isomorphism classes of irreducible smooth representations of G on a complex vector space, hereafter called *irreps* of G. Let $\mathrm{Irr}^\circ(G)$ denote the subset consisting of isomorphism classes of supercuspidal irreps. Let $\mathrm{Irr}_g(G)$ be the subset consisting of isomorphism classes of generic irreps; the adjective 'generic' is well defined, since we assume that $\mathbf{G}$ is adjoint and split over k. Finally, define $\mathrm{Irr}_g^\circ(G) = \mathrm{Irr}_g(G) \cap \mathrm{Irr}^\circ(G)$ to be the set of isomorphism classes of generic supercuspidal irreps.

Let $\hat{G}$ denote the complex dual group of $\mathbf{G}$; thus, $\hat{G}$ is a semisimple, simply connected complex Lie group. A *parameter* for G is a continuous homomorphism η: $W_k \to \hat{G}$ such that $\eta(w)$ is semisimple for all $w \in W_k$. We do not require the extra structure provided by the Weil–Deligne group here. A parameter η is called *cuspidal* if $\mathrm{Im}(\eta)$ is not contained in any proper parabolic subgroup of $\hat{G}$. Let $\mathrm{Par}(G)$ denote the set of parameters and $\mathrm{Par}^\circ(G)$ the set of cuspidal parameters for G. Note that $\hat{G}$ acts on the sets $\mathrm{Par}(G)$ and $\mathrm{Par}^\circ(G)$ by conjugation, denoted Ad.

An expectation of the local Langlands conjectures (refined by Arthur [Art89] and Vogan [Vog93]) is that there is a 'natural' bijective parameterization

$$\Phi(G) : \mathrm{Irr}_g^\circ(G) \to \frac{\mathrm{Par}^\circ(G)}{\mathrm{Ad}(\hat{G})},$$

whereby the generic supercuspidal irreps of G are parameterized precisely by the $\hat{G}$-conjugacy classes of cuspidal parameters. Indeed, suppose that π is a generic supercuspidal irrep of G. It is expected that π has an *Arthur* parameter whose restriction is a *Langlands* parameter:

$$\alpha : W_k \times \mathrm{SL}_2^{(1)}(\mathbb{C}) \times \mathrm{SL}_2^{(2)}(\mathbb{C}) \to \hat{G},$$
$$\lambda : W_k \times \mathrm{SL}_2^{(1)}(\mathbb{C}) \to \hat{G},$$

where the isomorphic groups $\mathrm{SL}_2^{(1)}(\mathbb{C})$ and $\mathrm{SL}_2^{(2)}(\mathbb{C})$ are written with different superscripts for the reader to distinguish them. The Aubert involution [Aub95] sends π to its contragredient $\tilde{\pi}$. On the other hand, it was conjectured by Hiraga [Hir04], and proven for $G = \mathrm{SO}_{2n+1}$ by Ban and Zhang [BZ05], that the Arthur parameter $\tilde{\alpha}$ of $\tilde{\pi}$ is obtained from α by swapping the two SL_2-factors.

If $\alpha(\mathrm{SL}_2^{(1)}(\mathbb{C}))$ were not trivial, then Hiraga's conjecture would imply that $\tilde{\alpha}(\mathrm{SL}_2^{(2)}(\mathbb{C})) \neq 1$, so that $\tilde{\pi}$ belongs to a non-tempered Arthur packet. But, as the generic member of a non-tempered Arthur packet, $\tilde{\pi}$ should arise as a Langlands quotient, from a parabolic subgroup which can be described from $\tilde{\alpha}(\mathrm{SL}_2^{(2)}(\mathbb{C}))$. This contradicts the fact that $\tilde{\pi}$ is supercuspidal.

Hence, it is expected that a generic supercuspidal irrep π of G has an Arthur parameter which is trivial on *both* copies of SL_2; its Langlands parameter should be elliptic and trivial on $\mathrm{SL}_2^{(1)}$, yielding the expectation of the map $\Phi(G)$ described above. We expect $\Phi(G)$ to be bijective since every L-packet with elliptic parameter should contain exactly one generic irrep [Art89, § 6].

1.2 The dichotomy

When $\mathbf{G}_2$ is a simple split algebraic group of type G_2 over k, $\hat{G}_2 = G_2(\mathbb{C})$ is the simple complex Lie group of type G_2. In this case, the Langlands conjectures predict that the generic supercuspidal irreps of G_2 are parameterized by $\hat{G}_2$-conjugacy classes of cuspidal parameters. However, the latter can be related to classical groups as follows.

Let $\mathbb{O}$ denote an octonion algebra (also called a Cayley algebra) over $\mathbb{C}$. Let $\mathbb{O}_\circ$ denote the subset of trace zero octonions, and realize $G_2(\mathbb{C})$ as the group of $\mathbb{C}$-algebra automorphisms of $\mathbb{O}$. Thus, we find an embedding $G_2(\mathbb{C}) \hookrightarrow \mathrm{SO}_7(\mathbb{C}) = \mathrm{SO}(\mathbb{O}_\circ, N)$, where N denotes the quadratic norm form on $\mathbb{O}_\circ$. As $G_2(\mathbb{C})$ is simply connected, this embedding extends to an embedding $G_2(\mathbb{C}) \hookrightarrow \mathrm{Spin}_7(\mathbb{C})$. As $\mathrm{Spin}_7(\mathbb{C})$ is the complex dual group to PGSp_6, we find a natural map

$$\mathrm{Par}^\circ(G_2) \to \mathrm{Par}(\mathrm{PGSp}_6).$$

To determine when the image of a cuspidal parameter for G_2 is a *cuspidal* parameter for PGSp_6, we discuss the maximal parabolic subgroups of $G_2(\mathbb{C})$ and $\mathrm{Spin}_7(\mathbb{C})$. A *nil-space* in $\mathbb{O}_\circ$ is a linear subspace $V \subset \mathbb{O}_\circ$ such that, for all $\alpha, \beta \in V$, $\alpha \cdot \beta = 0$. An *isotropic subspace* in $\mathbb{O}_\circ$ is a linear subspace $V \subset \mathbb{O}_\circ$ such that $N(\alpha) = 0$ for all $\alpha \in V$. While for one-dimensional subspaces of $\mathbb{O}_\circ$, nil-spaces coincide with isotropic spaces, this does not hold in higher dimension.

It is known that every maximal parabolic subgroup of $G_2(\mathbb{C})$ is the stabilizer of a one-dimensional or two-dimensional nil-space in $\mathbb{O}_\circ$ (Aschbacher [Asc87, Theorem 3]). It is also known that every maximal parabolic subgroup of $\mathrm{Spin}_7(\mathbb{C})$ is the stabilizer of a one-, two-, or three-dimensional isotropic subspace in $\mathbb{O}_\circ$.

Proposition 1.1. *Suppose that P is a maximal parabolic subgroup of $\mathrm{Spin}_7(\mathbb{C})$. Then either $P \cap G_2(\mathbb{C})$ is contained in a maximal parabolic subgroup of $G_2(\mathbb{C})$ or $P \cap G_2(\mathbb{C})$ is contained in a subgroup of $G_2(\mathbb{C})$ isomorphic to $\mathrm{SL}_3(\mathbb{C})$.*

Proof. There are three cases to consider, depending on whether P stabilizes a one-, two-, or three-dimensional isotropic subspace $V \subset \mathbb{O}_\circ$.

Case $\dim(V) = 1$. If $\dim(V) = 1$, then any vector in V has norm zero and trace zero, from which it follows that any vector $\alpha \in V$ satisfies $\alpha^2 = 0$. It follows that V is a nil-space in $\mathbb{O}_\circ$. Thus, $P \cap G_2(\mathbb{C})$ is the maximal parabolic subgroup of $G_2(\mathbb{C})$ stabilizing this nil-space.

Case $\dim(V) = 2$. If $\dim(V) = 2$, then every vector $\alpha \in V$ satisfies $\alpha^2 = 0$. If V is a nil-space, then $P \cap G_2(\mathbb{C})$ is the maximal parabolic subgroup of $G_2(\mathbb{C})$ stabilizing this nil-space. If V is not a nil-space, then there exists a basis $\{\alpha, \beta\} \subset V$ such that $\alpha \cdot \beta = \gamma \neq 0$. It follows that $V \cdot V \subset \mathbb{C}\gamma$. Therefore, if $g \in P \cap G_2(\mathbb{C})$, then g stabilizes not only V, but also the line spanned by γ.

Observe that $\gamma^2 = (\alpha\beta) \cdot (\alpha\beta) = \alpha(\beta\alpha)\beta$ by Moufang identities, and $\beta\alpha = -\alpha\beta$ since $(\alpha + \beta)^2 = 0$. Hence, $\gamma^2 = 0$. Therefore, $P \cap G_2(\mathbb{C})$ is contained in the maximal parabolic subgroup stabilizing the nil-line $\mathbb{C}\gamma$.

Case $\dim(V) = 3$. If $\dim(V) = 3$, then we begin by choosing a basis $\{\alpha, \beta, \gamma\}$ of V. There are two possibilities to consider. First, if $\gamma \in \mathbb{C}(\alpha \cdot \beta)$, then $V \cdot V \subset \mathbb{C}\gamma$, and $\gamma^2 = 0$. In this

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case, $P \cap G_2(\mathbb{C})$ stabilizes the nil-line $\mathbb{C}\gamma$, and hence is contained in a maximal parabolic subgroup of $G_2(\mathbb{C})$.

If $\gamma \notin \mathbb{C}(\alpha \cdot \beta)$, then $[\alpha, \beta, \gamma] \neq 0$, where the bracket denotes the associator:

$$[\alpha, \beta, \gamma] = (\alpha\beta)\gamma - \alpha(\beta\gamma).$$

In this case, we find that $[V, V, V] \subset \mathbb{C} \cdot [\alpha, \beta, \gamma]$. Therefore, $P \cap G_2(\mathbb{C})$ stabilizes the line $\mathbb{C} \cdot [\alpha, \beta, \gamma]$. The stabilizer of a line in $G_2(\mathbb{C})$ is either a maximal parabolic subgroup (if the line is a nil-line) or else a subgroup isomorphic to $\mathrm{SL}_3(\mathbb{C})$. Thus, $P \cap G_2(\mathbb{C})$ is contained in a maximal parabolic subgroup of $G_2(\mathbb{C})$ or else is contained in a subgroup isomorphic to $\mathrm{SL}_3(\mathbb{C})$. $\square$

Proposition 1.2. *Suppose that Q is a proper parabolic subgroup of $\mathrm{SL}_3(\mathbb{C})$. Then, for any embedding of $\mathrm{SL}_3(\mathbb{C})$ in $G_2(\mathbb{C})$, the image of Q is contained in a maximal parabolic subgroup of $G_2(\mathbb{C})$.*

Proof. By the theory of Borel and De Siebenthal [BD49], every embedding of the full rank subgroup $\mathrm{SL}_3(\mathbb{C})$ in $G_2(\mathbb{C})$ arises from a pair of long roots in the root system of type G_2. It follows that a parabolic subgroup $Q \subset \mathrm{SL}_3(\mathbb{C})$ arises from a single long root in the root system of type G_2; it follows that Q will be contained in the maximal parabolic subgroup of $G_2(\mathbb{C})$ corresponding to this long root. $\square$

The previous propositions now yield the following dichotomy for parameters.

Theorem 1.3. *Suppose that $\eta \in \mathrm{Par}^\circ(G_2)$ is a cuspidal parameter for G_2. Let η' be the associated parameter for PGSp_6 obtained by composing η with the inclusion $G_2(\mathbb{C}) \hookrightarrow \mathrm{Spin}_7(\mathbb{C})$. Then either $\eta' \in \mathrm{Par}^\circ(\mathrm{PGSp}_6)$, i.e., η' is a cuspidal parameter, or else there exists a cuspidal parameter $\eta'' \in \mathrm{Par}^\circ(\mathrm{PGL}_3)$ such that η is obtained from η'' via an inclusion $\mathrm{SL}_3(\mathbb{C}) \hookrightarrow G_2(\mathbb{C})$.*

Proof. If η' is not a cuspidal parameter, then there exists a maximal parabolic subgroup $P \subset \mathrm{Spin}_7(\mathbb{C})$ such that $\mathrm{Im}(\eta') \subset P$. It follows that $\mathrm{Im}(\eta) \subset P \cap G_2$. Since η was assumed cuspidal, we find that $P \cap G_2$ is not contained in any maximal parabolic subgroups of G_2. It follows from Proposition 1.1 that P is the stabilizer of a three-dimensional isotropic subspace of $\mathbb{O}_\circ$, and $P \cap G_2(\mathbb{C})$ is contained in a subgroup isomorphic to $\mathrm{SL}_3(\mathbb{C})$.

Hence, if $\eta' \notin \mathrm{Par}^\circ(\mathrm{PGSp}_6)$, then we find that there exists an embedding $\iota : \mathrm{SL}_3(\mathbb{C}) \hookrightarrow G_2(\mathbb{C})$ and a parameter $\eta'' \in \mathrm{Par}(\mathrm{PGL}_3)$ such that $\eta = \iota \circ \eta''$. If η'' were not cuspidal, its image would be contained in a maximal parabolic subgroup of $G_2(\mathbb{C})$ by Proposition 1.2, contradicting the cuspidality of η. Hence, $\eta'' \in \mathrm{Par}^\circ(\mathrm{PGL}_3)$. $\square$

This theorem demonstrates that to each $\eta \in \mathrm{Par}^\circ(G_2)$, one may associate a cuspidal parameter $\eta' \in \mathrm{Par}^\circ(\mathrm{PGSp}_6)$ or else a cuspidal parameter $\eta'' \in \mathrm{Par}^\circ(\mathrm{PGL}_3)$. Since all embeddings of $G_2(\mathbb{C})$ in $\mathrm{Spin}_7(\mathbb{C})$ are $\mathrm{Spin}_7(\mathbb{C})$-conjugate, we find that η' is uniquely determined (up to $\mathrm{Spin}_7(\mathbb{C})$-conjugacy) by η (up to $G_2(\mathbb{C})$-conjugacy).

Similarly, a cuspidal parameter $\eta \in \mathrm{Par}^\circ(G_2)$, which composes to yield a non-cuspidal parameter for PGSp_6, yields a cuspidal parameter $\eta'' \in \mathrm{Par}^\circ(\mathrm{PGL}_3)$ unique up to $G_2(\mathbb{C})$-conjugacy. Note that all embeddings of $\mathrm{SL}_3(\mathbb{C})$ into $G_2(\mathbb{C})$ are $G_2(\mathbb{C})$-conjugate; moreover, the $G_2(\mathbb{C})$-conjugacy class of a cuspidal parameter η determines the cuspidal parameter η'' uniquely, up to $\mathrm{SL}_3(\mathbb{C})$-conjugacy *and* outer automorphism. Namely, the outer automorphism of $\mathrm{SL}_3(\mathbb{C})$ sending g to $(g^{\mathsf{T}})^{-1}$ is realized by conjugating by an element of $G_2(\mathbb{C})$. The normalizer $N(\mathrm{SL}_3(\mathbb{C}))$ in $G_2(\mathbb{C})$ is generated by $\mathrm{SL}_3(\mathbb{C})$ and an element inducing this outer automorphism.

Putting these observations together, we find the following theorem.

Theorem 1.4 (Dichotomy of parameters). *There is a natural injective dichotomy for the set of cuspidal parameters for G_2, modulo $G_2(\mathbb{C})$-conjugacy:*

$$\frac{\mathrm{Par}^\circ(G_2)}{\mathrm{Ad}(G_2(\mathbb{C}))} \hookrightarrow \frac{\mathrm{Par}^\circ(\mathrm{PGSp}_6)}{\mathrm{Ad}(\mathrm{Spin}_7(\mathbb{C}))} \sqcup \frac{\mathrm{Par}^\circ(\mathrm{PGL}_3)}{\mathrm{Ad}(N(\mathrm{SL}_3(\mathbb{C})))}.$$

The image of this dichotomy can also be characterized. First, we observe the following.

Proposition 1.5. *Suppose that $\eta'' \in \mathrm{Par}^\circ(\mathrm{PGL}_3)$. Then, for any embedding $\iota : \mathrm{SL}_3(\mathbb{C}) \hookrightarrow G_2(\mathbb{C})$, $\iota \circ \eta'' \in \mathrm{Par}^\circ(G_2)$.*

Proof. It is clear that $\iota \circ \eta'' \in \mathrm{Par}(G_2)$. If P is a maximal parabolic subgroup of G_2, then P stabilizes a nil-line in $\mathbb{O}_\circ$ or a nil-plane in $\mathbb{O}_\circ$. As a representation of $\mathrm{SL}_3(\mathbb{C})$, the vector space $\mathbb{O}_\circ$ decomposes into the direct sum of two irreducible three-dimensional representations, and one trivial representation arising from a $\mathrm{SL}_3(\mathbb{C})$-fixed line in $\mathbb{O}_\circ$. Since there is no nil-line or nil-plane fixed by $\mathrm{SL}_3(\mathbb{C})$, we find that $P \cap \mathrm{SL}_3(\mathbb{C})$ fixes a line or plane in one of the irreducible three-dimensional representations of $\mathrm{SL}_3(\mathbb{C})$. Hence, $P \cap \mathrm{SL}_3(\mathbb{C})$ is contained in a maximal parabolic subgroup of $\mathrm{SL}_3(\mathbb{C})$. The proposition follows immediately. □

We find that the natural dichotomy for cuspidal parameters for G_2 includes all cuspidal parameters for PGL_3. However, not all parameters for PGSp_6 occur in this dichotomy. Perhaps the most convenient way of characterizing the parameters for PGSp_6 is through the following.

Proposition 1.6. *Suppose that $\eta' \in \mathrm{Par}^\circ(\mathrm{PGSp}_6)$. Let $L(\eta', \mathrm{Spin}, s)$ denote the Artin–Weil L-function associated to η' and the eight-dimensional spin representation of $\mathrm{Spin}_7(\mathbb{C})$. Then $L(\eta', \mathrm{Spin}, s)$ has a pole at $s = 0$ if and only if the image of η' is contained in a subgroup of $\mathrm{Spin}_7(\mathbb{C})$ isomorphic to $G_2(\mathbb{C})$.*

Proof. Let V be an eight-dimensional vector space, on which $\mathrm{Spin}_7(\mathbb{C})$ acts via the spin representation. The order of the pole of $L(\eta', \mathrm{Spin}, s)$ at $s = 0$ is precisely the multiplicity of the trivial representation of W_k for its action on V. Thus, we find that $L(\eta', \mathrm{Spin}, s)$ has a pole at $s = 0$ if and only if V has a non-zero vector fixed by W_k.

Now, the stabilizer of any non-zero vector $v \in V$ in $\mathrm{Spin}_7(\mathbb{C})$ is either a proper parabolic subgroup of $\mathrm{Spin}_7(\mathbb{C})$ or else a group isomorphic to $G_2(\mathbb{C})$. Since we assume that η' is a cuspidal parameter, its image in not contained in any proper parabolic subgroups of $\mathrm{Spin}_7(\mathbb{C})$. Thus, $L(\eta', \mathrm{Spin}, s)$ has a pole at $s = 0$ if and only if $\eta'(W_k)$ lies in an embedded $G_2(\mathbb{C})$ in $\mathrm{Spin}_7(\mathbb{C})$. □

Define $\mathrm{Par}^\circ_{\mathrm{Spin}}(\mathrm{PGSp}_6)$ to be the set of cuspidal parameters η' for PGSp_6, for which $L(\eta', \mathrm{Spin}, s)$ has a pole at $s = 0$. Then we find the following perfect dichotomy of parameters.

Theorem 1.7. *There is a bijective dichotomy for the set of cuspidal parameters for G_2, modulo $G_2(\mathbb{C})$-conjugacy:*

$$\frac{\mathrm{Par}^\circ(G_2)}{\mathrm{Ad}(G_2(\mathbb{C}))} \leftrightarrow \frac{\mathrm{Par}^\circ_{\mathrm{Spin}}(\mathrm{PGSp}_6)}{\mathrm{Ad}(\mathrm{Spin}_7(\mathbb{C}))} \sqcup \frac{\mathrm{Par}^\circ(\mathrm{PGL}_3)}{\mathrm{Ad}(N(\mathrm{SL}_3(\mathbb{C})))}.$$

1.3 Dichotomy for irreps of G_2

The dichotomy for parameters in Theorem 1.7 suggests, via the local Langlands conjectures, a dichotomy for the generic supercuspidal irreps of G_2. Recall that $\mathrm{Irr}^\circ_g(G)$ denotes the set of isomorphism classes of generic supercuspidal irreps of a (semisimple, adjoint, split) group G.

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Define $\mathrm{Irr}^\circ_{g,\mathrm{Spin}}(\mathrm{PGSp}_6)$ to be the subset of $\mathrm{Irr}^\circ_g(\mathrm{PGSp}_6)$, consisting of those irreps σ for which Shahidi's degree-eight L-function $L(\tau, \mathrm{Spin}, s)$ has a pole at $s = 0$. The main result of this paper is the following.

THEOREM 1.8. *Dual pair correspondences in the simple split adjoint groups E_6 and E_7 determine a dichotomy function Δ, which is bijective when $p \neq 2$ and injective when $p = 2$:*

$$\Delta : \mathrm{Irr}^\circ_g(G_2) \to \mathrm{Irr}^\circ_{g,\mathrm{Spin}}(\mathrm{PGSp}_6) \sqcup \frac{\mathrm{Irr}^\circ_g(\mathrm{PGL}_3)}{\mathrm{Contra}},$$

where Contra *denotes the equivalence relation given by contragredience.*

The existence of such a bijection is directly implied by Langlands conjectures and the dichotomy of parameters in Theorem 1.7. The realization of this bijection through theta correspondences is a result of additional interest, and follows many previous realizations of 'Langlands functoriality' in theta correspondences. Conversely, this result can be used to parameterize the generic, supercuspidal representations of G_2 over a p-adic field, using known and perhaps soon-to-be known parameterizations for PGL_3 and PGSp_6.

