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THE TENSOR PRODUCT OF DISTRIBUTIVE LATTICES II

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In this paper we continue our study of the tensor product of distributive lattices which was begun in (2). We obtain a representation of the tensor product as a ring of sets and we describe a simple way to construct the tensor product of finite distributive lattices.

1. Preliminaries

For terminology and basic results of lattice theory and universal algebra, consult Birkhoff (1) and Grätzer (3), (4). If $a_1, \ldots, a_n$ are elements of a lattice, the join of $a_1, \ldots, a_n$ is written as $a_1 + \cdots + a_n$ or $\Sigma_{i=1}^n a_i$, and the meet of $a_1, \ldots, a_n$ is written as $a_1 \cdots a_n$ or $\Pi_{i=1}^n a_i$. The smallest and largest elements of a lattice, if they exist, are denoted by 0 and 1 respectively. We denote by 2 the two-element lattice consisting of 0 and 1.

2. Elementary properties of the tensor product

We summarize some of the results that were established in our previous work (2).

Definition 2.1. Let A, B and C be distributive lattices. A function $f: A \times B \to C$ is a *bihomomorphism* if the functions $g_a: B \to C$ defined by $g_a(b) = f(a, b)$ and $h_b: A \to C$ defined by $h_b(a) = f(a, b)$ are homomorphisms for each $a \in A$ and $b \in B$.

Definition 2.2. Let A and B be distributive lattices. A distributive lattice C is a *tensor product* of A and B in the category of distributive lattices if there is a bihomomorphism $f: A \times B \to C$ such that C is generated by $f(A \times B)$ and for any distributive lattice D and any bihomomorphism $g: A \times B \to D$ there is a homomorphism $h: C \to D$ satisfying $g = hf$.

Note that since $f(A \times B)$ generates C, the homomorphism h is necessarily unique.

Theorem 2.3. *Let A and B be distributive lattices. Then a tensor product of A and B in the category of distributive lattices exists and is unique up to isomorphism.*

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The tensor product of A and B is denoted by $A \otimes B$ and the image of (a, b) under the bihomomorphism $f: A \times B \to A \otimes B$ is written as $a \otimes b$. In this notation $A \otimes B$ is the distributive lattice generated by the elements $a \otimes b$ $(a \in A, b \in B)$, subject to the bihomomorphic conditions $(a_1 + a_2) \otimes b = (a_1 \otimes b) + (a_2 \otimes b)$, $(a_1 a_2) \otimes b = (a_1 \otimes b)(a_2 \otimes b)$, $a \otimes (b_1 + b_2) = (a \otimes b_1) + (a \otimes b_2)$, and $a \otimes (b_1 b_2) = (a \otimes b_1)(a \otimes b_2)$ for all $a, a_1, a_2 \in A$ and $b, b_1, b_2 \in B$. Every element of $A \otimes B$ can be written in the form $\Sigma_{i=1}^{n} \Pi_{j=1}^{k} (a_{ij} \otimes b_{ij})$ for some $a_{ij} \in A$ and $b_{ij} \in B$, $i = 1, \ldots, n$, $j = 1, \ldots, k$.

3. Isomorphism theorems

It is well known that if A is a distributive lattice and $f: A \to 2$, then f is a homomorphism from A onto 2 if and only if the inverse image of 1 is a prime filter of A. Now let A and B be distributive lattices and let g be a bihomomorphism from $A \times B$ onto 2. We shall characterise those subsets of $A \times B$ which are the inverse images of 1 under such a bihomomorphism.

Definition 3.1. Let A and B be distributive lattices. A non-empty, proper subset C of $A \times B$ is called a *prime bi-filter* of $A \times B$ if for all $a, a_1, a_2 \in A$ and $b, b_1, b_2 \in B$, $(a_1, b_1) \leqslant (a_2, b_2)$ and $(a_1, b_1) \in C$ implies $(a_2, b_2) \in C$; (a_1, b), $(a_2, b) \in C$ implies $(a_1 a_2, b) \in C$; (a, b_1), $(a, b_2) \in C$ implies $(a, b_1 b_2) \in C$; $(a_1 + a_2, b) \in C$ implies $(a_1, b) \in C$ or $(a_2, b) \in C$; and $(a, b_1 + b_2) \in C$ implies $(a, b_1) \in C$ or $(a, b_2) \in C$.

We note that if g is a bihomomorphism from $A \times B$ onto 2 then the set $C = \{(a, b) \in A \times B \mid g(a, b) = 1\}$ is a prime bi-filter. Conversely, if C is a prime bi-filter then the function $g: A \times B \to 2$ defined by $g(a, b) = 1$ if and only if $(a, b) \in C$ is a bihomomorphism from $A \times B$ onto 2. Thus the above definition is an appropriate one.

Let $\mathcal{P}$ be the set of all prime bi-filters of $A \times B$. The set $\mathcal{P}$ is partially ordered by set inclusion. If L is a distributive lattice, we denote by $S(L)$ the set of all prime filters of L, partially ordered by inclusion.

Theorem 3.2. *Let A and B be distributive lattices. Then the set $\mathcal{P}$ of all prime bi-filters of $A \times B$ is isomorphic to $S(A \otimes B)$.*

Proof. Let P be a prime filter of $A \otimes B$ and let h_P be the homomorphism from $A \otimes B$ onto 2 determined by P. Thus h_P is defined by $h_P(\Sigma_i \Pi_j (a_{ij} \otimes b_{ij})) = 1$ if and only if $\Sigma_i \Pi_j (a_{ij} \otimes b_{ij})$ is in P. Let $g_P: A \times B \to 2$ be the bihomomorphism defined by $g_P(a, b) = h_P(a \otimes b)$ and let $C_P = \{(a, b) \in A \times B \mid g_P(a, b) = 1\}$. Then C_P is a prime bi-filter of $A \times B$. The correspondence $P \to C_P$ is clearly well-defined, one-to-one, and onto. It is easy to see that if P and Q are prime filters of $A \otimes B$, then $P \subseteq Q$ if and only if $C_P \subseteq C_Q$. Thus the correspondence $P \to C_P$ is an isomorphism between $S(A \otimes B)$ and $\mathcal{P}$.

We now consider the algebraic structure of $S(A \otimes B)$. If A and B are arbitrary distributive lattices then in general $S(A \otimes B)$ is not a lattice. For example, let A be the four-element lattice $\{0, a, b, 1\}$ with a and b incomparable and let B be the two-element chain 2. Let $C = \{(a, 1), (1, 1)\}$ and $D = \{(b, 1), (1, 1)\}$. Then C and D are prime bi-filters of $A \times B$ and C and D have no lower bound in $\mathcal{P}$. (The only possibility for a lower bound is the set $E = \{(1, 1)\}$, but $(1, 1) = (a + b, 1)$ so that if E were a prime bi-filter then $(a, 1)$ or $(b, 1)$ would have to be in E as well.) Thus $\mathcal{P}$ is not a lattice and so by Theorem 3.2, $S(A \otimes B)$ is not a lattice.

In view of our example, the following affirmative result concerning the structure of $S(A \otimes B)$ seems to be as good as one can expect.

Theorem 3.3. *If A and B are chains, then $S(A \otimes B)$ is a distributive lattice.*

Proof. Let A and B be chains. Then a subset C of $A \times B$ is a prime bi-filter if and only if C is a non-empty, proper subset and for all $a_1, a_2 \in A$ and $b_1, b_2 \in B$, $(a_1, b_1) \leq (a_2, b_2)$ and $(a_1, b_1) \in C$ implies $(a_2, b