Specifically, the local Langlands conjectures have been proven for PGL_3 (for GL_3 in fact) by Henniart [Hen84a], in the following sense.

PROPOSITION 1.9. *There is a natural (compatible with L-functions and ϵ-factors, among other properties) bijection*

$$\Phi(\mathrm{PGL}_3) : \frac{\mathrm{Irr}^\circ_g(\mathrm{PGL}_3)}{\mathrm{Contra}} \to \frac{\mathrm{Par}^\circ(\mathrm{PGL}_3)}{\mathrm{Ad}(N(\mathrm{SL}_3(\mathbb{C})))}.$$

In particular, the contragredient on irreps corresponds to the change in parameter given by the outer automorphism of $\mathrm{SL}_3(\mathbb{C})$.

While parts of the local Langlands conjectures are open for PGSp_6, it appears likely that the following will be proven in the not so distant future.

CONJECTURE 1.10. There is a bijection

$$\Phi(\mathrm{PGSp}_6) : \mathrm{Irr}^\circ_g(\mathrm{PGSp}_6) \to \frac{\mathrm{Par}^\circ(\mathrm{PGSp}_6)}{\mathrm{Ad}(\mathrm{Spin}_7(\mathbb{C}))},$$

in which Shahidi's degree-eight spin L-function on irreps corresponds to the Artin–Weil degree-eight L-function associated to the spin representation of $\mathrm{Spin}_7(\mathbb{C})$.

The main theorem of this paper implies the following theorem.

THEOREM 1.11. *Assuming a parameterization $\Phi(\mathrm{PGSp}_6)$ satisfying the previous conjecture, and assuming that $p \neq 2$, there is a bijective parameterization*

$$\Phi(G_2) : \mathrm{Irr}^\circ_g(G_2) \to \frac{\mathrm{Par}^\circ(G_2)}{\mathrm{Ad}(G_2(\mathbb{C}))}.$$

Of course, there are further properties of this parameterization $\Phi(G_2)$ that should be proven; for example, one hopes that $\Phi(G_2)$ is compatible with L-functions and ϵ-factors of various twists.

2. Structure theory

There are many constructions of exceptional Lie algebras and algebraic groups. The construction of Allison [All79] using structurable algebras [All78] (with similarities to earlier constructions of

Kantor [Kan72]) is well suited to some needs of this paper. The construction of Koecher [Koe67] using Jordan algebras is well suited to other needs of this paper. We recall these constructions of Lie algebras, and associated algebraic groups, in this section. The constructions here are valid whenever k is a field of characteristic zero (and, most likely, when $\mathrm{char}(k) \neq 2, 3$).

2.1 Composition, Jordan, and structurable algebras

2.1.1 *Composition algebras.*

Definition 2.1. A composition algebra (sometimes called a Hurwitz algebra) over k is a pair (C, N), where C is a k-algebra and $N : C \to k$ is a non-degenerate quadratic form which satisfies $N(xy) = N(x)N(y)$ for all $x, y \in C$.

Given a composition algebra (C, N) over k, we write N also for the associated symmetric bilinear form:

$$N(x, y) = N(x + y) - (N(x) + N(y)).$$

The *standard involution* on C is given by

$$\bar{x} = N(x, 1) - x.$$

The *norm* and *trace* can be recovered from the standard involution:

$$N(x) = x\bar{x} \quad \text{and} \quad \mathrm{Tr}(x) = x + \bar{x}.$$

According to classification results originating with Hurwitz, composition algebras over k have dimension one, two, four, or eight as vector spaces over k. A composition algebra of dimension eight will be called a *Cayley algebra*. Composition algebras of dimensions one and two are commutative and associative. Composition algebras of dimension four are associative. Composition algebras of dimension eight are alternative: if C is a Cayley algebra, and $x, y \in C$, then

$$(xx)y = x(xy) \quad \text{and} \quad (yx)x = y(xx).$$

Although Cayley algebras are non-associative, the map $(x, y, z) \mapsto \mathrm{Tr}(xyz)$ defines a *trilinear form* on a Cayley algebra C; the associative law is not required here since

$$\mathrm{Tr}(x(yz)) = \mathrm{Tr}((xy)z) \quad \text{for all } x, y, z \in C.$$

2.1.2 *Algebras with involution.* Suppose that A is a k-algebra with involution (denoted $a \mapsto \bar{a}$). For $x, y, z \in A$, we define the following: first, the left- and right-multiplication endomorphisms are defined by $L_x(y) = xy$ and $R_x(y) = yx$. Thus, $L_x, R_x \in \mathfrak{End}_k(A)$. Also, $[x, y] = xy - yx$ is the commutator, and $[x, y, z] = (xy)z - x(yz)$ is the associator. The involution yields a ternary composition

$$\{x, y, z\} = (x\bar{y})z + (z\bar{y})x - (z\bar{x})y.$$

This ternary composition yields the endomorphism $V_{x,y} \in \mathfrak{End}_k(A)$, given by $V_{x,y}(z) = \{x, y, z\}$. Finally, define the endomorphism $T_x \in \mathfrak{End}_k(A)$, given by $T_x = V_{x,1}$. Then

$$T_x = L_x + R_{x - \bar{x}}.$$

Given a k-algebra A with involution, one may consider the *hermitian* and *skew-hermitian* elements of A. The skew-hermitian (or trace zero) elements of A are

$$A_\circ = \{a \in A \text{ such that } a + \bar{a} = 0\}.$$

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The hermitian elements of A are denoted

$$A_+ = \{a \in A \text{ such that } a = \bar{a}\}.$$

As a k-vector space, one may clearly decompose A as a direct sum: $A = A_\circ \oplus A_+$.

There is a natural alternating $A_\circ$-valued k-bilinear form on A, defined by

$$\langle x, y \rangle = x\bar{y} - y\bar{x} = (x\bar{y}) - \overline{x\bar{y}}.$$

From this form, one may construct the two-step nilpotent Lie algebra

$$\mathfrak{h}(A, A_\circ) = A \oplus A_\circ,$$

whose brackets are given by

$$[(x, r), (y, s)] = (0, \langle x, y \rangle) = (0, x\bar{y} - y\bar{x}) \quad \text{for all } x, y \in A, r, s \in A_\circ.$$

One may also directly construct a two-step unipotent algebraic group

$$\mathbf{H}(A, A_\circ) = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & \bar{x} \\ 0 & 0 & 1 \end{pmatrix} : x, z \in A \operatorname{Tr}(z) = N(x) \right\},$$

where composition is given by the usual rules for matrix multiplication and the composition in the algebra A.

2.1.3 *Jordan algebras.* Let C be a composition algebra over k. Without reviewing the general theory of Jordan algebras, we mention and describe the Jordan algebra J_C of Hermitian-symmetric three-by-three matrices with entries in C:

$$_C = \left\{ \begin{pmatrix} a & \gamma & \bar{\beta} \\ \bar{\gamma} & b & \alpha \\ \beta & \bar{\alpha} & c \end{pmatrix} : a, b, c \in k, \alpha, \beta, \gamma \in C \right\}.$$

On J_C, there is the Jordan composition

$$j_1 \circ j_2 = \tfrac{1}{2} \cdot (j_1 j_2 + j_2 j_1),$$

where ordinary matrix multiplication is used on the right-hand side above.

But more importantly for our purposes are the quadratic adjoint, cubic determinant, and cross product. The quadratic adjoint is defined by (following notation of [Kru07, § 2.4])

$$\begin{pmatrix} a & \gamma & \bar{\beta} \\ \bar{\gamma} & b & \alpha \\ \beta & \bar{\alpha} & c \end{pmatrix}^\sharp = \begin{pmatrix} bc - N(\alpha) & \bar{\beta}\bar{\alpha} - c\gamma & \gamma\alpha - b\bar{\beta} \\ \alpha\beta - c\bar{\gamma} & ca - N(\beta) & \bar{\gamma}\bar{\beta} - a\alpha \\ \bar{\alpha}\bar{\gamma} - b\beta & \beta\gamma - a\bar{\alpha} & ab - N(\gamma) \end{pmatrix}.$$

The cross product is the linearization of this quadratic adjoint:

$$j_1 \times j_2 = (j_1 + j_2)^\sharp - (j_1^\sharp + j_2^\sharp).$$

There exists a unique cubic form $\mathsf{N} : J_C \to k$, for which

$$j \times j^\sharp = \mathsf{N}(j) \cdot j \quad \text{for all } j \in J_C.$$

There is a natural non-degenerate trace pairing

$$T(j, j') = \operatorname{Tr}(j \circ j').$$

2.1.4 *Structurable algebras.* We define and discuss *structurable algebras* here, following the foundational work of Allison [All78] very closely.

Definition 2.2. A k-algebra A with involution is called a *structurable algebra* if, for all $x, y, z \in A$, the following (quartic polynomial) identity holds:

$$[T_z, V_{x,y}] = V_{T_z x, y} - V_{x, T_{\bar{z}} y}.$$

Such an algebra satisfies

$$[r, x, y] = [x, y, r] = -[x, r, y] \quad \text{for all } x, y \in A, r \in A_\circ.$$

Let $\mathfrak{Der}(A)$ denote the Lie algebra over k, consisting of derivations of A which commute with the involution. These are k-endomorphisms D of A, which satisfy the following identities:

$$D(xy) = (Dx)y + x(Dy) \quad \text{and} \quad D(\bar{x}) = \overline{Dx} \quad \text{for all } x, y \in A.$$

Important examples of structurable algebras include tensor products of composition algebras. These have been studied extensively by Allison [All88], who proved the following proposition.

Proposition 2.3. *Suppose that B and C are composition algebras. Then $B \otimes_k C$, with the tensor product algebra structure and involution, is a structurable algebra.*

When $A = B \otimes_k C$ is a tensor product of two composition algebras, as above, one may check directly that

$$A_\circ = (B_\circ \otimes_k k) \oplus (k \otimes_k C_\circ) \cong B_\circ \oplus C_\circ.$$

In this way $\mathbf{H}(A, A_\circ)$ has central subgroup $B_\circ \oplus C_\circ$ and abelian quotient $B \otimes_k C$.

Another important example of a structurable algebra, from [All78, § 8], is given by a construction of Freudenthal. From a composition algebra C, and the resulting Jordan algebra J_C, consider the k-vector space

$$F_C = \left\{ \begin{pmatrix} a & j \\ j' & d \end{pmatrix} : a, d \in k \text{ and } j, j' \in J_C \right\}.$$

This space has a natural k-algebra structure given by

$$\begin{pmatrix} a_1 & j_1 \\ j_1' & d_1 \end{pmatrix} \cdot \begin{pmatrix} a_2 & j_2 \\ j_2' & d_2 \end{pmatrix} = \begin{pmatrix} a_1 a_2 + T(j_1, j_2') & a_1 j_2 + d_2 j_1 + j_1' \times j_2' \\ a_2 j_1' + a_2 j_2' + j_1 \times j_2 & T(j_2, j_1') + d_1 d_2 \end{pmatrix}.$$

An involution on F_C is given by

$$\overline{\begin{pmatrix} a & j \\ j' & d \end{pmatrix}} = \begin{pmatrix} d & j \\ j' & a \end{pmatrix}.$$

In [All78], Allison proved (in fact, he proved much more) the following proposition.

Proposition 2.4. *If C is any composition algebra, then F_C, with product and involution given above, is a structurable algebra.*

Note that the trace zero elements of F_C form a one-dimensional subspace

$$(F_C)_\circ = \left\{ \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} : a \in k \right\}.$$

2.2 Lie algebras

From Jordan algebras and structurable algebras, we may follow constructions of Tits–Koecher and Allison to construct certain Lie algebras over k. We review these constructions here.

2.2.1 *Lie algebras from Jordan algebras.* Suppose that J is a semisimple Jordan algebra. Then constructions of Tits, Kantor, or Koecher [Koe67] (whom we follow here) yield a graded Lie algebra

$$\mathfrak{g}_J = \mathfrak{g}_J^{(-1)} \oplus \mathfrak{g}_J^{(0)} \oplus \mathfrak{g}_J^{(1)},$$

where $\mathfrak{g}_J^{(0)} = \mathfrak{Str}(J)$ is the subalgebra of $\mathfrak{End}_k(J)$ generated by derivations of J and left Jordan multiplications L_j (for $j \in J$) and $\mathfrak{g}_J^{(\pm 1)}$ is identified with J as a k-vector space. The Lie bracket on $\mathfrak{g}_J$ is given by the following.

- For all $j \in J$, let $\alpha_\pm(j)$ denote the element of $\mathfrak{g}_J^{(\pm 1)}$ associated to j. The Lie algebras $\mathfrak{g}_J^{(\pm)}$ are abelian, i.e.,

$$[\alpha_+(j), \alpha_+(j')] = [\alpha_-(j), \alpha_-(j')] = 0 \quad \text{for all } j, j' \in J.$$

- For all $X \in \mathfrak{g}_J^{(0)}$ and all $j \in J$, we define Lie brackets by

$$[X, \alpha_+(j)] = \alpha_+(X(j)), \text{recalling that } X \in \mathfrak{Str}(J) \subset \mathfrak{End}(J).$$

  Also, we define

$$[X, \alpha_-(j)] = \alpha_-(-X^*(j)),$$

  where X^* denotes the adjoint endomorphism of J, with respect to the trace pairing on J.

- For all $j, j' \in J$, we define

$$[\alpha_+(j), \alpha_-(j')] = 2(L_{j \circ j'} + [L_j, L_{j'}]) \in \mathfrak{Str}(J) = \mathfrak{g}_J^{(0)}.$$

In this way, the Lie algebra $\mathfrak{g}_J$ is naturally endowed with a parabolic subalgebra $\mathfrak{p}_J = \mathfrak{g}_J^{(0)} \oplus \mathfrak{g}_J^{(1)}$ with abelian nilradical $\mathfrak{u}_J = \mathfrak{g}_J^{(1)} = J$.

2.2.2 *Lie algebras from structurable algebras.* Suppose that A is a structurable algebra. Following Allison [All79], let $\mathfrak{Strl}(A)$ be the k-subspace of $\mathfrak{End}_k(A)$ spanned by $\mathfrak{Der}(A)$ and endomorphisms of the form T_a for $a \in A$. Then $\mathfrak{Strl}(A)$ is a Lie subalgebra of $\mathfrak{End}_k(A)$, and contains $\mathfrak{Der}(A)$ as a Lie subalgebra. Given $X \in \mathfrak{Strl}(A)$, $X \in \mathfrak{Der}(A)$ if and only if $X(1) = 0$.

Many elements of $\mathfrak{Strl}(A)$ arise from 'inner' endomorphisms, i.e., endomorphisms arising directly from the composition and involution on A.

- For all $r \in A$, $T_r \in \mathfrak{Strl}(A)$ by definition.
- For all $x, y \in A$, define a derivation of A by

$$D_{x,y}(z) = \tfrac{1}{3}[[x, y] + [\bar{x}, \bar{y}], z] + [z, y, x] - [z, \bar{x}, \bar{y}],$$

  for all $z \in A$. From [All79, § 1], $D_{x,y} \in \mathfrak{Der}(A) \subset \mathfrak{Strl}(A) \subset \mathfrak{End}_k(A)$.

- For all $x, y \in A$, one has

$$V_{x,y} = \tfrac{1}{3} T_{2xy + \bar{y}x - \bar{x}y + y\bar{x}} + D_{x,\bar{y}}.$$

  Hence, $V_{x,y} \in \mathfrak{Strl}(A)$.

- For all $r, s \in A_\circ$,

$$L_r L_s = T_{rs} - V_{r,s}.$$

  Hence, $L_r L_s \in \mathfrak{Strl}(A)$.

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Following [All78, All79], we write $\mathfrak{Instrl}(A)$ for the subspace of $\mathfrak{Strl}(A)$ spanned by $V_{x,y}$ for all $x, y \in A$. We write $\mathfrak{Inder}(A)$ for the subspace of $\mathfrak{Der}(A)$ spanned by $D_{x,y}$ for all $x, y \in A$. Then $\mathfrak{Instrl}(A)$ is an ideal in $\mathfrak{Strl}(A)$, and $\mathfrak{Inder}(A)$ is an ideal in $\mathfrak{Der}(A)$. The subspace $\mathfrak{L}(A)$ spanned by $L_r L_s$ for all $r, s \in A_\circ$ is an ideal in $\mathfrak{Strl}(A)$, and there is a chain of inclusions

$$\mathfrak{L}(A) \subset \mathfrak{Instrl}(A) \subset \mathfrak{Strl}(A).$$

For all $X \in \mathfrak{Strl}(A)$, define X^ϵ and X^δ by

$$X^\epsilon = X - T_{X(1)+\overline{X(1)}} \quad \text{and} \quad X^\delta = X + R_{\overline{X(1)}}.$$

Then $X \mapsto X^\epsilon$ is an automorphism of the Lie algebra $\mathfrak{Strl}(A)$ of order two. The element $X^\delta \in \mathfrak{End}_k(A)$ preserves the subspace $A_\circ \subset A$, and the resulting map $X \mapsto X^\delta$ is a Lie algebra representation:

$$\mathfrak{Strl}(A) \to \mathfrak{End}_k(A_\circ).$$

From a structurable algebra A, Allison [All79] constructed a Lie algebra, with similarities to earlier work of Kantor [Kan72]. This Lie algebra, $\mathfrak{g}_A$, is constructed with a $\mathbb{Z}$-grading, vanishing outside degrees $-2, -1, 0, 1, 2$. In these degrees, the Lie algebra is constructed as follows.

- In degree ± 2, we define $\mathfrak{g}_A^{(\pm 2)} = A_\circ$. For all $r \in A_\circ$, we write $\zeta_\pm(r)$ for the corresponding element of $\mathfrak{g}_A^{(\pm 2)}$.
- In degree ± 1, we define $\mathfrak{g}_A^{(\pm 1)} = A$. For all $x \in A$, we write $\eta_\pm(x)$ for the corresponding element of $\mathfrak{g}_A^{(\pm 1)}$.
- In degree zero, we define $\mathfrak{g}_A^{(0)} = \mathfrak{Instrl}(A)$.

The brackets on the Lie algebra $\mathfrak{g}_A = \bigoplus_{i=-2}^{2} \mathfrak{g}_A^{(i)}$ are defined by the following identities:

- the space $\mathfrak{u}_A = \mathfrak{g}_A^{(1)} \oplus \mathfrak{g}_A^{(2)} = A \oplus A_\circ$ is identified as a Lie algebra with $\mathfrak{h}(A, A_\circ)$. In other words,
$$[\eta_+(x) + \zeta_+(r), \eta_+(y) + \zeta_+(s)] = \zeta_+(x\bar{y} - y\bar{x}),$$
for all $x, y \in A$ and $r, s \in A_\circ$. The bracket on $\mathfrak{g}_A^{(-1)} \oplus \mathfrak{g}_A^{(-2)}$ is defined in the same way:
$$[\eta_-(x) + \zeta_-(r), \eta_-(y) + \zeta_-(s)] = \zeta_-(x\bar{y} - y\bar{x});$$
- the elements $X \in \mathfrak{g}_A^{(0)} = \mathfrak{Instrl}(A)$ are endomorphisms of the k-vector space A. For such elements, X^δ is an endomorphism of the k-vector space $A_\circ$. Hence, for all $X \in \mathfrak{g}_A^{(0)}$, it makes sense to define
$$[X, \eta_+(x) + \zeta_+(r)] = \eta_+(X(x)) + \zeta_+(X^\delta(r)).$$
Recalling that ϵ is an automorphism of $\mathfrak{Instrl}(A)$ of order two, it makes sense to define
$$[X, \eta_-(x) + \zeta_-(r)] = \eta_-(X^\epsilon(x)) + \zeta_-(X^{\epsilon\delta}(r));$$
- for $x, y \in A$ and $r, s \in A_\circ$, define
$$[\eta_+(x) + \zeta_+(r), \eta_-(y) + \zeta_-(s)] = -\eta_-(sx) + (V_{x,y} + L_r L_s) + \eta_+(ry).$$

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These identities suffice to determine the Lie algebra structure on all of $\mathfrak{g}_A$. Note that $\mathfrak{g}_A$ is naturally endowed with a parabolic subalgebra

$$\mathfrak{p}_A = \mathfrak{g}_A^{(0)} \oplus \mathfrak{g}_A^{(1)} \oplus \mathfrak{g}_A^{(2)},$$

with unipotent radical $\mathfrak{u}_A$ with center $\mathfrak{z}_A$. Furthermore, $\mathfrak{z}_A$ is identified with $A_\circ$, and $\mathfrak{u}_A/\mathfrak{z}_A$ is identified with A.

2.3 Algebraic groups

Consider a Jordan algebra J, and the Koecher Lie algebra $\mathfrak{g}_J$ constructed earlier. Define an algebraic group $\mathbf{G}_J$ over k as the algebraic subgroup of $\mathbf{GL}(\mathfrak{g}_J)$ preserving the Lie bracket and a Killing form. The three-term grading on $\mathfrak{g}_J$ yields a parabolic subgroup $\mathbf{P}_J$ with abelian unipotent radical $\mathbf{U}_J$, whose k-points are identified with J itself.

If $J \subset K$ is an embedding of Jordan algebras (i.e., J and K are Jordan algebras, and J is embedded as a sub-k-algebra of K), then $\mathfrak{g}_J$ is naturally a graded Lie subalgebra of $\mathfrak{g}_K$. This follows quickly from the fact, proven by Jacobson [Jac49], that all derivations of the semisimple Jordan algebras considered are inner derivations; hence, these derivation algebras extend to derivations of larger semisimple Jordan algebras.

Since $\mathbf{G}_K$ is an algebraic group with Lie algebra $\mathfrak{g}_K$, and $\mathfrak{g}_J$ is a semisimple Lie subalgebra of $\mathfrak{g}_K$, there are an algebraic subgroup $\mathbf{G}'_J \subset \mathbf{G}_K$ and an isogeny $\iota : \mathbf{G}'_J \to \mathbf{G}_J$ (where $\mathbf{G}_J$ is the adjoint algebraic group associated to $\mathfrak{g}_J$). Let $\mathbf{P}'_J = \iota^{-1}(\mathbf{P}_J)$ and let $\mathbf{U}'_J$ be the neutral component of $\iota^{-1}(\mathbf{U}_J)$.

The embedding of algebraic groups $\mathbf{G}'_J \subset \mathbf{G}_K$, is *compatible with parabolics*:

$$\mathbf{P}_K \cap \mathbf{G}'_J = \mathbf{P}'_J \quad \text{and} \quad \mathbf{U}_K \cap \mathbf{G}'_J = \mathbf{U}'_J.$$

Similarly, consider a structurable algebra A, and Allison's Lie algebra $\mathfrak{g}_A$ constructed previously. Define an algebraic group $\mathbf{G}_A$ over k as the algebraic subgroup of $\mathbf{GL}(\mathfrak{g}_A)$ preserving the Lie bracket and a Killing form. The five-term grading on $\mathfrak{g}_A$ yields a parabolic subgroup $\mathbf{P}_A$ with two-step unipotent radical $\mathbf{U}_A \supset \mathbf{Z}_A$. The k-points of the center $\mathbf{Z}_A$ can be identified with $A_\circ$, and the k-points of the quotient $\mathbf{U}_A/\mathbf{Z}_A$ can be identified with A itself.

If $A \subset B$ is an embedding of structurable algebras (i.e., A and B are structurable algebras, and A is embedded as a sub-k-algebra with involution into B), then $\mathfrak{g}_A$ is naturally a graded Lie subalgebra of $\mathfrak{g}_B$ (since elements of $\mathfrak{Instrl}(A) \subset \mathfrak{End}_k(A)$ extend naturally to elements of $\mathfrak{Instrl}(B) \subset \mathfrak{End}_k(B)$). As before, one obtains an algebraic subgroup $\mathbf{G}'_A \subset \mathbf{G}_B$ together with an isogeny $\iota : \mathbf{G}'_A \to \mathbf{G}_A$. This embedding is *compatible with parabolics*:

$$\mathbf{P}_B \cap \mathbf{G}'_A = \mathbf{P}'_A, \quad \mathbf{U}_B \cap \mathbf{G}'_A = \mathbf{U}'_A, \quad \mathbf{Z}_B \cap \mathbf{G}'_A = \mathbf{Z}'_A.$$

2.3.1 *Automorphisms of composition algebras.* Fix a 'complete chain' of composition algebras $k \subset K \subset B \subset C$, where K, B, C are composition algebras of k-dimension two, four, eight, respectively. Some interesting algebraic groups arise as automorphism groups of extensions of composition algebras. Namely, if $H \subset E$ is an embedding of composition algebras over k, then let $\mathbf{Aut}_{E/H}$ denote the algebraic subgroup of $\mathbf{GL}(E)$ preserving the algebra structure and fixing the subalgebra H element-wise. For example, $\mathbf{Aut}_{C/k}$ is an absolutely simple group of type G_2 and $\mathbf{Aut}_{C/K}$ is a simply connected absolutely simple group of type A_2. And, $\mathbf{Aut}_{B/k}$ is an adjoint absolutely simple group of type A_1 and $\mathbf{Aut}_{C/B}$ is a simply connected absolutely simple group of type A_1.

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2.3.2 *Groups from Jordan algebras.* The chain of composition algebras $k \subset K \subset B \subset C$ yields a chain of Jordan algebras $J_k \subset J_K \subset J_B \subset J_C$. The associated algebraic groups $\mathbf{G}_J$ with parabolic subgroup $\mathbf{P}_J = \mathbf{L}_J \mathbf{U}_J$ are tabulated below.

Composition algebra	k	K	B	C
Dimension of J	6	9	15	27
Type of $\mathbf{G}_J$	C_3	A_5	D_6	E_7
Type of Levi $\mathbf{L}_J$	A_2	$\mathsf{A}_2 \times \mathsf{A}_2$	A_5	E_6

Given an embedding $H \subset E$ of composition algebras, we find an embedding of Jordan algebras $J_H \subset J_E$, and a subgroup $\mathbf{G}'_{J_H}$ of $\mathbf{G}_{J_E}$ together with an isogeny $\mathbf{G}'_{J_H} \to \mathbf{G}_{J_H}$. Moreover, the subgroup $\mathbf{G}'_{J_H}$ commutes with $\mathbf{Aut}_{E/H}$, naturally embedded in $\mathbf{G}_{J_E}$. In this way we find many commuting pairs of subgroups. We label them only by their type, leaving the precise determination of isogeny type up to the reader.

H	E	$\mathbf{Aut}_{E/H} \times \mathbf{G}'_{J_H}$	$\mathbf{G}_{J_E}$
k	C	$\mathsf{G}_2 \times \mathsf{C}_3$	E_7
K	C	$\mathsf{A}_2 \times \mathsf{A}_5$	E_7
k	B	$\mathsf{A}_1 \times \mathsf{C}_3$	E_6
B	C	$\mathsf{A}_1 \times \mathsf{E}_6$	E_7

2.3.3 *Tensor products of composition algebras.* The chain of Hurwitz algebras yields embeddings of structurable algebras, from which we examine

$$k \otimes B \subset k \otimes C \subset K \otimes C \subset B \otimes C \subset C \otimes C.$$

This yields embeddings (up to isogeny) of algebraic groups $\mathbf{G}_A$, compatible with two-step parabolic subgroups $\mathbf{P}_A = \mathbf{L}_A \mathbf{U}_A$. We tabulate some possibilities in the following.

A	$k \otimes B$	$k \otimes C$	$K \otimes C$	$B \otimes C$	$C \otimes C$
Type of $\mathbf{G}_A$	C_3	F_4	E_6	E_7	E_8
Type of Levi $\mathbf{L}_A$	$\mathsf{A}_1 \times \mathsf{A}_1$	B_3	$\mathsf{A}_1 \times \mathsf{A}_2 \times \mathsf{A}_2$	$\mathsf{D}_5 \times \mathsf{A}_1$	D_7
Dimension of $\mathbf{U}_A/\mathbf{Z}_A$	4	8	16	32	64
Dimension of $\mathbf{Z}_A$	3	7	8	10	14

This construction also realizes some well-known dual reductive pairs. Consider three composition algebras H, H', E, such that $H \subset E$. Then $\mathbf{Aut}_{E/H}$ naturally acts on the Lie algebra $\mathfrak{g}_{E \otimes H'}$ and $\mathbf{Aut}_{E/H}$ fixes the elements of the subalgebra $\mathfrak{g}_{H \otimes H'}$. This yields a homomorphism of algebraic groups:

$$\mathbf{Aut}_{E/H} \times \mathbf{G}'_{H \otimes H'} \hookrightarrow \mathbf{G}_{E \otimes H'}.$$

In particular, we find many commuting pairs of subgroups.

H	E	H'	$\mathbf{Aut}_{E/H} \times \mathbf{G}'_{H \otimes H'}$	$\mathbf{G}_{E \otimes H'}$
k	C	C	$\mathsf{G}_2 \times \mathsf{F}_4$	E_8
K	C	C	$\mathsf{A}_2 \times \mathsf{E}_6$	E_8
B	C	C	$\mathsf{A}_1 \times \mathsf{E}_7$	E_8
k	C	B	$\mathsf{G}_2 \times \mathsf{C}_3$	E_7
B	C	k	$\mathsf{A}_1 \times \mathsf{C}_3$	F_4

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While such exceptional dual pairs occur often in the literature, this construction is convenient for at least two reasons: first, it gives dual pairs of non-split subgroups which may be otherwise difficult to construct. Second, the embeddings are compatible with a distinguished parabolic subgroup, which is convenient later for computation of Jacquet modules.

2.3.4 *Freudenthal structurable algebras.* Finally, we recall that associated to the chain of composition algebras $k \subset K \subset B \subset C$, there is a chain of Jordan algebras $J_k \subset J_K \subset J_B \subset J_C$, and thus a chain of structurable algebras of Freudenthal type:

$$F_k \subset F_K \subset F_B \subset F_C.$$

Each one of these structurable algebras h as a one-dimensional subspace of trace zero elements. Allison's construction yields embeddings of algebraic groups (up to some isogeny)

$$\mathbf{G}'_{F_k} \subset \mathbf{G}'_{F_K} \subset \mathbf{G}'_{F_B} \subset \mathbf{G}_{F_C},$$

compatible with two-step 'Heisenberg' parabolic subgroups $\mathbf{P}_F = \mathbf{L}_F \mathbf{U}_F$. We tabulate the possibilities in the following.

Jordan algebra	J_k	J_K	J_B	J_C
Dimension of F	14	20	32	56
Type of $\mathbf{G}_F$	F_4	E_6	E_7	E_8
Type of Levi $\mathbf{L}_F$	C_3	A_5	D_6	E_7

3. Theta correspondence

The main result to be proven in this paper is a bijective dichotomy:

$$\mathrm{Irr}_g^\circ(G_2) \leftrightarrow \mathrm{Irr}_{g,\mathrm{Spin}}^\circ(\mathrm{PGSp}_6) \sqcup \frac{\mathrm{Irr}_g^\circ(\mathrm{PGL}_3)}{\mathrm{Contra}}.$$

In this section, we begin the proof of this main result. We use theta correspondences in E_6 and E_7 to describe maps for the above dichotomy. We begin with a generic supercuspidal irrep τ of G_2.

- We will define $\overrightarrow{\Theta}_6(\tau)$, a representation of PGL_3, and $\overrightarrow{\Theta}_7(\tau)$, a representation of PGSp_6.
- If $\overrightarrow{\Theta}_6(\tau) = 0$, then $\overrightarrow{\Theta}_7(\tau)$ has a unique generic supercuspidal irreducible subrepresentation.
- Otherwise, and *if* $p \neq 2$, then $\overrightarrow{\Theta}_6(\tau)$ has a unique, up to contragredience, generic supercuspidal irreducible subrepresentation. Even if $p = 2$, $\overrightarrow{\Theta}_6(\tau)$ is a multiplicity-free supercuspidal representation of PGL_3.

By establishing these facts, we establish a map in this section, when $p \neq 2$:

$$\Delta : \mathrm{Irr}_g^\circ(G_2) \to \mathrm{Irr}_g^\circ(\mathrm{PGSp}_6) \sqcup \frac{\mathrm{Irr}_g^\circ(\mathrm{PGL}_3)}{\mathrm{Contra}},$$

where $\Delta(\tau)$ is either the unique (up to isomorphism) generic supercuspidal subrepresentation of $\overrightarrow{\Theta}_7(\tau)$ or the unique (up to isomorphism and contragredience) generic supercuspidal subrepresentation of $\overrightarrow{\Theta}_6(\tau)$.

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3.1 Minimal representations

Let Π_6 and Π_7 denote the minimal representations of the adjoint simple split groups E_6 and E_7, respectively (we refer to [GS05] for definitions and properties of minimal representations). Let σ be a supercuspidal irrep of PGSp_6, let τ be a supercuspidal irrep of G_2, and let ρ be a supercuspidal irrep of PGL_3. We define the following:

$$\overleftarrow{\Theta}_7(\sigma) = \mathrm{Hom}_{\mathrm{PGSp}_6}(\sigma, \Pi_7) \quad \text{and} \quad \overrightarrow{\Theta}_7(\tau) = \mathrm{Hom}_{G_2}(\tau, \Pi_7).$$

Of course, we view $\overleftarrow{\Theta}_7(\sigma)$ as a representation of G_2, and $\overrightarrow{\Theta}_7(\tau)$ as a representation of PGSp_6, via the dual pair (see § 2.3.3):

$$\mathbf{PGSp_6} \times \mathbf{G}_2 \to \mathbf{E}_7.$$

Observe here that we consider embeddings of σ and τ as subrepresentations rather than the more commonly used quotients; however, the injectivity and projectivity of supercuspidals in the category of smooth representations imply that nothing is lost. Note that $\sigma \boxtimes \overleftarrow{\Theta}_7(\sigma)$ is naturally a $(\mathrm{PGSp}_6, \sigma)$-isotypic subspace of Π_7, and $\overrightarrow{\Theta}_7(\tau) \boxtimes \tau$ is naturally a (G_2, τ)-isotypic subspace of Π_7.

Similarly, we define

$$\overleftarrow{\Theta}_6(\rho) = \mathrm{Hom}_{\mathrm{PGL}_3}(\rho, \Pi_6) \quad \text{and} \quad \overrightarrow{\Theta}_6(\tau) = \mathrm{Hom}_{G_2}(\tau, \Pi_6).$$

Here, we view $\overleftarrow{\Theta}_6(\rho)$ as a representation of G_2 and $\overrightarrow{\Theta}_6(\tau)$ as a representation of PGL_3, via the dual pair

$$\mathbf{PGL}_3 \times \mathbf{G}_2 \hookrightarrow \mathbf{E}_6.$$

Observe that $\rho \boxtimes \overleftarrow{\Theta}_6(\rho)$ is naturally a (PGL_3, ρ)-isotypic subspace of Π_6, and $\overrightarrow{\Theta}_6(\tau) \boxtimes \tau$ is naturally a (G_2, τ)-isotypic subspace of Π_6.

3.2 Whittaker functionals

Let N_2 and N_3 be the unipotent radicals of Borel subgroups of G_2 and PGSp_6, respectively. Let $\psi_2 : N_2 \to \mathbb{C}^\times$ and $\psi_3 : N_3 \to \mathbb{C}^\times$ be generic (principal) characters. Since G_2 and PGSp_6 are of adjoint type, these characters are unique up to conjugation by the tori of the respective Borel subgroups. For this reason, τ and σ are unambiguously called *generic* (rather than ψ_2-generic and ψ_3-generic) if $\tau_{N_2,\psi_2} \neq 0$ and $\sigma_{N_3,\psi_3} \neq 0$, respectively.

More generally, when $\mathbf{G}$ is a split adjoint semisimple group over k, and π is a smooth representation of G, we write $\mathrm{Wh}_G(\pi)$ for the space of Whittaker functionals on π, with respect to some maximal unipotent subgroup $\mathbf{N}$ of $\mathbf{G}$ and principal character ψ of N:

$$\mathrm{Wh}_G(\pi) = \mathrm{Hom}_N(\pi, \psi).$$

Thus, τ is called generic if $\mathrm{Wh}_{G_2}(\tau) \neq 0$ and σ is called generic if $\mathrm{Wh}_{\mathrm{PGSp}_6}(\sigma) \neq 0$.

It is important to recall a few equivalent formulations of Whittaker functionals and genericity. While well known, a good treatment can be found in the work of Casselman and Shalika [CS80]. First, since $\pi_{N,\psi}$ is the maximal quotient on which N acts via ψ, we find canonical isomorphisms

$$\mathrm{Wh}_G(\pi) = \mathrm{Hom}_N(\pi, \psi) \cong \mathrm{Hom}_N(\pi_{N,\psi}, \psi) \cong \mathrm{Hom}_{\mathbb{C}}(\pi_{N,\psi}, \mathbb{C}).$$

In particular, $\dim(\mathrm{Wh}_G(\pi)) = \dim(\pi_{N,\psi})$ if one of these vector spaces is finite dimensional.

Next, by Frobenius reciprocity, observe that

$$\mathrm{Wh}_G(\pi) = \mathrm{Hom}_N(\pi, \psi) \cong \mathrm{Hom}_G(\pi, \mathrm{Ind}_N^G \psi).$$

If π is a generic irrep of G, and so $\mathrm{Wh}_G(\pi)$ is non-zero, then π embeds as a subrepresentation of $\mathrm{Ind}_N^G \psi$. The image of π via such an embedding is uniquely determined by π; it is called the Whittaker model of π.

On the other hand, we often consider the *Gelfand–Graev* representation $\text{c-Ind}_N^G \psi$; since this is a submodule of $\mathrm{Ind}_N^G \psi$, we find an injective linear map

$$\mathrm{Hom}_G(\pi, \text{c-Ind}_N^G \psi) \hookrightarrow \mathrm{Hom}_G(\pi, \mathrm{Ind}_N^G \psi) \cong \mathrm{Wh}_G(\pi).$$

In particular, the only irreps of G which occur as *subrepresentations* of a Gelfand–Graev representation are generic irreps, and moreover the uniqueness of Whittaker models implies that

$$\dim \mathrm{Hom}_G(\pi, \text{c-Ind}_N^G \psi) \leqslant 1,$$

for any irrep π of G.

While perhaps not all generic irreps occur as subrepresentations of the Gelfand–Graev representation, we can say more about generic supercuspidal irreps. Corollary 6.5 of [CS80] directly implies the following.

PROPOSITION 3.1. *Suppose that π is a generic supercuspidal irrep of G. Then π occurs as a subrepresentation of* $\text{c-Ind}_N^G \psi$.

Namely, the Whittaker model of a generic supercuspidal irrep of G, *a priori* a G-submodule of $\mathrm{Ind}_N^G \psi$, is in fact a G-submodule of $\text{c-Ind}_N^G \psi$.

3.3 Useful facts

We will be proving that certain smooth representations of G_2 have no generic supercuspidal subrepresentations. To this end, it is useful to have a few criteria that exclude such representations of G_2.

PROPOSITION 3.2. *Let π be a smooth irrep of G_2. Let $\mathbf{H}$ be a subgroup of $\mathbf{G}_2$, such that $\mathbf{H}$ is isomorphic to $\mathbf{SL}_3$ over an algebraic closure $\bar{k}$ of k. If $\pi_H \neq 0$ (there exists a non-zero H-invariant linear functional), then π is not generic.*

Proof. Every such A_2 subgroup $\mathbf{H}$ of $\mathbf{G}_2$ is conjugate over $\bar{k}$ (by the theory of Borel and De Siebenthal [BD49]). All such subgroups arise as stabilizers of quadratic subalgebras of $\mathbb{O}$. Lemma 4.10 of [GS98] now implies the result. $\square$

For $n \geqslant 4$, consider the commuting pair of split groups over k:

$$\mathbf{B}_3 \times \mathbf{B}_{n-4} \hookrightarrow \mathbf{D}_n,$$

where $\mathbf{B}_3 = \mathbf{SO}_7$, $\mathbf{B}_{n-4} = \mathbf{SO}_{2n-7}$, and $\mathbf{D}_n = \mathbf{SO}_{2n}$ are split classical groups labeled by their type. We regard $\mathbf{B}_0$ as the trivial group. Embed $\mathbf{G}_2$ into $\mathbf{B}_3$ via the action of $\mathbf{G}_2$ on $\mathbb{O}_\circ$.

PROPOSITION 3.3. *Let Π_n denote the minimal representation of D_n for $n \geqslant 4$. Then, as a smooth representation of G_2, Π_n does not have any generic supercuspidal subrepresentations.*

Proof. We prove this by induction on n. For the base step, when $n = 4$, the proposition follows directly from [HMS98, Corollary 5.2].

When $n > 4$, consider a maximal parabolic subgroup $\mathbf{P} = \mathbf{MN}$ of $\mathbf{D}_n$ whose Levi component $\mathbf{M}$ satisfies

$$\mathbf{G}_2 \subset \mathbf{B}_3 \subset \mathbf{D}_{n-1} \subset \mathbf{M} \cong \mathbf{GL}_1 \times \mathbf{SO}_{2n-2}.$$

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The adjoint representation of $\mathbf{M}$ on $\mathbf{N}$ is a standard representation of $\mathbf{GL}_1 \times \mathbf{SO}_{2n-2}$; N is a $(2n-2)$-dimensional vector space over k with non-degenerate symmetric bilinear form. Let $\Omega \subset N$ be the set of isotropic vectors in N. By [MS97, Theorem 1.1], there is a filtration of the minimal representation Π_n, as a representation of P:

$$0 \to C_c^\infty(\Omega) \to \Pi_n \to (\Pi_{n-1} \otimes |\det|^{1/(2n-2)}) \oplus |\det|^{(n-2)/(2n-2)} \to 0.$$

By induction, the minimal representation Π_{n-1} of D_{n-1} does not support any generic supercuspidal representations of G_2. The character $|\det|^{(n-2)/(2n-2)}$ supports nothing but the trivial representation of G_2.

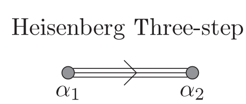
Finally, the representation $C_c^\infty(\Omega)$ of G_2 arises from the action of G_2 on the set of isotropic vectors in N. The stabilizer of such a vector in G_2 is a subgroup of type A_2 as discussed in the previous proposition, a subgroup isomorphic to $[Q, Q]$ for a maximal parabolic $\mathbf{Q} \subset \mathbf{G}_2$, or else all of G_2. By the previous proposition, no generic supercuspidal irreps of G_2 have vectors fixed by an A_2 subgroup. No supercuspidal irreps have vectors fixed by $[Q, Q]$. No non-trivial irreps have vectors fixed by all of G_2. Hence, no generic supercuspidal irreps of G_2 occur (as subrepresentations) in the restriction of Π_n to G_2. □

3.4 Analysis of the correspondences

Here, we begin the analysis of the theta correspondences in E_6 and E_7, focusing on generic supercuspidal representations. We start with the following proposition, which is primarily a consequence of results in the literature.

Proposition 3.4. *Let σ be a generic supercuspidal irrep of* PGSp_6. *Then $\overleftarrow{\Theta}_7(\sigma)$ is a supercuspidal and multiplicity-free representation of G_2. Every irreducible subrepresentation of $\overleftarrow{\Theta}_7(\sigma)$ is generic.*

Proof. First, we prove that $\overleftarrow{\Theta}_7(\sigma)$ is supercuspidal. There are two maximal parabolic subgroups (up to conjugacy) of $\mathbf{G}_2$ which must be considered.



(Heisenberg) Suppose first that $\mathbf{Q}_2 = \mathbf{L}_2\mathbf{U}_2$ is the Heisenberg parabolic subgroup of $\mathbf{G}_2$. If $\overleftarrow{\Theta}_7(\sigma)_{U_2} \neq 0$, then σ occurs in $(\Pi_7)_{U_2}$. The structure of $(\Pi_7)_{U_2}$ as a $\mathrm{PGSp}_6 \times L_2$-module has been described in [MS97, Theorem 7.6]. More precisely, one can pick a maximal parabolic subgroup $\mathbf{Q}_7 = \mathbf{L}_7\mathbf{U}_7$ in $\mathbf{E}_7$ such that $\mathbf{Q}_7 \cap \mathbf{G}_2 = \mathbf{Q}_2$ and $\mathbf{PGSp}_6 \times \mathbf{L}_2$ is contained in the Levi factor $\mathbf{L}_7$ (using the construction of § 2.3.4). Then we have a natural map

$$(\Pi_7)_{U_2} \to (\Pi_7)_{U_7}.$$

By [MS97, Theorem 7.6], the kernel of this map does not support any supercuspidal representations of PGSp_6. In particular, σ must occur in $(\Pi_7)_{U_7}$. By the same result of [MS97], the representation $(\Pi_7)_{U_7}$, as a representation of $\mathbf{L}_7$, has constituents with wave front set supported in the closure of the minimal nilpotent orbit; the constituents are essentially a minimal representation and a trivial representation of $\mathbf{L}_7$. Note that $\mathbf{L}_7$ is a split reductive group $\mathbf{GSpin}_{12}$ of type D_6.

The dual pair $\mathrm{PGL}_2 \times \mathrm{PGSp}_6$ in a group of type D_6 is addressed in [Sav94, § 8], and no generic supercuspidal representations of PGSp_6 can occur. Thus, no generic supercuspidal irreps of PGSp_6 occur in $(\Pi_7)_{U_7}$. Therefore, $\overleftarrow{\Theta}_7(\sigma)_{U_2} = 0$.

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(Three-step) Now, suppose that $\mathbf{Q}_2 = \mathbf{L}_2\mathbf{U}_2$ is the three-step parabolic subgroup of $\mathbf{G}_2$. The structure of $(\Pi_7)_{U_2}$ as a $\mathrm{PGSp}_6 \times L_2$-module has been described in [SG99, Proposition 6.8]. If $\overleftarrow{\Theta}_7(\sigma)_{U_2} \neq 0$, then σ occurs in $(\Pi_7)_{U_2}$.

One can pick a maximal parabolic subgroup $\mathbf{Q}_7 = \mathbf{L}_7\mathbf{U}_7$ in $\mathbf{E}_7$ such that $\mathbf{Q}_7 \cap \mathbf{G}_2 = \mathbf{Q}_2$ and $\mathbf{PGSp}_6 \times \mathbf{L}_2$ is contained in the Levi factor $\mathbf{L}_7$. Such a parabolic subgroup is discussed and called P_1 in [SG99, § 4]. Then we have a natural map

$$(\Pi_7)_{U_2} \to (\Pi_7)_{U_7}.$$

The results of [SG99, Proposition 6.8] imply that the kernel does not support any supercuspidal representations of PGSp_6. In particular, if σ occurs in $(\Pi_7)_{U_2}$, then σ occurs in $(\Pi_7)_{U_7}$. $\mathbf{L}_7$ is isogenous to $\mathrm{GL}_2 \times \mathrm{PGL}_6$.

By considering the Iwahori-fixed vectors, any L_7 constituent of the representation $(\Pi_7)_{U_7}$ is an Iwahori-spherical representation of $\mathrm{GL}_2 \times \mathrm{PGL}_6$ associated to the reflection or trivial representation of the Iwahori Hecke algebra of PGL_6. Thus, $(\Pi_7)_{U_7}$, as a representation of PGL_6, has all constituents appearing in degenerate principal series representations. Such degenerate principal series representations restrict to degenerate principal series representations of PGSp_6, which are not generic.

It follows that σ cannot occur in $(\Pi_7)_{U_7}$. Therefore, $\overleftarrow{\Theta}_7(\sigma)_{U_2} = 0$.

Thus, $\overleftarrow{\Theta}_7(\sigma)$ is a supercuspidal representation of G_2. It follows that $\overleftarrow{\Theta}_7(\sigma)$ is semisimple; a direct sum of supercuspidal irreps.

Next, we recall that $\mathrm{Wh}_{\mathrm{PGSp}_6}(\Pi_7) = (\Pi_7)_{N_3,\psi_3}$ is the Gelfand–Graev module for G_2 [GS04, Proposition 17]:

$$\mathrm{Wh}_{\mathrm{PGSp}_6}(\Pi_7) \cong \text{c-Ind}_{N_2}^{G_2}(\psi_2).$$

Since σ is a generic irreducible supercuspidal representation of PGSp_6, $\mathrm{Wh}_{\mathrm{PGSp}_6}(\sigma)$ is one dimensional, and the embedding $\sigma \boxtimes \overleftarrow{\Theta}_7(\sigma)$ into Π_7 gives an embedding of $\overleftarrow{\Theta}_7(\sigma)$ into the Gelfand–Graev module for G_2.

Since generic (and only generic) supercuspidal irreps appear as subrepresentations of the Gelfand–Graev module, and each appears with multiplicity one, we have shown that $\overleftarrow{\Theta}_7(\sigma)$ is a multiplicity-free (though, at this point, possibly empty) direct sum of generic supercuspidal irreps of G_2. □

To summarize the previous proposition, we have found that if σ is a generic supercuspidal irrep of PGSp_6, then

$$\overleftarrow{\Theta}_7(\sigma) = \bigoplus_{i \in I} \tau_i,$$

where the right-hand side denotes a (possibly empty and possibly infinite) direct sum of distinct (pairwise non-isomorphic) generic supercuspidal irreps of G_2.

Next, we consider $\overrightarrow{\Theta}_6(\tau)$, when τ is a generic supercuspidal irrep of G_2, using the same methods as the previous proposition.

PROPOSITION 3.5. *Let τ be a generic supercuspidal irrep of G_2. Then $\overrightarrow{\Theta}_6(\tau)$ is a supercuspidal and multiplicity-free representation of* PGL_3.

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Proof. First, we demonstrate that $\overrightarrow{\Theta}_6(\tau)$ is supercuspidal. There are two maximal parabolic subgroups (up to conjugacy) of $\mathbf{PGL}_3$ which must be considered.

(Plane stabilizer) Let $\mathbf{Q}_2 = \mathbf{L}_2\mathbf{U}_2$ be the maximal parabolic subgroup of $\mathbf{PGL}_3$ stabilizing a plane in the standard (projective) representation on k^3. There exists a parabolic subgroup $\mathbf{Q}_6 = \mathbf{L}_6\mathbf{U}_6$ of $\mathbf{E}_6$ for which $\mathbf{Q}_6 \cap \mathbf{PGL}_3 = \mathbf{Q}_2$ and $\mathbf{U}_6 \cap \mathbf{PGL}_3 = \mathbf{U}_2$.

Theorem 4.3 of [MS97] describes $(\Pi_6)_{U_2}$ as a $\mathrm{GL}_2 \times G_2$-module; in particular, the kernel of $(\Pi_6)_{U_2} \to (\Pi_6)_{U_6}$ does not support any supercuspidal representations of G_2. It follows that $\overrightarrow{\Theta}_6(\tau)_{U_2} \boxtimes \tau$ is a $(\mathrm{GL}_2 \times G_2)$-submodule of

$$(\Pi_6)_{U_6} \cong (\Pi_5 \otimes |\det|) \oplus (1 \otimes |\det|^2),$$

where Π_5 is the minimal representation of the Levi component $\mathbf{L}_6$ of type D_5. But no generic supercuspidal representations of G_2 occur in the restriction of the minimal (or trivial) representation of Spin_{10} by Proposition 3.3. Thus, $\overrightarrow{\Theta}_6(\tau)_{U_2} = 0$.

(Line stabilizer) Now let $\mathbf{Q}_2' = \mathbf{L}_2'\mathbf{U}_2'$ be the maximal parabolic subgroup of $\mathbf{PGL}_3$ stabilizing a line in the standard representation. Although $\mathbf{Q}_2'$ is not conjugate to a plane-stabilizing parabolic $\mathbf{Q}_2$, there exists an outer automorphism of PGL_3 which exchanges these two types of maximal parabolic subgroups. Furthermore, this outer automorphism extends to an outer automorphism of E_6. The uniqueness of the minimal representation of E_6 now demonstrates that $\overrightarrow{\Theta}_6(\tau)_{U_2'} = 0$ as well.

Hence, we find that $\overrightarrow{\Theta}_6(\tau)$ is supercuspidal. Let $\mathbf{N}_2'$ denote the unipotent radical of a Borel subgroup of $\mathbf{PGL}_3$, and let ψ_2' be a generic character of N_2'. Let $\mathbf{N}_2$ be the unipotent radical of a Borel subgroup of $\mathbf{G}_2$. By [GS04, Proposition 17], it is known that the G_2-Whittaker functionals of Π_6 yield the Gelfand–Graev representation of PGL_3:

$$\mathrm{Wh}_{G_2}(\Pi_6) = (\Pi_6)_{N_2,\psi_2} \cong \text{c-Ind}_{N_2'}^{\mathrm{PGL}_3} \mathbb{C}_{\psi_2'}.$$

Thus, since τ is a generic supercuspidal irrep of G_2, the same arguments as in Proposition 3.4 imply that $\overrightarrow{\Theta}_6(\tau)$ is a multiplicity-free semisimple representation of PGL_3: $\overrightarrow{\Theta}_6(\tau)$ is a direct sum of pairwise non-isomorphic (automatically generic) supercuspidal irreps. □

It is more complicated to analyze $\overrightarrow{\Theta}_7(\tau)$ when τ is a generic supercuspidal irrep of G_2, since $\overrightarrow{\Theta}_7(\tau)$ may or may not be supercuspidal as a representation of PGSp_6. But we may consider the maximal supercuspidal (as a representation of PGSp_6) submodule $\overrightarrow{\Theta}_7^\circ(\tau)$, which fits into a split short exact sequence:

$$0 \to \overrightarrow{\Theta}_7^\circ(\tau) \to \overrightarrow{\Theta}_7(\tau) \to \overrightarrow{\Theta}_7^{ns}(\tau) \to 0.$$

Proposition 3.6. *Let $\mathbf{Q}_3$ denote the Siegel parabolic subgroup of $\mathbf{PGSp}_6$ (a maximal parabolic subgroup with abelian unipotent radical). Then the PGSp_6-module $\overrightarrow{\Theta}_7^{ns}(\tau)$ is a submodule of $\mathrm{Ind}_{Q_3}^{\mathrm{PGSp}_6} \overrightarrow{\Theta}_6(\tau) \otimes |\det|$. In particular, $\overrightarrow{\Theta}_7^{ns}(\tau)$ is a (possibly empty and possibly infinite) direct sum of finite-length representations of PGSp_6. If $\overrightarrow{\Theta}_6(\tau) = 0$, then $\overrightarrow{\Theta}_7(\tau)$ is supercuspidal.*

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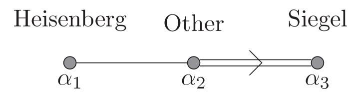
Proof. We consider the Jacquet modules of $\overrightarrow{\Theta}_7(\tau)$, for the three (conjugacy classes of) maximal parabolic subgroups in $\mathbf{PGSp_6}$.



The global analogues of the following computations are carried out in Case (4) of the proof of Theorem 3.1 of [GRS97].

(Heisenberg) First, let $\mathbf{Q}_3 = \mathbf{L}_3\mathbf{U}_3$ be the 'Heisenberg parabolic', whose Levi component $\mathbf{L}_3$ is a split group $\mathrm{GSpin}_5 \cong \mathrm{GSp}_4$. We find that $\sigma_{U_3} \boxtimes \tau$ is a quotient of $(\Pi_7)_{U_3}$ as representations of $L_3 \times G_2$. The unipotent group $\mathbf{U}_3$ is five dimensional, with one-dimensional center $\mathbf{Z}_3$; there exists a parabolic subgroup $\mathbf{Q}_7 = \mathbf{L}_7\mathbf{U}_7$ of $\mathbf{E}_7$ such that $\mathbf{L}_7$ is isomorphic to $\mathbf{GSpin}_{12}$, and $\mathbf{U}_7$ is a Heisenberg group of dimension 33 (with one-dimensional center $\mathbf{Z}_7$). Furthermore, one may choose this parabolic subgroup in such a way that $\mathbf{Q}_7 \cap \mathbf{PGSp_6} = \mathbf{Q}_3$, $\mathbf{U}_7 \cap \mathbf{PGSp_6} = \mathbf{U}_3$, and $\mathbf{Z}_7 = \mathbf{Z}_3$. Furthermore, this gives an embedding $\mathbf{L}_3 \times \mathbf{G}_2 \hookrightarrow \mathbf{L}_7 = \mathbf{GSpin}_{12}$.

Now, $\overrightarrow{\Theta}_7(\tau)_{U_3} \boxtimes \tau$ is a subrepresentation of $(\Pi_7)_{U_3}$. To study $(\Pi_7)_{U_3}$, we examine a commutative diagram with exact rows and columns.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & W \arrow[r] \arrow[d] & (\Pi_7)_{Z_3} \arrow[r] \arrow[d] & (\Pi_7)_{U_3} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_c^\infty(\Omega) \arrow[r] & (\Pi_7)_{Z_7} \arrow[r] \arrow[d] & (\Pi_7)_{U_7} \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Here, Ω denotes the smallest non-trivial L_7-orbit in the 32-dimensional vector space U_7/Z_7; this can be identified with the 15-dimensional quotient GSpin_{12}/Q_6, where Q_6 is a minuscule maximal parabolic subgroup (with Levi subgroup of type A_5) of GSpin_{12}. Geometrically, Ω can be viewed as a Grassmannian of isotropic 6-spaces in the 12-dimensional standard representation V of Spin_{12}.

From [MS97, Theorem 6.1], the kernel of $(\Pi_7)_{Z_7} \to (\Pi_7)_{U_7}$ can be identified, as a Q_7-module, with $C_c^\infty(\Omega)$. We are led to consider the action and orbits of $G_2 \times \mathrm{Spin}_5$ on Ω. It helps to study the action of $\mathrm{Spin}_7 \times \mathrm{Spin}_5$ on Ω. Here, the embedding of $\mathrm{Spin}_7 \times \mathrm{Spin}_5$ in Spin_{12} corresponds to a decomposition $V = V_7 \oplus V_5$ of the standard representation of Spin_{12}.

Such actions were studied by Kudla [Kud86, Proposition 3.4]. If $\omega \in \Omega$ corresponds to an isotropic 6-space Λ_ω, then the projection of Λ_ω onto V_7 is at least one dimensional. It follows that ω is stabilized by some maximal parabolic subgroup Q of Spin_7.

It follows that the stabilizer S_ω of ω in G_2 contains a maximal parabolic subgroup of G_2, or S_ω contains a subgroup of type A_2 (by the arguments of Proposition 1.1). If S_ω contains a maximal parabolic subgroup of G_2, then $C_c^\infty(G_2/S_\omega)$ does not support any supercuspidal representations of G_2. If S_ω contains a subgroup of type A_2, then $C_c^\infty(G_2/S_\omega)$ does not support any generic supercuspidal representations of G_2 by Proposition 3.2. Thus, $C_c^\infty(\Omega)$ does not support any generic supercuspidal representations of G_2.

Since $\mathrm{Ker}((\Pi_7)_{U_3} \to (\Pi_7)_{U_7})$ is a quotient of $C_c^\infty(\Omega)$ (by the snake lemma), we find that $\overrightarrow{\Theta}_7(\tau)_{U_3} \boxtimes \tau$ is a subrepresentation of $(\Pi_7)_{U_7}$. But this implies that τ occurs in the restriction of the minimal representation of Spin_{12} or else $\overrightarrow{\Theta}_7(\tau)_{U_3} = 0$. By Proposition 3.3, no generic supercuspidal representations of G_2 occur in this restriction. It follows that $\overrightarrow{\Theta}_7(\tau)_{U_3} = 0$.

(Other) Next, let $\mathbf{Q}_3 = \mathbf{L}_3\mathbf{U}_3$ denote the 'other parabolic', with $\mathbf{L}_3$ isogenous to $\mathrm{GL}_2 \times \mathrm{SL}_2$. The unipotent radical $\mathbf{U}_3$ has three-dimensional center $\mathbf{Z}_3$ and four-dimensional quotient $\mathbf{U}_3/\mathbf{Z}_3$. Z_3 can be identified with the space $M_\circ$ of two-by-two matrices with trace zero, and U_3/Z_3 can be identified with the space M of all two-by-two matrices.

We find that $\overrightarrow{\Theta}_7(\tau)_{U_3} \boxtimes \tau$ is a subrepresentation of $(\Pi_7)_{U_3}$, as representations of $L_3 \times G_2$.

There exists a parabolic subgroup $\mathbf{Q}_7 = \mathbf{L}_7\mathbf{U}_7$ such that $\mathbf{L}_7$ is isogenous to $\mathrm{GSpin}_{10} \times \mathrm{SL}_2$, $\mathbf{U}_7$ is a two-step unipotent group with 10-dimensional center $\mathbf{Z}_7$, $\mathbf{Q}_7 \cap \mathbf{PGSp}_6 = \mathbf{Q}_3$, and $\mathbf{L}_7 \cap \mathbf{PGSp}_6 = \mathbf{L}_3$. Z_7 can be identified with the space $M_\circ \oplus \mathbb{O}_\circ$ of pairs (m, ω), and U_7/Z_7 can be identified with the (32-dimensional) space $M \otimes \mathbb{O}$. This arises from the construction of § 2.3.3. This parabolic arises in a similar computation in [SG99], and our $\mathbf{Q}_7$ corresponds to the parabolic called P and associated to the vertex α_4 in [SG99].

There are natural short exact sequences which we describe and analyze below:

$$0 \to C_c^\infty(\Omega, \mathcal{S}) \to (\Pi_7) \to (\Pi_7)_{Z_7} \to 0,$$
$$0 \to C_c^\infty(\Omega') \to (\Pi_7)_{Z_7} \to (\Pi_7)_{U_7} \to 0.$$

Here, Ω is the set of non-trivial characters ω of Z_7 for which $(\Pi_7)_{Z_7,\omega} \neq 0$. On Ω, $\mathcal{S}$ is a $\mathbf{Q}_7$-equivariant sheaf whose fibre over $\omega \in \Omega$ is an irreducible representation of U_7 with central character corresponding to ω. One can compare this to [SG99, § 6].

Similarly, Ω' is the set of non-trivial characters of U_7 for which $(\Pi_7)_{U_7,\omega} \neq 0$. Identifying characters of U_7 with $\mathbb{O} \otimes M$, a minuscule representation of L_7, Ω' can be identified with the quotient L_7/P_6, where $\mathbf{P}_6$ is a minuscule parabolic subgroup of $\mathbf{L}_7$.

Taking U_3 coinvariants in each of the short exact sequences, we are led to consider $C_c^\infty(\Omega, \mathcal{S})_{U_3}$ and $C_c^\infty(\Omega')_{U_3}$. In the first case, we find that $C_c^\infty(\Omega, \mathcal{S})_{U_3}$ is a quotient of $C_c^\infty(\Omega, \mathcal{S})_{Z_3}$. We compute

$$C_c^\infty(\Omega, \mathcal{S})_{Z_3} \cong C_c^\infty(\Omega^{\perp Z_3}, \mathcal{S}),$$

where $\Omega^{\perp Z_3}$ can be identified:

$$\begin{aligned}\Omega^{\perp Z_3} &= \{(m, \omega) \in M_\circ \oplus \mathbb{O}_\circ : N(\omega) - N(m) = 0 \text{ and } m = 0\} \\ &= \{\omega \in \mathbb{O} : \omega^2 = 0\}.\end{aligned}$$

It follows that

$$C_c^\infty(\Omega, \mathcal{S})_{Z_3} \cong \mathrm{Ind}_{P_\omega}^{G_2} \mathcal{S}_\omega,$$

where $\mathcal{S}_\omega$ is the fibre of $\mathcal{S}$ over $\omega \in \mathbb{O}$, which satisfies $\omega^2 = 0$, and P_ω is the maximal parabolic subgroup of G_2 stabilizing ω. The representation $\mathcal{S}_\omega$ of P_ω factors through the Levi quotient $L_\omega \cong \mathrm{GL}_2$ of P_ω. It follows that $C_c^\infty(\Omega, \mathcal{S})_{Z_3}$ and hence $C_c^\infty(\Omega, \mathcal{S})_{U_3}$ does not support any supercuspidal representations of G_2.

Next we are led to consider $C_c^\infty(\Omega')$. Every point of Ω' corresponds to an isotropic 5-plane Λ in $\mathbb{O}_\circ \oplus M_\circ$ (the standard representation of $\mathrm{GSpin}_{10} \subset L_7$), since GSpin_{10} acts via the spin representation on U_7/Z_7. The projection of Λ onto $\mathbb{O}_\circ$ is at least two dimensional; hence, Λ is stabilized by a maximal parabolic subgroup of $\mathrm{Spin}_7 \subset \mathrm{Spin}_{10}$. Hence, Λ is stabilized by a maximal parabolic subgroup of G_2, or by a subgroup of type A_2 in G_2. It follows that $C_c^\infty(\Omega')$ does not support any generic supercuspidal representations of G_2 using Propositions 3.2 and 3.3.

By the snake lemma argument as before, we find that $\overrightarrow{\Theta}_7(\tau)_{U_3} \boxtimes \tau$ occurs as a subrepresentation of $(\Pi_7)_{U_7}$. The representation $(\Pi_7)_{U_7}$ of L_7 has wave front set supported in the minimal orbit. If $\overrightarrow{\Theta}_7(\tau)_{U_3}$ were non-trivial, then τ would occur in a theta correspondence $G_2 \times (\mathrm{Spin}(3) \times \mathrm{SL}_2) \subset \mathrm{GSpin}_{10} \times \mathrm{SL}_2$. But no generic supercuspidal representations of G_2 occur in such a correspondence, by Proposition 3.3. Hence, $\overrightarrow{\Theta}_7(\tau)_{U_3} = 0$.

(Siegel) Finally, let $\mathbf{Q}_3 = \mathbf{L}_3\mathbf{U}_3$ denote the 'Siegel parabolic', with $\mathbf{L}_3 \cong \mathbf{GL}_3$. We find that $\overrightarrow{\Theta}_7(\tau)_{U_3} \boxtimes \tau$ is a subrepresentation of $(\Pi_7)_{U_3}$, as representations of $\mathrm{GL}_3 \times G_2$. Let $\mathbf{Q}_7$ denote a maximal parabolic subgroup of $\mathbf{E}_7$ whose Levi component has derived subgroup $\mathbf{E}_6$, such that $\mathbf{Q}_7 \cap \mathbf{PGSp}_6 = \mathbf{Q}_3$. These embeddings and parabolics arise from the construction of § 2.3.2. By [MS97, Theorem 5.3], the kernel of $(\Pi_7)_{U_3} \twoheadrightarrow (\Pi_7)_{U_7}$ does not support any supercuspidal representations of G_2. It follows that $\overrightarrow{\Theta}_7(\tau)_{U_3} \boxtimes \tau$ is a subrepresentation of $(\Pi_7)_{U_7}$.

By [MS97, Theorem 5.3] again, there is a $G_2 \times \mathrm{GL}_3$-module isomorphism

$$(\Pi_7)_{U_7} \cong (\Pi_6 \otimes |\det|) \oplus (1 \otimes |\det|^2).$$

Taking (G_2, τ)-isotypic components, we find an isomorphism of GL_3-modules:

$$\overrightarrow{\Theta}_7(\tau)_{U_3} \cong \overrightarrow{\Theta}_6(\tau) \otimes |\det|.$$

For the rest of the proof, let $\mathbf{Q}_3 = \mathbf{L}_3\mathbf{U}_3$ denote the Siegel parabolic subgroup of $\mathbf{PGSp}_6$. The previous computations and Frobenius reciprocity yield a morphism of PGSp_6-modules:

$$\overrightarrow{\Theta}_7(\tau) \to \mathrm{Ind}_{Q_3}^{\mathrm{PGSp}_6} \overrightarrow{\Theta}_6(\tau) \otimes |\det|.$$

Moreover, the kernel of this morphism is a submodule of $\overrightarrow{\Theta}_7(\tau)$ whose U_3-coinvariants vanish. But since all other (with respect to the Heisenberg parabolic and 'other' parabolic) Jacquet modules of $\overrightarrow{\Theta}_7(\tau)$ vanish, the kernel of this morphism is a supercuspidal PGSp_6-submodule of $\overrightarrow{\Theta}_7(\tau)$. Conversely, every supercuspidal PGSp_6-submodule of $\overrightarrow{\Theta}_7(\tau)$ is contained in the kernel of the morphism, since supercuspidals do not occur as subrepresentations of parabolically induced representations.

It follows that there is an injective morphism of PGSp_6-modules:

$$\overrightarrow{\Theta}_7^{ns}(\tau) \hookrightarrow \mathrm{Ind}_{Q_3}^{\mathrm{PGSp}_6} \overrightarrow{\Theta}_6(\tau) \otimes |\det|.$$

The previous proposition implies that there exists a set of pairwise non-isomorphic supercuspidal irreps $\{\rho_i\}_{i \in I}$ of PGL_3, such that

$$\overrightarrow{\Theta}_6(\tau) \cong \bigoplus_{i \in I} \rho_i.$$

It follows that there is an injective morphism of PGSp_6-modules:

$$\overrightarrow{\Theta}_7^{ns}(\tau) \hookrightarrow \bigoplus_{i \in I} \mathrm{Ind}_{Q_3}^{\mathrm{PGSp}_6} \rho \otimes |\det|.$$

Although there may be an infinite number of summands on the right-hand side above, only finitely many lie in any given Bernstein component for PGSp_6. We find that $\overrightarrow{\Theta}_7^{ns}(\tau)$ is a (possibly infinite and possibly empty) direct sum of finite-length representations of PGSp_6. Moreover, if $\overrightarrow{\Theta}_6(\tau) = 0$, then $\overrightarrow{\Theta}_7^{ns}(\tau)$ vanishes, and so $\overrightarrow{\Theta}_7(\tau)$ is supercuspidal. □

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To synthesize the previous propositions, we find that for any generic supercuspidal irrep τ of G_2, there are a set $\{\sigma_j\}_{j\in J}$ of supercuspidal irreps of PGSp_6, a set $\{\rho_i\}_{i\in I}$ of supercuspidal irreps of PGL_3, and a set of finite-length PGSp_6 modules $\{\pi_i\}_{i\in I}$ satisfying

$$\overrightarrow{\Theta}_6(\tau) \cong \bigoplus_{i\in I} \rho_i,$$

$$\overrightarrow{\Theta}_7(\tau) \cong \bigoplus_{j\in J} \sigma_j \oplus \bigoplus_{i\in I} \pi_i \quad \text{and} \quad \pi_i \subset \mathrm{Ind}_{Q_3}^{\mathrm{PGSp}_6} \rho_i \quad \text{for all } i \in I.$$

The above decomposition refines the decomposition of $\overrightarrow{\Theta}_7(\tau)$ into supercuspidal and non-supercuspidal parts:

$$\overrightarrow{\Theta}{}_7^{\circ}(\tau) \cong \bigoplus_{j\in J} \sigma_j \quad \text{and} \quad \overrightarrow{\Theta}{}_7^{ns}(\tau) \cong \bigoplus_{i\in I} \pi_i.$$

Proposition 3.7. *Let τ be a generic supercuspidal irrep of G_2. Let $\{\sigma_j\}_{j\in J}$, $\{\rho_i\}_{i\in I}$, and $\{\pi_i\}_{i\in I}$ be the representations of PGSp_6, PGL_3, and PGSp_6 in the above decomposition. Then $\overrightarrow{\Theta}_7(\tau)$ is non-trivial (so $I \sqcup J \neq \emptyset$). Moreover, exactly one of the following statements holds.*

(i) *There exists exactly one $j \in J$ such that σ_j is generic. There does not exist $i \in I$ such that π_i is generic.*

(ii) *There exists exactly one $i \in I$ such that π_i is generic. There does not exist $j \in J$ such that π_j is generic.*

Proof. For τ a generic supercuspidal irrep of G_2, τ occurs with multiplicity one in the Gelfand–Graev module:

$$\dim(\mathrm{Hom}_{G_2}(\tau, \text{c-Ind}_{N_2}^{G_2} \psi_2)) = 1.$$

But, using [GS04, Proposition 17] again,

$$\mathrm{Wh}_{\mathrm{PGSp}_6}(\Pi_7) = (\Pi_7)_{N_3,\psi_3} \cong \text{c-Ind}_{N_2}^{G_2}(\psi_2).$$

Thus, we find that

$$\dim(\mathrm{Hom}_{G_2}(\tau, (\Pi_7)_{N_3,\psi_3})) = \dim(\mathrm{Hom}_{G_2}(\tau, \Pi_7))_{N_3,\psi_3} = 1.$$

Thus, $\mathrm{Wh}_{\mathrm{PGSp}_6}(\overrightarrow{\Theta}_7(\tau))$ is one dimensional. In particular, $\overrightarrow{\Theta}_7(\tau)$ is non-trivial.

Now, we apply the decomposition

$$\overrightarrow{\Theta}_7(\tau) \cong \bigoplus_{j\in J} \sigma_j \oplus \bigoplus_{i\in I} \pi_i \quad \text{and} \quad \pi_i \subset \mathrm{Ind}_{Q_3}^{\mathrm{PGSp}_6} \rho_i \quad \text{for all } i \in I.$$

Taking Whittaker functionals, we find that

$$\mathrm{Wh}_{\mathrm{PGSp}_6}(\overrightarrow{\Theta}_7(\tau)) \cong \bigoplus_{j\in J} \mathrm{Wh}_{\mathrm{PGSp}_6}(\sigma_j) \oplus \bigoplus_{i\in I} \mathrm{Wh}_{\mathrm{PGSp}_6}(\pi_i).$$

Since the left-hand side is one dimensional, precisely one summand on the right-hand side is one dimensional and all other summands on the right-hand side vanish. The result follows immediately. $\square$

When the residue characteristic p is odd, the representations $\mathrm{Ind}_{Q_3}^{\mathrm{PGSp}_6}(\rho \otimes |\det|)$ are irreducible and generic, whenever ρ is a supercuspidal irrep of PGL_3. This significantly simplifies the analysis of the theta correspondence, in the following way.

Proposition 3.8. *Suppose that $p \neq 2$. Let τ be a generic supercuspidal irrep of G_2. Then, if $\overrightarrow{\Theta}_6(\tau) \neq 0$, $\overrightarrow{\Theta}_6(\tau)$ has a unique irreducible subrepresentation, up to contragredience: $\overrightarrow{\Theta}_6(\tau) = \rho \oplus \tilde{\rho}$ for some supercuspidal irrep ρ of* PGL_3.

Proof. If ρ is an irreducible subrepresentation of $\overrightarrow{\Theta}_6(\tau)$ (and hence ρ is generic and supercuspidal), then $\rho \boxtimes \tau$ occurs as a quotient (by the injectivity and projectivity of supercuspidals) of the minimal representation Π_6 of the adjoint group E_6. But we have seen that if $\mathbf{Q}_3 = \mathbf{L}_3\mathbf{U}_3$ is the Siegel parabolic subgroup of PGSp_6, then there is a surjective map of GL_3-modules:

$$\overrightarrow{\Theta}_7(\tau)_{U_3} \twoheadrightarrow (\rho \otimes |\det|).$$

By Frobenius reciprocity, we find a non-trivial map of PGSp_6-modules:

$$\overrightarrow{\Theta}_7(\tau) \to \mathrm{Ind}_{Q_3}^{\mathrm{PGSp}_6}(\rho \otimes |\det|).$$

Let π denote this induced representation, $\pi = \mathrm{Ind}_{Q_3}^{\mathrm{PGSp}_6}(\rho \otimes |\det|)$. Since $p \neq 2$, the representation ρ is *not* self-contragredient, and so π is an irreducible generic representation of PGSp_6.

Thus, π must be the *unique* generic summand of $\overrightarrow{\Theta}_7(\tau)$ in the decomposition

$$\overrightarrow{\Theta}_7(\tau) \cong \bigoplus_{j \in J} \sigma_j \oplus \bigoplus_{i \in I} \pi_i.$$

By the geometric lemma and Frobenius reciprocity (using the fact that ρ is supercuspidal), the only representations of GL_3 which parabolically induce to give this representation π of PGSp_6 are ρ and its contragredient $\tilde{\rho}$. Hence, $\overrightarrow{\Theta}_6(\tau)$ contains a unique irreducible subrepresentation up to contragredience, and this irrep and its contragredient are supercuspidal. This demonstrates that

$$\rho \subset \overrightarrow{\Theta}_6(\tau) \subset \rho \oplus \tilde{\rho}.$$

Lastly, note that the map $\overrightarrow{\Theta}_7(\tau) \to \pi$ is surjective, from which it follows that the map

$$\overrightarrow{\Theta}_7(\tau)_{U_3} \to \pi_{U_3}$$

is also surjective. But both $\rho \otimes |\det|$ and $\tilde{\rho} \otimes |\det|$ occur in π_{U_3}. Since $\overrightarrow{\Theta}_7(\tau)_{U_3} \cong \overrightarrow{\Theta}_6(\tau) \otimes |\det|$, we find that both ρ and $\tilde{\rho}$ occur in $\overrightarrow{\Theta}_6(\tau)$. □

Using the previous propositions, we find (regardless of residue characteristic) the following.

Theorem 3.9. *Suppose that τ is a generic supercuspidal irrep of G_2. Then either there exists a unique generic supercuspidal irreducible subrepresentation σ of $\overrightarrow{\Theta}_7(\tau)$ or else there exists a unique, up to contragredience, generic supercuspidal irreducible subrepresentation ρ of $\overrightarrow{\Theta}_6(\tau)$ for which the generic summand of $\mathrm{Ind}_{Q_3}^{\mathrm{PGSp}_6}(\rho \otimes |\det|)$ occurs in $\overrightarrow{\Theta}_7(\tau)$.*

In this way, the theta correspondences yield a map:

$$\Delta : \mathrm{Irr}_g^\circ(G_2) \to \mathrm{Irr}_g^\circ(\mathrm{PGSp}_6) \sqcup \frac{\mathrm{Irr}_g^\circ(\mathrm{PGL}_3)}{\mathrm{Contra}},$$
$$\tau \mapsto \sigma \quad \textit{or} \quad \{\rho, \tilde{\rho}\}.$$

When $p \neq 2$, we find that the dichotomy map is given somewhat simply by

$$\Delta(\tau) = \begin{cases} \sigma & \text{if } \overrightarrow{\Theta}_6(\tau) = 0, \\ \{\rho, \tilde{\rho}\} & \text{if } \overrightarrow{\Theta}_6(\tau) \neq 0. \end{cases}$$

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However, when $p=2$, it is possible *a priori* that a self-contragredient supercuspidal irrep ρ occurs as a summand of $\Theta_6(\tau)$, the *non-generic* summand π of $\mathrm{Ind}_{Q_3}^{\mathrm{PGSp}_6} \rho \otimes |\det|$ occurs as a summand of $\overrightarrow{\Theta}_7(\tau)$, and still a generic supercuspidal representation of PGSp_6 occurs as a summand of $\overrightarrow{\Theta}_7(\tau)$. We cannot yet exclude such a strange possibility.

4. Shalika functionals

4.1 The Shalika subgroup

It is convenient hereafter to view $\mathbf{GSp_6}$ in the traditional way, as a group of symplectic similitudes. We let $\mathbf{M_2}$ denote the abelian unipotent algebraic group of two-by-two matrices (under addition); if g is a matrix, we write g^{T} for its transpose.

Let $I = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, and

$$J_3 = \left(\begin{array}{c|c|c} 0 & 0 & J \\ \hline 0 & J & 0 \\ \hline J & 0 & 0 \end{array}\right).$$

Let $\mathbf{GSp_6}$ be the algebraic group of symplectic similitudes:

$$\mathbf{GSp}_6 = \{g \in \mathbf{GL}_6 : gJ_3 g^{\mathsf{T}} = \mathrm{sim}(g)\cdot J_3 \text{ for some } \mathrm{sim}(g) \in \mathbf{GL}_1\}.$$

The resulting character $\mathrm{sim} : \mathbf{GSp_6} \to \mathbf{GL}_1$ is called the similitude character.

Let $\mathbf{Q}_3 = \mathbf{L}_3 \mathbf{U}_3$ be the maximal parabolic subgroup of $\mathbf{GSp_6}$, with Levi component

$$\mathbf{L}_3 = \left\{ \left(\begin{array}{c|c|c} g & 0 & 0 \\ \hline 0 & h & 0 \\ \hline 0 & 0 & \det(g^{-1}h)\cdot g \end{array}\right) : g, h \in \mathbf{GL}_2 \right\}$$

and unipotent radical

$$\mathbf{U}_3 = \left\{ \left(\begin{array}{c|c|c} I & X & Z \\ \hline 0 & I & Y \\ \hline 0 & 0 & I \end{array}\right) : X, Y, Z \in \mathbf{M}_2, XJ + JY^{\mathsf{T}} = 0, ZJ + JZ^{\mathsf{T}} = -XJX^{\mathsf{T}} \right\}.$$

The center of $\mathbf{U}_3$ is three dimensional,

$$\mathbf{Z}_3 = \left\{ \left(\begin{array}{c|c|c} I & 0 & Z \\ \hline 0 & I & 0 \\ \hline 0 & 0 & I \end{array}\right) : ZJ + JZ^{\mathsf{T}} = 0 \right\}.$$

There is an isomorphism of unipotent groups $\mathbf{U}_3/\mathbf{Z}_3 \to \mathbf{M}_2$, given by

$$\left(\begin{array}{c|c|c} I & X & Z \\ \hline 0 & I & Y \\ \hline 0 & 0 & I \end{array}\right) \mapsto X.$$

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There is also an isomorphism of reductive groups $\mathbf{L}_3 \to \mathbf{GL}_2 \times \mathbf{GL}_2$, given by

$$\left(\begin{array}{c|c|c} g & 0 & 0 \\ \hline 0 & h & 0 \\ \hline 0 & 0 & \det(g^{-1}h)\cdot g \end{array}\right) \mapsto (g,h).$$

With these identifications, the conjugation action of $\mathbf{L}_3$ on $\mathbf{U}_3/\mathbf{Z}_3$ is given by

$$(g,h)\cdot X = gXh^{-1}.$$

Let $\Delta : \mathbf{GL}_2 \to \mathbf{GL}_2 \times \mathbf{GL}_2 \cong \mathbf{L}_3$ denote the diagonal embedding (there should be no risk of confusing this Δ with the dichotomy map in other sections). If $g \in \mathbf{GL}_2$, then $\Delta(g)$ is identified with an element of $\mathbf{L}_3 \subset \mathbf{GSp}_6$:

$$\Delta(g) = \left(\begin{array}{c|c|c} g & 0 & 0 \\ \hline 0 & g & 0 \\ \hline 0 & 0 & g \end{array}\right).$$

Then we write $\mathbf{S}$ for the 'Shalika subgroup':

$$\mathbf{S} = \Delta(\mathbf{GL}_2) \ltimes \mathbf{U}_3 \subset \mathbf{Q}_3.$$

Observe also that the Shalika subgroup has another interpretation: if A is any k-algebra, consider the degenerate cubic A-algebra $A[\epsilon]/\langle \epsilon^3 \rangle$. Then there is a natural inclusion (of codimension one):

$$\mathbf{S}(A) \subset \mathbf{GL}_2(A[\epsilon]/\langle \epsilon^3 \rangle).$$

Define a character ψ_3 of U_3 by $\psi_3(u) = \psi_k(-\mathrm{Tr}(X))$ (for a matrix $u \in U_3$ projecting to $X \in M_2 \cong U_3/Z_3$). $\Delta(\mathrm{GL}_2)$ is precisely the centralizer of the character ψ_3 in L; hence, the character ψ_3 can be extended uniquely to a character ψ_S of S such that $\psi_S(\Delta(g)) = 1$ for all $g \in \mathrm{GL}_2$.

When σ is a smooth representation of GSp_6, we define the space of *Shalika functionals* by

$$\mathrm{Sh}(\sigma) = \mathrm{Hom}_S(\sigma, \psi_S).$$

Note that, if σ has a non-zero Shalika functional, then the central character of σ is trivial. The main goal of this section is to demonstrate that for supercuspidal irreps σ of PGSp_6, $\dim(\mathrm{Sh}(\sigma)) \leqslant 1$; the 'uniqueness' of Shalika functionals.

Our methods are similar to many other papers; we mention the work of Jacquet and Rallis [JR96], who proved uniqueness of Shalika models for GL_{2n}. The k-points of their 'Shalika subgroup' can be identified with $\mathrm{GL}_n(k[\epsilon]/\langle \epsilon^2 \rangle)$. While their Shalika functionals are related to a degenerate quadratic algebra, ours are related to a degenerate cubic algebra.

4.2 Double cosets

If $g \in \mathrm{GSp}_6$, then its transpose g^{T} is also an element of GSp_6, and the transpose is an involution (anti-automorphism of order two) of $\mathbf{GSp}_6$. If $\mathbf{H} \subset \mathbf{G}$ is an algebraic subgroup, we write $\mathbf{H}^{\mathsf{T}}$ for its transpose.

We will require an explicit description of the double cosets $\mathbf{Q}_3^{\mathsf{T}} \backslash \mathbf{GSp}_6 / \mathbf{Q}_3$ as well as $\mathbf{S}^{\mathsf{T}} \backslash \mathbf{GSp}_6 / \mathbf{S}$. As $\mathbf{Q}_3$ is a maximal parabolic subgroup of $\mathbf{GSp}_6$, the first is a routine computation; it suffices to find representatives for double cosets in the Weyl group of type C_3, modulo the parabolic subgroup of type $\mathsf{A}_1 \times \mathsf{A}_1$. For this, we define elements of $\mathbf{GSp}_6$ corresponding to simple

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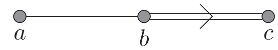
root reflections a, b, c (though we refrain from identifying a maximal torus, Borel subgroup, etc):



$$a = \left(\begin{array}{c|c|c} J & 0 & 0 \\ \hline 0 & I & 0 \\ \hline 0 & 0 & J \end{array}\right), \quad c = \left(\begin{array}{c|c|c} I & 0 & 0 \\ \hline 0 & J & 0 \\ \hline 0 & 0 & I \end{array}\right), \quad b = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

Proposition 4.1. *The algebraic variety* $\mathbf{GSp}_6$ *can be decomposed as a finite disjoint union*

$$\mathbf{GSp}_6 = \bigsqcup_{\sigma \in \Sigma} \mathbf{Q}_3^{\mathsf{T}} \sigma \mathbf{Q}_3,$$

where

$$\Sigma = \{1, b, bcb, bacb, bcabacb\}.$$

Proof. The non-trivial shortest representatives for double cosets in the Weyl group are given by words in a, b, c, which begin and end with b. These can be found by direct computation, using the relations in the Coxeter group. $\square$

Define an embedding η of $\mathbf{GL}_2$ into $\mathbf{L}_3$ by

$$\eta(g) = \left(\begin{array}{c|c|c} I & 0 & 0 \\ \hline 0 & g & 0 \\ \hline 0 & 0 & \det(g) \cdot I \end{array}\right).$$

Then it can be easily verified that

$$\mathbf{L}_3 = \Delta(\mathbf{GL}_2)\eta(\mathbf{GL}_2) = \eta(\mathbf{GL}_2)\Delta(\mathbf{GL}_2).$$

The previous proposition now implies the following corollary.

Corollary 4.2. *The algebraic variety* $\mathbf{GSp}_6$ *can be decomposed as a disjoint union*

$$\mathbf{GSp}_6 = \bigsqcup_{\sigma \in \Sigma} \mathbf{S}^{\mathsf{T}} \eta(\mathbf{GL}_2) \sigma \eta(\mathbf{GL}_2) \mathbf{S}.$$

Let $\mathbf{R} = \mathbf{Q}_3^{\mathsf{T}} \sigma \mathbf{Q}_3$ be a double coset in $\mathbf{GSp}_6$. Then we find that

$$\mathbf{R}^{\mathsf{T}} = \mathbf{Q}_3^{\mathsf{T}} (\sigma^{\mathsf{T}}) \mathbf{Q}_3 = \mathbf{R}.$$

If $s \in S$, then we define a character ψ_S^{T} of S^{T} by

$$\psi_S^{\mathsf{T}}(s) = \psi_S(s^{\mathsf{T}}).$$

4.3 Distributions

If X is a subset of GSp_6 and $X = S^{\mathsf{T}} X S$, then there is a natural action $\ell \times \rho$ of $S^{\mathsf{T}} \times S$ on $C_c^\infty(X)$, given by

$$[\ell(s)\rho(t)f](x) = [\rho(t)\ell(s)f](x) = f(s^{-1}xt),$$

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for all $s \in S^\mathsf{T}$, $t \in S$, $x \in X$, $f \in C_c^\infty(X)$. If T is a distribution on X, i.e., T is a linear functional on $C_c^\infty(X)$, then we say that T is (S, ψ, T)-invariant if, for all $s \in S^\mathsf{T}$, $t \in S$, $f \in C_c^\infty(X)$,

$$T((\ell(s)\rho(t)f) = \psi_S^\mathsf{T}(s)\psi_S(t^{-1})T(f).$$

We frequently apply the following restrictions on the support of such distributions.

(R1) If $s \in S$, $g \in G$, $gsg^{-1} \in S^\mathsf{T}$, and $\psi_S(s) \neq \psi_S^\mathsf{T}(gsg^{-1})$, then the coset $S^\mathsf{T} gS$ does not support any (S, ψ, T)-invariant distributions.

(R2) If $s \in S$, $g \in G$, $g^{-1}s^\mathsf{T} g \in S$, and $\psi_S^\mathsf{T}(s^\mathsf{T}) \neq \psi_S(g^{-1}s^\mathsf{T} g)$, then the coset $S^\mathsf{T} gS$ does not support any (S, ψ, T)-invariant distributions.

These restrictions follow directly from Bernstein's localization principle; this method is used often in the study of Shalika and Whittaker models, and we point to the recent work of Jiang *et al.* [JNQ08] for an example similar in spirit.

We will also apply the following criterion to prove transpose-invariance of distributions.

(TI) If $g \in G$, and there exist $s_1, s_2 \in S \cap S^\mathsf{T} = \Delta(\mathrm{GL}_2)$ such that $s_1 g s_2 = g^\mathsf{T}$, then any (S, ψ, T)-invariant distribution on $S^\mathsf{T} gS$ is also transpose-invariant.

Following the methods of Gelfand–Kazhdan [GK75], we prove the following.

THEOREM 4.3. *Let $R = Q_3^\mathsf{T} \sigma Q_3$ be a double coset in* GSp_6. *Suppose that T is a (S, ψ, T)-invariant distribution on R. Then T is transpose-invariant.*

Proof. We prove this theorem, by analyzing the five cosets $Q_3^\mathsf{T} \sigma Q_3$ individually. We whittle down the support of such a distribution T using the restrictions (R1) and (R2), and prove transpose-invariance using criterion (TI).

Case $\sigma = 1$. For $\sigma = 1$, we are led to consider distributions T on $R = Q_3^\mathsf{T} \cdot Q_3 = U_3^\mathsf{T} \cdot L_3 \cdot U_3$. Since L_3 normalizes both U_3 and U_3^T, the (S, ψ, T)-invariant distributions T on R are in natural correspondence with distributions on L_3 which are $\Delta(\mathrm{GL}_2)$ bi-invariant.

Thus, we are led to consider the orbits for the action α of $\Delta(\mathrm{GL}_2) \times \Delta(\mathrm{GL}_2) \cong \mathrm{GL}_2 \times \mathrm{GL}_2$ on $L_3 \cong \mathrm{GL}_2 \times \mathrm{GL}_2$, given by

$$\alpha(g, h)(x, y) = (gxh, gyh).$$

Clearly, every element $(x, y) \in L_3 \cong \mathrm{GL}_2 \times \mathrm{GL}_2$ is in the same orbit as $(1, x^{-1}y)$. Furthermore, we find that for all $g \in \mathrm{GL}_2$, $(1, x^{-1}y)$ is in the same orbit as $(1, gx^{-1}yg^{-1})$. Finishing this analysis, we find that the orbits of $\mathrm{GL}_2 \times \mathrm{GL}_2$ on L_3 are in natural bijection with the orbits of GL_2 on GL_2 by conjugation. Furthermore, this bijection is compatible with the transpose (on L_3 and on GL_2).

It follows that the $(\mathrm{GL}_2 \times \mathrm{GL}_2)$-invariant distributions on L_3 are in bijection with the GL_2-invariant distributions on GL_2 (for the conjugation action). Since every element of GL_2 is conjugate to its transpose, we find that conjugation-invariant distributions on GL_2 are also transpose-invariant. It follows that $(\mathrm{GL}_2 \times \mathrm{GL}_2)$-invariant distributions on L_3 are also transpose-invariant, finishing this case.

Case $\sigma = b$. For $\sigma = b$, we first whittle down the support of (S, ψ, T)-invariant distributions T on $R = Q_3^\mathsf{T} \sigma Q_3$. Consider a general (S^T, S) coset representative in R: $g = \eta(u)\sigma\eta(v)$. We require

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explicit forms for the entries of u and v:

$$u = \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix}, \quad v = \begin{pmatrix} v_1 & v_2 \\ v_3 & v_4 \end{pmatrix}.$$

We must consider two cases.

Case $u_3 = 0$. If $u_3 = 0$, then choose λ_1, λ_2 so that

$$\psi_k(\lambda_1 v_1 + \lambda_2 v_2) \neq 1,$$

using the fact that v is non-singular. Define

$$x = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = \begin{pmatrix} \lambda_1 v_1 & \lambda_1 v_2 \\ \lambda_2 v_1 & \lambda_2 v_2 \end{pmatrix} \quad \text{and} \quad s = \left(\begin{array}{c|c|c} I & x & 0 \\ \hline 0 & I & J x^{\mathsf{T}} J^{-1} \\ \hline 0 & 0 & I \end{array}\right).$$

We compute

$$gsg^{-1} = \begin{pmatrix} 1 & \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & u_1\lambda_2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -u_1\lambda_2 & 1 & \lambda_1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We find that

$$\psi_S(s) = \psi_k(x_1 + x_4) = \psi_k(\lambda_1 v_1 + \lambda_2 v_2) \neq 1, \quad \psi_S^{\mathsf{T}}(gxg^{-1}) = \psi_k(0) = 1.$$

By criterion (R1), $R = S^{\mathsf{T}} g S$ does not support any (S, ψ, T)-invariant distributions.

Case $u_3 \neq 0$. Suppose that $u_3 \neq 0$. If $v_1 \neq 0$, then define

$$x = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = \begin{pmatrix} \lambda_1 v_1 & \lambda_1 v_2 \\ \lambda_2 v_1 & \lambda_2 v_2 \end{pmatrix},$$

where λ_1 and λ_2 are chosen in such a way that $\lambda_2 \neq 0$ and

$$x_1 + x_4 = \lambda_1 v_1 + \lambda_2 v_2 = 0.$$

Then, since v is non-singular, we find that

$$x_4 v_3 - x_3 v_4 = \lambda_2 v_2 v_3 - \lambda_2 v_1 v_4 = -\lambda_2 \det(v) \neq 0.$$

Simplifying,

$$gsg^{-1} = \begin{pmatrix} 1 & \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 u_1 & 1 & 0 & 0 & 0 \\ 0 & -\lambda_2 u_3 & 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda_2 u_3 & -\lambda_2 u_1 & 1 & \lambda_1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

By scaling the vector (λ_1, λ_2) if necessary, we find that

$$\psi_S(s) = \psi_k(x_1 + x_4) = \psi_k(0) = 1, \quad \psi_S^{\mathsf{T}}(gsg^{-1}) = \psi_k(-\lambda_2 u_3) \neq 1.$$

By criterion (R1), $R = S^{\mathsf{T}} g S$ does not support any (S, ψ, T)-invariant distributions.

If $v_1 = 0$ and $v_2 \neq -u_3$, then we may choose λ_2 such that $\psi(-\lambda_2 u_3) \neq \psi(\lambda_2 v_2)$. From this, it follows that $\psi_S^{\mathsf{T}}(gsg^{-1}) \neq \psi_S(s)$. It follows that $R = S^{\mathsf{T}} gS$ does not support any (S, ψ, T)-invariant distributions.

We find that all (S, ψ, T)-invariant distributions T must be supported on cosets $S^{\mathsf{T}} gS$ for which $g = \eta(u)\sigma\eta(v)$ with $v_1 = 0$, $u_3 \neq 0$, and $v_2 = -u_3$. Applying (R2) instead of (R1), we can whittle down the support further (in a symmetric way), and we find that all (S, ψ, T)-invariant distributions T must be supported on

$$X = \{S^{\mathsf{T}} \eta(u)\sigma\eta(v)S : u_1 = v_1 = 0, v_2 = -u_3\}.$$

Now, if $g = \eta(u)\sigma\eta(v)$, $u_1 = v_1 = 0$, and $v_2 = -u_3$, consider the elements $z, y \in S$ given by

$$z = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & u_4 v_3 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad y = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & v_4 u_2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then we find that

$$y^{\mathsf{T}} gz = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & v_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -u_2 v_2 v_3 & 0 \\ 0 & v_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -u_2 v_2 v_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \det(uv) \end{pmatrix}.$$

Observing that $y^{\mathsf{T}} gz$ is equal to its transpose, and $\psi_S(y) = \psi_S(z) = 1$, we find that (S, ψ, T)-invariant distributions on X are also transpose-invariant.

Case $\sigma = bcb$. For $\sigma = bcb$, consider a general coset representative $g = \eta(u)\sigma\eta(v)$. With u, v as before, define $w = uv$, so that

$$w = \begin{pmatrix} w_1 & w_2 \\ w_3 & w_4 \end{pmatrix} = \begin{pmatrix} u_1 v_1 + u_2 v_3 & u_1 v_2 + u_2 v_4 \\ u_3 v_1 + u_4 v_3 & u_3 v_2 + u_4 v_4 \end{pmatrix}.$$

If $(w_3, w_4) \neq (-\det(v), 0)$, then there exist x_3, x_4 such that

$$\psi_k\left(\frac{w_4 x_3 - w_3 x_4}{\det(v)}\right) \neq 1 \quad \text{and} \quad \psi_k(x_4) = 1.$$

In this case, we set $x_1 = x_2 = 0$ and x_3, x_4 satisfying the above conditions. Define as in the previous case

$$x = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \quad \text{and} \quad s = \left(\begin{array}{c|c|c} I & x & 0 \\ \hline 0 & I & Jx^{\mathsf{T}} J^{-1} \\ \hline 0 & 0 & I \end{array}\right).$$

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Then we compute

$$gsg^{-1}=\begin{pmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&\dfrac{w_2x_3-w_1x_4}{\det(v)}&1&0&0&0\\0&\dfrac{w_4x_3-w_3x_4}{\det(v)}&0&1&0&0\\0&0&-\dfrac{w_4x_3-w_3x_4}{\det(v)}&\dfrac{w_2x_3-w_1x_4}{\det(v)}&1&0\\0&0&0&0&0&1\end{pmatrix}.$$

We find that

$$\psi_S(s)=\psi_k(x_4)=1,\quad \psi_S^{\mathsf{T}}(gsg^{-1})=\psi_k\left(\frac{w_4x_3-w_3x_4}{\det(v)}\right)\neq 1.$$

By (R1), it follows that if T is a (S,ψ,T)-invariant distribution on R, then T is supported on (S^{T},S)-cosets of the form $S^{\mathsf{T}}\eta(u)\sigma\eta(v)S$ for $u,v\in\mathrm{GL}_2$ satisfying

$$uv=\begin{pmatrix}*&\det(u)\\-\det(v)&0\end{pmatrix}. \tag{1}$$

For such u,v, we compute

$$g=\begin{pmatrix}1&0&0&0&0&0\\0&0&0&0&-\det(v)&0\\0&0&*&\det(u)&0&0\\0&0&-\det(v)&0&0&0\\0&\det(u)&0&0&0&0\\0&0&0&0&0&\det(uv)\end{pmatrix}.$$

A direct computation yields

$$\Delta\begin{pmatrix}\det(v)&0\\0&-\det(u)\end{pmatrix}g\Delta\begin{pmatrix}\det(v)&0\\0&-\det(u)\end{pmatrix}^{-1}=g^{\mathsf{T}}.$$

By criterion (TI), we find that all (S,ψ,T)-invariant distributions on these cosets are also transpose-invariant.

Case $\sigma=bacb$. For $\sigma=bacb$, consider a general coset representative $g=\eta(u)\sigma\eta(v)$ with u,v as before.

First, if $u_1=0$, then we may choose x_1,x_2,x_3,x_4 such that

$$\psi_k(x_1+x_4)\neq 1\quad\text{and}\quad v_2x_1-v_1x_2=0.$$

For this choice, there exists λ such that $x_1=\lambda v_1$ and $x_2=\lambda v_2$. Define

$$x=\begin{pmatrix}x_1&x_2\\0&x_4\end{pmatrix},\qquad s=\left(\begin{array}{c|c|c}I&x&0\\\hline 0&I&Jx^{\mathsf{T}}J^{-1}\\\hline 0&0&I\end{array}\right).$$

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Then we compute

$$gsg^{-1} = \begin{pmatrix} 1 & \frac{-v_1x_4}{\det(v)} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ u_3\lambda & 0 & 0 & 1 & 0 & 0 \\ \frac{x_4v_3}{\det(u^{-1}v)} & 0 & 0 & 0 & 1 & \frac{-v_1x_4}{\det(v)} \\ 0 & -\frac{x_4v_3}{\det(u^{-1}v)} & u_3\lambda & 0 & 0 & 1 \end{pmatrix}.$$

We find that $gsg^{-1} \in S^{\mathsf{T}}$,

$$\psi_S(s) = \psi_k(x_1 + x_4) \neq 1, \quad \psi_S^{\mathsf{T}}(gsg^{-1}) = \psi_k(0) = 1.$$

By (R1), the double coset $S^{\mathsf{T}}\eta(u)\sigma\eta(v)S$ does not support any (S, ψ, T)-invariant distributions.

Next suppose that $u_1 \neq 0$. Choose λ such that $\psi_k(u_1\lambda) \neq 1$. Define

$$x = \begin{pmatrix} \lambda v_1 & \lambda v_2 \\ 0 & -\lambda v_1 \end{pmatrix} \quad \text{and} \quad s = \left(\begin{array}{c|c|c} I & x & 0 \\ \hline 0 & I & Jx^{\mathsf{T}}J^{-1} \\ \hline 0 & 0 & I \end{array}\right).$$

We compute

$$gsg^{-1} = \begin{pmatrix} 1 & \frac{\lambda v_1^2}{\det(v)} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ u_1\lambda & 0 & 1 & 0 & 0 & 0 \\ u_3\lambda & 0 & 0 & 1 & 0 & 0 \\ \frac{-\lambda v_1v_3}{\det(u^{-1}v)} & 0 & 0 & 0 & 1 & \frac{\lambda v_1^2}{\det(v)} \\ 0 & \frac{\lambda v_1v_3}{\det(u^{-1}v)} & u_3\lambda & -u_1\lambda & 0 & 1 \end{pmatrix}.$$

We find that $gsg^{-1} \in S^{\mathsf{T}}$,

$$\psi_S(s) = \psi_k(x_1 + x_4) = \psi_k(0) = 1,$$

$$\psi_S^{\mathsf{T}}(gsg^{-1}) = \psi_k(u_1\lambda) \neq 1.$$

By (R1), the double coset $S^{\mathsf{T}}\eta(u)\sigma\eta(v)S$ does not support any (S, ψ, T)-invariant distributions.

Case $\sigma = bcabacb$. Suppose that $g = \eta(u)\sigma\eta'(v)$ for $u, v \in \mathrm{GL}_2$, where $\eta'(v) = \Delta(v)\eta(v)^{-1}$; we find it convenient to use slightly different coset representatives here, using η' instead of η. There are two cases to consider.

First, suppose that $u\det(v)^{-1} = -v \in \mathrm{GL}_2$. Then we find that

$$g = \left(\begin{array}{c|c|c} 0 & 0 & -u \\ \hline 0 & u & 0 \\ \hline -u & 0 & 0 \end{array}\right).$$

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Note that there exists $\gamma \in \mathrm{GL}_2$ such that $\gamma u \gamma^{-1} = u^{\mathsf{T}}$. It follows that

$$\Delta(\gamma) g \Delta(\gamma)^{-1} = g^{\mathsf{T}}.$$

By (TI), any (S, ψ, T)-invariant distribution on $S^{\mathsf{T}} g S$ will be transpose-invariant.

Next, suppose that $u \det(v)^{-1} \neq -v$. Then we may choose $X \in M_2$ such that

$$\psi_k(-\mathrm{Tr}(X)) \neq \psi_k(\mathrm{Tr}(\det(v)^{-1} u X v^{-1})).$$

Define an element $s \in S$ by

$$s = \left(\begin{array}{c|c|c} I & X & 0 \\ \hline 0 & I & J X^{\mathsf{T}} X \\ \hline 0 & 0 & I \end{array}\right).$$

Then we find that

$$gsg^{-1} = \left(\begin{array}{c|c|c} I & 0 & 0 \\ \hline \det(v)^{-1} u J X^{\mathsf{T}} J v^{-1} & I & 0 \\ \hline 0 & \det(u) v X u^{-1} & I \end{array}\right).$$

We find that $gsg^{-1} \in S^{\mathsf{T}}$ and

$$\psi_S(s) = \psi_k(-\mathrm{Tr}(X)),$$
$$\psi_S(gsg^{-1}) = \psi_k(\mathrm{Tr}(\det(v)^{-1} u X v^{-1})).$$

By (R1), the coset $S^{\mathsf{T}} g S$ does not support any (S, ψ, T)-invariant distributions. $\square$

With this technical work done, we can state the following theorem.

Theorem 4.4. *Suppose that σ is a supercuspidal irrep of* GSp_6. *Then the space of Shalika functionals for σ is at most one dimensional:*

$$\dim(\mathrm{Sh}(\sigma)) \leqslant 1.$$

Proof. Our previous results on distributions, with the methods of Gelfand, Kazhdan, and Bernstein, imply that the pair (GSp_6, S) is a Gelfand pair, in the sense of [Gro91, Condition 4.1] (though we work with the character ψ_S of S rather than the trivial representation of S). To be precise, for an irrep σ of GSp_6, with contragredient $\tilde{\sigma}$, we find (cf. [Gro91, Proposition 4.2]) that

$$\dim(\mathrm{Sh}(\sigma)) \cdot \dim(\mathrm{Sh}(\tilde{\sigma})) \leqslant 1.$$

So, it remains to check that σ has a non-vanishing Shalika functional if and only if $\tilde{\sigma}$ has a non-vanishing Shalika functional.

Since S is a unimodular subgroup of GSp_6, there is a non-degenerate GSp_6-invariant pairing:

$$\text{c-Ind}_S^{\mathrm{GSp}_6} \psi_S \times \text{c-Ind}_S^{\mathrm{GSp}_6} \tilde{\psi}_S \to \mathbb{C},$$

given by integration of functions on $S \backslash G$:

$$\langle f_1, f_2 \rangle = \int_{S \backslash G} f_1(g) f_2(g)\, dg.$$

Now, if σ is a supercuspidal irrep of GSp_6 with non-vanishing Shalika functional, then σ occurs as a subrepresentation of $\text{c-Ind}_S^{\mathrm{GSp}_6} \psi_S$. The non-degeneracy of the pairing above (and the injectivity of supercuspidals) implies that $\tilde{\sigma}$ occurs as a subrepresentation of $\text{c-Ind}_S^{\mathrm{GSp}_6} \tilde{\psi}_S$. It follows that $\tilde{\sigma}$ has a non-vanishing Shalika functional, with respect to the character $\tilde{\psi}_S$. But since $\tilde{\psi}_S$ and ψ_S are conjugate (via an element of GSp_6) characters of S, we find that $\mathrm{Sh}(\tilde{\sigma}) \neq 0$. □

4.4 Theta correspondence

The importance of Shalika functionals in the theta correspondence is the following.

LEMMA 4.5. *Suppose that σ is a generic supercuspidal irrep of PGSp_6. Then there is a linear isomorphism*

$$\mathrm{Wh}_{G_2}(\overleftarrow{\Theta}_7(\sigma)) = \overleftarrow{\Theta}_7(\sigma)_{N_2,\psi_2} \cong \mathrm{Sh}(\sigma).$$

Proof. Here, let $\mathbf{N}_2$ be the unipotent radical of a Borel subgroup of $\mathbf{G}_2$ and ψ_2 a principal character of N_2. Let $\mathbf{Q}_2 = \mathbf{L}_2\mathbf{U}_2$ be a maximal parabolic subgroup of $\mathbf{G}_2$ such that $\mathbf{U}_2$ is contained in $\mathbf{N}_2$ and $\mathbf{N}_2/\mathbf{U}_2$ corresponds to a short simple root.

Then $\mathrm{Wh}_{G_2}(\Pi_7) = (\Pi_7)_{N_2,\psi_2}$ can be computed in two stages:

$$(\Pi_7)_{N_2,\psi_2} = ((\Pi_7)_{U_2,\psi_2})_{N_2,\psi_2}.$$

Lemma 2.9 on p. 213 in [GS98] shows how to compute the coinvariants of Π_7 with respect to any character of U_2. The characters of U_2 are parameterized by cubic k-algebras, and the restriction of ψ_2 to U_2 corresponds to the degenerate cubic algebra $k[\epsilon]/\langle \epsilon^3 \rangle$.

Let $S^\circ \subseteq S$ be the semidirect product of GL_2 with $U_3^\circ \subseteq U_3$, where U_3° contains the center Z_3 and U_3°/Z_3 corresponds to trace zero matrices in $U_3/Z_3 \cong M_2(k)$. Then

$$(\Pi_7)_{U_2,\psi_2} \cong \text{c-Ind}_{S^\circ}^{\mathrm{GSp}_6}(\mathbb{C}).$$

Under this identification, one observes that the action of N_2 on $(\Pi_7)_{U_2,\psi_2}$ (which restricts to the character ψ_2 on U_2) is identified with the action of S/S° by left translation on $\text{c-Ind}_{S^\circ}^{\mathrm{GSp}_6}(\mathbb{C})$. This implies that

$$\mathrm{Wh}_{G_2}(\Pi_7) = (\Pi_7)_{N_2,\psi_2} \cong \text{c-Ind}_S^{\mathrm{GSp}_6}(\psi_S),$$

as representations of GSp_6.

Applying $\mathrm{Hom}_{\mathrm{GSp}_6}(\sigma, \cdot)$ to both sides above, we find that

$$\mathrm{Wh}_{G_2}(\overleftarrow{\Theta}_7(\sigma)) = \overleftarrow{\Theta}_7(\sigma)_{N_2,\psi_2} \cong \mathrm{Sh}(\sigma).$$ □

Since $\overleftarrow{\Theta}_7(\sigma)$ is multiplicity-free and supercuspidal, and every subrepresentation is generic, we immediately find the following proposition.

PROPOSITION 4.6. *Suppose that σ is a generic supercuspidal irrep of PGSp_6. Then $\overleftarrow{\Theta}_7(\sigma)$ is non-zero if and only if $\mathrm{Sh}(\sigma) \neq 0$. Moreover, if $\overleftarrow{\Theta}_7(\sigma) \neq 0$, then $\overleftarrow{\Theta}_7(\sigma)$ is a generic supercuspidal irrep of G_2.*

Proof. This proposition directly follows from the previous lemma, and the 'uniqueness of Shalika functionals' of Theorem 4.4. □

4.5 Injectivity of dichotomy

We can now demonstrate the following.

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Theorem 4.7. *The dichotomy map is injective:*

$$\Delta : \mathrm{Irr}_g^\circ(G_2) \hookrightarrow \mathrm{Irr}_g^\circ(\mathrm{PGSp}_6) \sqcup \frac{\mathrm{Irr}_g^\circ(\mathrm{PGL}_3)}{\mathrm{Contra}}.$$

Proof. If two generic supercuspidal irreps τ, τ' of G_2 have the property that $\overrightarrow{\Theta}_6(\tau)$ and $\overrightarrow{\Theta}_6(\tau')$ have a common supercuspidal subrepresentation, then τ is isomorphic to τ' by [GS04, Theorem 19].

If two generic supercuspidal irreps τ, τ' of G_2 have the property that $\overrightarrow{\Theta}_7(\tau)$ and $\overrightarrow{\Theta}_7(\tau')$ have a common generic supercuspidal subrepresentation σ, then τ is isomorphic to τ' by Proposition 4.6, since both τ and τ' must be subrepresentations of the irrep $\overleftarrow{\Theta}_7(\sigma)$. □

5. L-functions and periods

Now that we have proven that the dichotomy map is injective, it remains to characterize its image. In fact, all supercuspidal irreps of PGL_3 occur in the theta correspondence with a generic supercuspidal irrep of G_2, by Gan and Savin [GS04, Theorem 19].

Proposition 5.1. *Suppose that ρ is a supercuspidal irrep of PGL_3. Then there exists a unique generic supercuspidal irrep τ of G_2 occurring in $\overleftarrow{\Theta}_6(\rho)$.*

This immediately implies the following.

Corollary 5.2. *The image of dichotomy in $\mathrm{Irr}_g^\circ(\mathrm{PGL}_3)$ includes all non-self-contragredient supercuspidal irreps. In particular, Δ surjects onto $\mathrm{Irr}_g^\circ(\mathrm{PGL}_3)$ when $p \neq 2$.*

On the other hand, the image of dichotomy in $\mathrm{Irr}_g^\circ(\mathrm{PGSp}_6)$ is so far only characterized as

$$\Delta(\mathrm{Irr}_g^\circ(G_2)) \cap \mathrm{Irr}_g^\circ(\mathrm{PGSp}_6) = \{\sigma \in \mathrm{Irr}_g^\circ(\mathrm{PGSp}_6) : \mathrm{Sh}(\sigma) \neq 0\}.$$

In this section, we demonstrate that the image of dichotomy can be described not only by the Shalika functional, but also by the degree-eight spin L-function. The goal of this section is to prove the following.

Theorem 5.3. *Suppose that σ is a generic supercuspidal irrep of PGSp_6. Let $L(\sigma, \mathrm{Spin}, s)$ denote Shahidi's L-function, associated to the eight-dimensional spin representation of $\mathrm{Spin}_7(\mathbb{C})$. Then $\mathrm{Sh}(\sigma) \neq 0$ if and only if $L(\sigma, \mathrm{Spin}, s)$ has a pole at $s = 0$.*

One direction in this theorem, that a non-vanishing Shalika functional implies that $L(\sigma, \mathrm{Spin}, s)$ has a pole at $s = 0$, follows from Shahidi's work, examination of a reducibility point, and properties of the minimal representation of E_8. The other direction relies on an integral representation for the spin L-function due to Bump–Ginzburg [BG92] and studied by Vo [Vo97]. We prove that these two incarnations of the spin L-function have the same poles, using global methods.

5.1 A reducibility point

Let $\mathbf{P}_4 = \mathbf{M}_4\mathbf{N}_4$ be the Heisenberg parabolic subgroup of $\mathbf{F}_4$, with Levi component $\mathbf{M}_4 \cong \mathbf{GSp}_6$, and sim the similitude character of $\mathbf{GSp}_6$. The modular character, for the adjoint action of M_4 on N_4, can then be expressed as

$$\delta_{P_4}(m) = |\mathrm{sim}(m)|^8.$$

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For σ a generic supercuspidal irrep of PGSp_6, consider the family of representations of F_4:

$$I(\sigma, s) = \mathrm{Ind}_{P_4}^{F_4}(\sigma \otimes |\mathrm{sim}|^{s+4}),$$

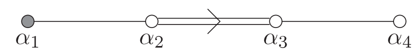
where sim is the similitude character of GSp_6. The normalization factor $|\mathrm{sim}|^4$ is chosen so that $I(\sigma, 0)$ is unitary when σ is unitary.

Let $L(\sigma, \mathrm{Spin}, s)$ be Shahidi's L-function, where Spin is the eight-dimensional representation of the dual Levi $\hat{\mathbf{M}}_4 \cong \mathbf{GSpin}_7(\mathbb{C})$ on the abelian quotient of the unipotent radical of the parabolic $\hat{\mathbf{P}}_4$ dual to $\mathbf{P}_4$. The following result is essentially due to Shahidi [Sha90].

LEMMA 5.4. *The L-function $L(\sigma, \mathrm{Spin}, s)$ has a pole at $s = 0$ if and only if $I(\sigma, -1)$ is reducible, in which case it has a composition series of length two. In this case, the unique irreducible submodule $J(\sigma)$ of $I(\sigma, -1)$ is not generic.*

Proof. To compute this reducibility point, we compute some constants discussed in [Sha90]. Let $\alpha_1, \ldots, \alpha_4$ denote the simple roots in a root system of type F_4, numbered as below.

$$P_4 = \mathrm{GSp}_6 \ltimes N_4$$



Let β denote the highest root, so that

$$\beta = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4.$$

Observe that the maximal parabolic subgroup $\mathbf{P}_4$ is associated to the root α_1, which is adjacent to $-\beta$ in the extended (affine) Dynkin diagram.

Let ρ_P denote the half-sum of the roots occurring in $\mathbf{N}_4$. Then $\rho_P = 4\beta$. It follows that

$$\tilde{\alpha}_1 = \langle \alpha_1^\vee, \rho_P \rangle^{-1} \cdot \rho_P = \langle \alpha_1^\vee, \beta \rangle^{-1} \beta = \beta.$$

Since β corresponds precisely to the similitude character of $\mathbf{M}_4 = \mathbf{GSp}_6$, it follows that $I(\sigma, s)$ is normalized as in Shahidi [Sha90]. The result now follows directly from [Sha90]; a helpful exposition of the results from Shahidi can be found in [Zha99, § 2]. □

To demonstrate a connection between non-vanishing of a theta correspondence and L-functions, we use a method of Muić–Savin [MS00] and consider a theta correspondence in a larger group. The following lemma plays a similar role in this section to [MS00, Proposition 4.1].

LEMMA 5.5. *Let Π_8 denote the minimal representation of E_8. Let ϕ_4 be a generic character of a maximal unipotent subgroup U_4 of F_4. Then*

$$\mathrm{Wh}_{F_4}(\Pi_8) = (\Pi_8)_{U_4,\phi_4} = 0.$$

Proof. We study the Whittaker functionals $\mathrm{Wh}_{F_4}(\Pi_8) = (\Pi_8)_{U_4,\phi_4}$ in stages:

$$(\Pi_8)_{U_4,\phi_4} = (((\Pi_8)_{N_4,\psi_4})_{N_3,\psi_3})_{U_2,\psi_2},$$

where N_4 is a 15-dimensional Heisenberg group in F_4, N_3 is a six-dimensional abelian unipotent subgroup of GSp_6, and U_2 is a maximal unipotent subgroup of SL_3.

Stage 1. The N_4, ψ_4 coinvariants. We view $\mathbf{F}_4$ here as the algebraic group associated to the 14-dimensional structurable algebra of Freudenthal type

$$F_k \cong k \oplus J_k \oplus J_k \oplus k.$$

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Similarly, we view $\mathbf{E}_8$ as the algebraic group associated to the 56-dimensional structurable algebra of Freudenthal type

$$F_{\mathbb{O}} \cong k \oplus J_{\mathbb{O}} \oplus J_{\mathbb{O}} \oplus k.$$

The construction of these algebras and groups follows § 2.3.4. As a result, $\mathbf{F}_4$ is endowed with a parabolic subgroup $\mathbf{P}_4 = \mathbf{M}_4\mathbf{N}_4$, and $\mathbf{E}_8$ contains a parabolic subgroup $\mathbf{P}_8 = \mathbf{M}_8\mathbf{N}_8$, such that:

(i) $\mathbf{N}_4$ and $\mathbf{N}_8$ are two-step unipotent groups with one-dimensional centers $\mathbf{Z}_4$ and $\mathbf{Z}_8$;

(ii) $\mathbf{N}_4/\mathbf{Z}_4$ is naturally identified with F_k, and $\mathbf{N}_8/\mathbf{Z}_8$ is naturally identified with $F_{\mathbb{O}}$;

(iii) the parabolics are aligned, in the sense that $\mathbf{P}_8 \cap \mathbf{F}_4 = \mathbf{P}_4$, $\mathbf{N}_8 \cap \mathbf{F}_4 = \mathbf{N}_4$, and $\mathbf{Z}_8 = \mathbf{Z}_4$.

Let ψ_4 denote the restriction of ϕ_4 to N_4. Then ψ_4 is in the minimal GSp_6-orbit in the space of characters of N_4. By conjugation, we may assume that ψ_4 corresponds to the element $\left(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right) \in F_k \cong k \oplus J_k \oplus J_k \oplus k$ (identified with N_4^-/Z_4^-).

The space $(\Pi_8)_{N_4,\psi_4}$ is a quotient of the kernel

$$\mathrm{Ker}((\Pi_8)_{Z_8} \to (\Pi_8)_{N_8}).$$

From [GS05, Corollary 11.12], this kernel can be identified with $C_c^\infty(\Omega)$, where Ω is the minimal non-zero M_8-orbit in the 56-dimensional minuscule representation N_8^-/Z_8^-.

Then the characters of N_8 which restrict to ψ_4 on N_4, and also are in the N_8/Z_8-support of Π_8, correspond to elements

$$\begin{pmatrix} 1 & j \\ j^\sharp & 0 \end{pmatrix} \in F_{\mathbb{O}} \cong k \oplus J_{\mathbb{O}} \oplus J_{\mathbb{O}} \oplus k,$$

where $j \in J_{\mathbb{O}}$, the entries of j and $j^\sharp$ are in $\mathbb{O}_\circ$, $j^\sharp$ is the quadratic adjoint of j, and $\mathsf{N}(j) = 0$. Here, we refer to [GS05, Proposition 11.2, § 10] for a description of the orbit Ω and Jordan algebras.

Thus, the representation $(\Pi_8)_{N_4,\psi_4}$ is identified with $C_c^\infty(\Omega^\perp)$, where

$$\Omega^\perp = \left\{ \begin{pmatrix} 0 & \alpha & -\beta \\ -\alpha & 0 & \gamma \\ \beta & -\gamma & 0 \end{pmatrix} : \begin{matrix} \alpha^2 = \beta^2 = \gamma^2 = 0, \\ \mathrm{Tr}(\alpha\beta) = \mathrm{Tr}(\beta\gamma) = \mathrm{Tr}(\gamma\alpha) = 0, \end{matrix} \ \mathrm{Tr}(\alpha\beta\gamma) = 0 \right\}.$$

Equivalently, we may view

$$\Omega^\perp = \{(\alpha, \beta, \gamma) \in \mathbb{O}_\circ^3 : \mathrm{Span}_k(\alpha, \beta, \gamma) \text{ is isotropic}, \mathrm{Tr}(\alpha\beta\gamma) = 0\}.$$

Stage 2. The N_3, ψ_3 coinvariants. Now we are led to consider

$$((\Pi_8)_{N_4,\psi_4})_{N_3,\psi_3} \cong C_c^\infty(\Omega^\perp)_{N_3,\psi_3}.$$

First, we describe the subgroup N_3 of GSp_6, and the character ψ_3. Here, $\mathbf{Q}_3 = \mathbf{M}_3\mathbf{N}_3$ denotes the 'Siegel parabolic' in GSp_6, whose derived subgroup is $\mathbf{M}_3' \cong \mathbf{SL}_3 \ltimes \mathbf{N}_3$. This derived subgroup $\mathbf{M}_3'$ stabilizes the character ψ_4 of N_4. The unipotent radical $\mathbf{N}_3$ of $\mathbf{Q}_3$ is abelian, and its k-points N_3 are identified naturally with the space J_k of symmetric three-by-three matrices with entries in k. Then N_3 acts on $\Omega^\perp$ in the following way:

$$\kappa \star j = j + (j^\sharp \times \kappa) \quad \text{for all } \kappa \in N_3 = J_k, j \in \Omega^\perp \subset J_{\mathbb{O}}.$$

In particular, we can compute

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \star \begin{pmatrix} 0 & \alpha & -\beta \\ -\alpha & 0 & \gamma \\ \beta & -\gamma & 0 \end{pmatrix} = \begin{pmatrix} 0 & \alpha & -\beta \\ -\alpha & 0 & \gamma - \alpha\beta \\ \beta & \gamma + \alpha\beta & 0 \end{pmatrix}$$

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or, in shorthand,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \star (\alpha, \beta, \gamma) = (\alpha, \beta, \gamma - \alpha\beta).$$

Let ψ_3 be the character of N_3 given by

$$\psi_3 \begin{pmatrix} a & r & s \\ r & b & t \\ s & t & c \end{pmatrix} = \psi_k(a).$$

We now decompose $\Omega^\perp$ into two subsets:

$$\Omega_1^\perp = \{(\alpha, \beta, \gamma) \in \Omega^\perp : \alpha\beta = 0\},$$
$$\Omega_2^\perp = \{(\alpha, \beta, \gamma) \in \Omega^\perp : \alpha\beta \neq 0\}.$$

We find almost immediately that $C_c^\infty(\Omega_1^\perp)_{N_3,\psi_3} = 0$: indeed, if $j \in \Omega_1^\perp$, $t \in k$ and $\psi_k(t) \neq 0$, then

$$\psi_3 \begin{pmatrix} t & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq 1 \quad \text{and} \quad \begin{pmatrix} t & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \star j = j.$$

In other words, $\Omega_1^\perp$ does not support any (N_3, ψ_3)-invariant distributions. It follows that

$$((\Pi_8)_{N_4,\psi_4})_{N_3,\psi_3} = C_c^\infty(\Omega^\perp)_{N_3,\psi_3} = C_c^\infty(\Omega_2^\perp)_{N_3,\psi_3}.$$

Note that if $j = (\alpha, \beta, \gamma) \in \Omega_2^\perp$, then $\alpha\beta$ is non-zero and orthogonal to α, β, γ, and itself (with respect to the trace pairing on $\mathbb{O}_\circ$. But the maximal dimension of an isotropic subspace in $\mathbb{O}_\circ$ is three, so there must exist $a, b, c \in k$, not all zero, such that

$$\alpha\beta = a\alpha + b\beta + c\gamma.$$

Multiplying through by α or by β, we find that

$$b\alpha\beta = -c\alpha\gamma,$$
$$a\alpha\beta = -c\gamma\beta.$$

Hence, $\mathrm{Span}_k(\alpha\beta, \beta\gamma, \gamma\alpha)$ is one dimensional. Note also that $c \neq 0$ in the above relations, since otherwise $\alpha\beta = 0$.

Stage 3. The U_2, ψ_2 coinvariants. Now we are led to consider the coinvariants

$$\mathrm{Wh}_{F_4}(\Pi_8) = (\Pi_8)_{U_4,\phi_4} = (((\Pi_8)_{N_4,\psi_4})_{N_3,\psi_3})_{U_2,\psi_2} = (C_c^\infty(\Omega_2^\perp)_{N_3,\psi_3})_{U_2,\psi_2}.$$

We describe the subgroup U_2 and character ψ_2 here. There is a chain of embeddings:

$$U_2 \subset \mathrm{SL}_3 \subset Q_3' = \mathrm{SL}_3 \ltimes J_k \subset Q_3 \subset \mathrm{GSp}_6 \subset Q_4 = \mathrm{GSp}_6 \ltimes F_k \subset F_4.$$

The resulting action of SL_3 on $F_k \cong k \oplus J_k \oplus J_k \oplus k$ is given by the formulas of Krutelevich [Kru07, § 3.1].

$$\gamma \cdot \begin{pmatrix} a & A \\ B & b \end{pmatrix} = \begin{pmatrix} a & \gamma A \gamma^\mathsf{T} \\ (\gamma^{-1})^\mathsf{T} B \gamma^{-1} & b \end{pmatrix} \quad \text{for all } \gamma \in \mathrm{SL}_3.$$

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In particular, let U_2 denote the standard maximal unipotent subgroup of this SL_3; the action of U_2 on $\Omega^\perp$ is given by

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \cdot (\alpha, \beta, \gamma) = (\alpha + (xy - z)\gamma + y\beta, \beta - x\gamma, \gamma).$$

We define ψ_2 to be the principal character of U_2 given by

$$\psi_2 \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \psi_k(x - y).$$

Together with the action of N_3, we find an action of $\mathrm{SL}_3 \ltimes N_3$ on $\Omega^\perp$: for all $\gamma \in \mathrm{SL}_3$ and all $\kappa \in N_3$,

$$\gamma \cdot (\kappa \star j) = (\gamma \kappa \gamma^\mathsf{T}) \star (\gamma \cdot j).$$

The subgroup U_2 of SL_3 stabilizes the character ψ_3 of N_3:

$$\psi_3(u \kappa u^\mathsf{T}) = \psi_3(\kappa) \quad \text{for all } u \in U_2, \kappa \in N_3.$$

Now, for $j = (\alpha, \beta, \gamma) \in \Omega_2^\perp$, so that $\alpha\beta \neq 0$, we find three possibilities.

Case $\alpha\gamma \neq 0$. If $\alpha\gamma \neq 0$, then there exists $x \in k$ such that $\alpha\beta - x\alpha\gamma = 0$. We find that $j = (\alpha, \beta, \gamma)$ is in the same U_2-orbit as $j' = (\alpha, \beta - x\gamma, \gamma)$, and $(\alpha)(\beta - x\gamma) = 0$. Thus, $j' \in \Omega_1^\perp$, and cannot be contained in the support of an (N_3, ψ_3)-invariant distribution by the result of Stage 2.

Case $\beta\gamma \neq 0$. If $\beta\gamma \neq 0$, then there exists $z \in k$ such that $\alpha\beta - z\gamma\beta = 0$. We find that $j = (\alpha, \beta, \gamma)$ is in the same U_2-orbit as $j' = (\alpha - z\gamma, \beta, \gamma)$, and $(\alpha - z\gamma)(\beta) = 0$. Such elements $j' \in \Omega_1^\perp$ cannot be in the support of an (N_3, ψ_3)-invariant distribution, again by the result of Stage 2.

Case $\alpha\gamma = \beta\gamma = 0$. If $\alpha\gamma = \beta\gamma = 0$, then we find that $a = b = 0$ in the linear dependence

$$\alpha\beta = a\alpha + b\beta + c\gamma.$$

Thus, $\alpha\beta = c\gamma$. Define an element j' in the N_3-orbit of j by

$$j' = \begin{pmatrix} c^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \star j = (\alpha, \beta, 0).$$

Then j' cannot be in the support of a (U_2, ψ_2)-invariant distribution since, for any $x \in k$ such that $\psi_k(x) \neq 1$, we have

$$\psi_2 \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \neq 1 \quad \text{and} \quad \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot j' = j'.$$

It follows that $(C_c^\infty(\Omega_2^\perp)_{N_3, \psi_3})_{U_2, \psi_2} = 0$, and so

$$\mathrm{Wh}_{F_4}(\Pi_8) = 0. \qquad \square$$

Now we can demonstrate a connection between a non-vanishing Shalika functional and a pole in Shahidi's L-function.

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THEOREM 5.6. *Suppose that σ is a generic supercuspidal irrep of PGSp_6, with non-zero Shalika functional. Let $\tau = \overleftarrow{\Theta}_7(\sigma)$, a generic supercuspidal irrep by Proposition 4.6. Then the following statements are true.*

(i) *The L-function $L(\sigma, \mathrm{Spin}, s)$ has a pole at $s = 0$.*

(ii) *If $J(\sigma)$ is the unique irreducible subrepresentation of $I(\sigma, -1)$, then $J(\sigma) \boxtimes \tau$ occurs as a quotient of the minimal representation Π_8 of E_8, restricted to the dual pair $F_4 \times G_2$.*

Proof. We use the Heisenberg parabolic subgroups $\mathbf{P}_4$, $\mathbf{P}_8$ of $\mathbf{F}_4$, $\mathbf{E}_8$, discussed in the previous result.

Recall that $\sigma \boxtimes \tau$ occurs as a quotient (or subrepresentation) of the minimal representation Π_7 of E_7. By [MS97, Theorem 6.1] (following [Sav94, Proposition 4.1], and not requiring any condition on residue characteristic), the Jacquet functor (along N_8) of the minimal representation Π_8 of E_8 can be identified as a representation of GE_7:

$$(\Pi_8)_{N_8} \cong (\Pi_7 \otimes |\det|^{3/28}) \oplus |\det|^{5/28}.$$

Since $N_4 \subset N_8$, $(\Pi_8)_{N_8}$ is a quotient of $(\Pi_8)_{N_4}$, as representations of $\mathrm{GSp}_6 \times G_2 \subset GE_7$. It follows that there is a surjective $\mathrm{GSp}_6 \times G_2$ intertwining map:

$$(\Pi_8)_{N_4} \twoheadrightarrow \Pi_7 \otimes |\det|^{3/28}.$$

Since $\sigma \boxtimes \tau$ occurs as a quotient of Π_7, restricted to $\mathrm{GSp}_6 \times G_2$, we find a surjective $\mathrm{GSp}_6 \times G_2$ intertwining map:

$$(\Pi_8)_{N_4} \twoheadrightarrow (\sigma \boxtimes \tau) \otimes |\det|^{3/28}.$$

It follows by Frobenius reciprocity that there is a non-trivial $F_4 \times G_2$ intertwining map:

$$\Pi_8 \to \mathrm{Ind}_{P_4}^{F_4}(\sigma \otimes |\det|^{3/28}) \boxtimes \tau.$$

In order to identify the restriction of det (the determinant for the action of GE_7 on a 56-dimensional space) to GSp_6, we consider the coroot $\alpha^\vee$ of $\mathbf{F}_4$, which satisfies

$$\alpha^\vee(t) \in Z(\mathrm{GSp}_6) \quad \text{for all } t \in k^\times \text{ and } \mathrm{sim}(\alpha^\vee(t)) = t^2.$$

This is the coroot of the $\mathbf{SL}_2$ which commutes with $\mathbf{Sp}_6 = [\mathbf{M}_4, \mathbf{M}_4]$ in $\mathbf{F}_4$. This $\mathbf{SL}_2$ is identified with the $\mathbf{SL}_2$ which commutes with $\mathbf{E}_7 = [\mathbf{M}_8, \mathbf{M}_8]$ in $\mathbf{E}_8$. Indeed, both copies of $\mathbf{SL}_2$ arise as $\mathbf{Aut}_{\mathbb{O}/M_2}$, embedded in $\mathbf{G}_{\mathbb{O}\otimes\mathbb{O}} \cong \mathbf{E}_8$ and in $\mathbf{G}_{k\otimes\mathbb{O}} \cong \mathbf{F}_4$. We refer to § 2.3.3 for a construction of these groups from tensor products of composition algebras.

The character det of GE_7, considered above, pairs with $\alpha^\vee$, in such a way that

$$\det(\alpha^\vee(t)) = t^{56} = \mathrm{sim}(\alpha^\vee(t))^{28}.$$

Indeed, $\alpha^\vee(t)$ acts on the 56-dimensional space $N_8/[N_8, N_8]$ by the scalar t, and the determinant is computed above. Comparing with the similitude character, for every element m of the subgroup $\mathrm{GSp}_6 \subset GE_7$, one has

$$|\det(m)|^{3/28} = |\mathrm{sim}(m)|^3.$$

Hence, we find a non-trivial $F_4 \times G_2$ intertwining map:

$$\Pi_8 \to \mathrm{Ind}_{P_4}^{F_4}(\sigma \otimes |\mathrm{sim}|^3) \boxtimes \tau = I(\sigma, -1) \boxtimes \tau.$$

Since $I(\sigma, -1)$ is generic, we find that $\mathrm{Wh}_{F_4}(I(\sigma, -1)) = I(\sigma, -1)_{U_4, \phi_4}$ is non-zero, where ϕ_4 is a generic character of U_4 as before. But $\mathrm{Wh}_{F_4}(\Pi_8) = (\Pi_8)_{U_4, \phi_4} = 0$ by the previous lemma.

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It follows that the image of the above intertwining map must be a proper submodule of $I(\sigma, -1) \boxtimes \tau$. Thus, we get both statements at once.

(i) $I(\sigma, -1)$ is reducible and, by the work of Shahidi, $L(\sigma, \mathrm{Spin}, s)$ has a pole at $s = 0$.

(ii) $J(\sigma) \boxtimes \tau$ occurs as a quotient of Π_8 (restricted from E_8 to $F_4 \times G_2$). □

5.2 Eisenstein series

Here, we review Eisenstein series on GL_2, as they are used in the construction of the spin L-function by Bump and Ginzburg [BG92]. Let F be a global field with adele ring $\mathbb{A}$. Following [GS88, p. 47], for every place v of F and s in $\mathbb{C}$, we define $V(s)$ to be the local unramified principal series representation of $\mathbf{GL}_2(F_v)$, unnormalized, so that the trivial representation is a submodule of $V(0)$ and a quotient of $V(1)$. Here, F_v is the completion of F at v; q_v will denote the cardinality of the residue field at v, if v is a finite place.

We have an intertwining operator $M_v(s) : V(s) \to V(1-s)$ defined by

$$M_v(s)(f_{v,s})(g) = \int_{N_v} f_{v,s}(wng)\, dn,$$

where $f_{v,s}$ is in $V(s)$. Let $f^0_{v,s}$ be the spherical vector in $V(s)$ normalized so that $f^0_{v,s}(1) = 1$. Then (see [GS88, p. 51])

$$M_v(s) f^0_{v,s} = \frac{L_v(2s-1)}{L_v(2s)} f^0_{v,s},$$

where $L_v(s) = (1 - q_v^{-s})^{-1}$. We normalize the operator $M_v(s)$ by defining

$$M^*_v(s) = \gamma_v(2s-1) \cdot M_v(s),$$

where $\gamma_v(s)$ is the γ-factor attached to the trivial representation of GL_1. In particular, $\gamma_v(s) = L_v(1-s)/L_v(s)$ for finite places v. (Note that, since $\prod_v \gamma_v(s) = 1$, this normalization has no effect globally.) An advantage of this normalization is that

$$M^*_v(1-s) \circ M^*_v(s) = \mathrm{Id}.$$

Moreover, we normalize the spherical vector by defining $f^*_{v,s} = L_v(2s) \cdot f^0_{v,s}$. The advantage of this normalization is that

$$M^*_v(s)(f^*_{v,s}) = f^*_{v,1-s}.$$

In order to define Eisenstein series, as in [GS88, p. 52], we define admissible sections $f_s = \otimes f_{v,s}$ as follows. Let S be a finite set of places containing all archimedean places. Then define $f_{v,s} = f^*_{v,s}$ for all $v \notin S$ and, for $v \in S$, we take $f_{v,s}$ to be one of the following two functions:

(i) $f_{v,s}$ is a constant section, i.e., its restriction to a maximal compact K_v does not depend on s;

(ii) $f_{v,s} = M^*_v(1-s)(g_{v,s})$, where $g_{v,s}$ is a constant section.

Note that if $f_{v,s}$ is defined by (ii), then $f_{v,s}$ has a pole at $s = 0$ with residue contained in the trivial submodule of $V(0)$. For an admissible section f_s, define Eisenstein series by

$$E(s, g, f_s) = \sum_{\gamma \in \mathbf{B}(F) \backslash \mathbf{GL}_2(F)} f_s(\gamma g),$$

where $\mathbf{B}$ is the standard Borel subgroup of upper-triangular matrices in $\mathbf{GL}_2$.

5.3 Zeta integrals

Let $\sigma = \bigotimes_v \sigma_v$ be a generic cuspidal automorphic representation of $\mathbf{GSp_6}(\mathbb{A})$. Then, for an admissible section f_s, we have a zeta integral

$$Z(s,\phi,f_s) = \int_{\mathbf{Z}(\mathbb{A})\mathbf{GL}_2(F)\backslash\mathbf{GL}_2(\mathbb{A})}\int_{\mathbf{U}(F)\backslash\mathbf{U}(\mathbb{A})}\phi(\Delta(g)u)\psi_U(u)E(s,g,f_s)\,du\,dg,$$

where ϕ is an automorphic form in the space of σ, f_s is an admissible section, $\mathbf{U}$ is the two-step unipotent radical of the Shalika subgroup $\mathbf{S}$, and $\mathbf{Z}$ is the center of $\mathbf{GL}_2$.

Let $W_\phi = \bigotimes_v W_v$ be the Whittaker function associated to ϕ. The global zeta integral can be rewritten as a product of local zeta integrals $\prod_v Z(s,W_v,f_{v,s})$, where the local factor $Z(s,W_v,f_{v,s})$ is

$$\int_{\mathbf{B}(F_v)\backslash\mathbf{GL}_2(F_v)}\int_{F_v^\times}\int_{F_v^2} W_v\left(\begin{pmatrix} y &&&&& \\ & y &&&& \\ && y &&& \\ & z & x & 1 && \\ && z && 1 & \\ &&&&& 1\end{pmatrix} w\Delta(g)\right)|y|^{s-3} f_{v,s}(g)\,dx\,dz\,d^\times y\,dg$$

where

$$w = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1\end{pmatrix}.$$

In contrast to the formula in [BG92], we do not have the factor $L_v(2s)$ as we have built it into the definition of $f_{v,s}$. If σ_v is supercuspidal and $f_{v,s}$ is a constant section, then the local zeta integral converges for all s. The following is of crucial interest to us: assume that v is finite and take $f_{v,s} = f^0_{v,s}$ in the local zeta integral. Since $f^0_{v,0}$ is the constant function on $\mathbf{GL}_2(F_v)$, the zeta integral at $s=0$ defines a Shalika functional.

The following is claimed as [BG92, Theorem 1].

PROPOSITION 5.7. *Assume that σ_v is unramified, and let W_v be the corresponding (spherical) Whittaker function. Then*

$$Z(s,W_v,f^*_{v,s}) = L(\sigma_v,\mathrm{Spin},s),$$

where the spin L-function on the right-hand side is given by the appropriate Euler factor from the Satake parameters of σ_v.

For every finite place v, we can now define a local γ-factor by

$$\gamma(\sigma_v,s)Z(s,W_v,f_{v,s}) = Z(1-s,W_v,M^*_v(s)(f_{v,s})).$$

The fact that the definition of $\gamma(\sigma_v,s)$ is independent of W_v and $f_{v,s}$, and that it is a rational function in q_v^s, was proved by Vo [Vo97]. Since $M^*_v(1-s)\circ M^*_v(s) = \mathrm{Id}$, we have a local functional equation

$$\gamma(\sigma_v,1-s)\gamma(\sigma_v,s) = 1,$$

for every finite place v. In particular, $\gamma(\sigma_v,s)$ has a pole at $s=1$ if and only if it has a zero at $s=0$.

Assume that σ_v is unramified. Since $M_v^*(s)(f_{v,s}^*) = f_{v,1-s}^*$, Proposition 5.7 implies that

$$\gamma(\sigma_v, s) = \frac{L(\sigma_v, \mathrm{Spin}, 1-s)}{L(\sigma_v, \mathrm{Spin}, s)}.$$

Proposition 5.8. *Let σ_v be a generic supercuspidal irrep of GSp_6. If $\gamma(\sigma_v, s)$ has a zero at $s=0$, then σ_v has a Shalika functional.*

Proof. Let $f_{v,s} = M_v^*(1-s)(g_{v,s})$, where $g_{v,s}$ is a constant section. Note that $f_{v,s}$ can have a pole at $s=0$ with the residue contained in the trivial subrepresentation of $V(0)$. Consider the functional equation

$$\gamma(\sigma_v, s) Z(s, W_v, f_{v,s}) = Z(1-s, W_v, M_v^*(s)(f_{v,s})) = Z(1-s, W_v, g_{v,s}).$$

Since σ_v is supercuspidal, the local zeta integral $Z(1-s, W_v, g_{v,s})$ converges for all s and can be arranged to be non-zero at $s=0$ by a result of Vo [Vo97, Proposition 10.4].

Thus, the functional equation and $\gamma(\sigma_v, 0) = 0$ imply that the zeta integral $Z(s, W_v, f_{v,s})$ has a pole at $s=0$, for some choice of $g_{v,s}$. After taking the residue of $f_{v,s}$ at $s=0$, the zeta integral gives a Shalika functional. □

Proposition 5.9. *Let σ be a generic supercuspidal representation of $\mathrm{GSp}_6 = \mathbf{GSp_6}(k)$ with trivial central character. Let $\gamma(\sigma, s)$ be the local factor defined above by means of zeta integrals. Let $\gamma'(\sigma, s)$ be the analogous local factor constructed by Shahidi [Sha90]. Then the poles and zeros of $\gamma(\sigma, s)$, counted with multiplicity, coincide with poles and zeros of $\gamma'(\sigma, s)$.*

Proof. The proof of this is global and uses the idea of [GRS99a]. Assume, as we may, that the global field F contains a place v such that $F_v \cong k$. Let Σ be a global generic cuspidal automorphic representation such that Σ_w is unramified for all finite places $w \neq v$ and $\Sigma_v \cong \sigma$.

The functional equation for Eisenstein series $E(s, g, f_s)$ [GJ79, p. 232] implies a functional equation of the global zeta integral:

$$Z(s, \phi, f_s) = Z(1-s, \phi, M_s(f_s)).$$

This in turn, implies that

$$\gamma(\Sigma_\infty, s)\gamma(\sigma, s) = \frac{L_S(\Sigma, \mathrm{Spin}, 1-s)}{L_S(\Sigma, \mathrm{Spin}, s)},$$

where $S = S_\infty \cup \{v\}$ is the set of places consisting of all archimedean places S_∞ and v, $L_S(\Sigma, \mathrm{Spin}, s)$ is the corresponding partial L-function, and

$$\gamma(\Sigma_\infty, s) = \prod_{w \in S_\infty} \frac{Z(1-s, W_w, M_{w,s}^*(f_{w,s}))}{Z(s, W_w, f_{w,s})}.$$

We have a similar global equation satisfied by Shahidi's γ-factors. Combining the two gives

$$\gamma(\Sigma_\infty, s)\gamma(\sigma, s) = \gamma'(\Sigma_\infty, s)\gamma'(\sigma, s).$$

Note that, as a consequence, $\gamma(\Sigma_\infty, s)$ does not depend on the choice of $f_{v,s}$.

We need to understand the locations of poles and zeros of $\gamma(\Sigma_\infty, s)$. Fortunately, in [Vo97, Proposition 12.1], Vo showed that for every archimedean place w and every s_0, one can pick a constant section $f_{w,s}$ such that $Z(s_0, W_w, f_{w,s_0}) \neq 0$. He also showed (see [Vo97, Proposition 11.1 and Lemma 11.5]) that the poles of the zeta integral for a constant section at archimedean places lie among the poles of $\Gamma(s_0 + s)$ for finitely many complex numbers s_0. Since the poles of $M_{w,s}^*(f_{w,s})$ are contained on the real axis, it follows that poles of $\gamma(\Sigma_\infty, s)$ are located on

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finitely many lines parallel to the real axis. The same is true for zeros, since

$$\gamma(\Sigma_\infty, s)\gamma(\Sigma_\infty, 1-s) = 1.$$

On the other hand, since $\gamma(\sigma, s)$ is a rational function in q^s, if s_0 is a zero or a pole then so is $s_0 + (2\pi i n/\log q)$ for every integer n. The same is true for Shahidi's factors; poles and zeros of $\gamma'(\Sigma_\infty, s)$ lie on finitely many lines parallel to the real axis, while the poles or zeros of $\gamma'(\sigma, s)$ lie on lines parallel to the imaginary axis. In view of the identity

$$\gamma(\Sigma_\infty, s)\gamma(\sigma, s) = \gamma'(\Sigma_\infty, s)\gamma'(\sigma, s),$$

it follows that poles and zeros of $\gamma(\sigma, s)$ must coincide with poles and zeros of $\gamma'(\sigma, s)$, as desired. □

We can now demonstrate Theorem 5.3, which is encompassed by the theorem below.

THEOREM 5.10. *Let σ be a generic supercuspidal irrep of* PGSp_6. *Then the following three conditions are equivalent:*

(i) *σ has a non-vanishing Shalika functional;*

(ii) *Shahidi's L-function $L(\sigma, \mathrm{Spin}, s)$ has a pole at $s = 0$;*

(iii) *the Bump–Ginzburg–Vo L-function $L(\sigma, \mathrm{Spin}, s)$ has a pole at $s = 0$.*

Proof. We prove the full circle of implications, from the results earlier in the section.

(i) implies (ii). If σ has a non-vanishing Shalika functional, then Shahidi's $L(\sigma, \mathrm{Spin}, s)$ has a pole at $s = 0$ by Theorem 5.6.

(ii) implies (iii). If Shahidi's L-function $L(\sigma, \mathrm{Spin}, s)$ has a pole at $s = 0$, then Shahidi's local factor $\gamma'(\sigma, s)$ has a zero at $s = 0$. By the previous proposition, the local factor for the Bump–Ginzburg–Vo L-function $\gamma(\sigma, s)$ must also have a zero at $s = 0$. It follows that the Bump–Ginzburg–Vo L-function $L(\sigma, \mathrm{Spin}, s)$ has a pole at $s = 0$.

(iii) implies (i). If the Bump–Ginzburg–Vo L-function $L(\sigma, \mathrm{Spin}, s)$ has a pole at $s = 0$, the local factor $\gamma(\sigma, s)$ has a zero at $s = 0$. Then Proposition 5.8 implies that σ has a non-vanishing Shalika functional. □

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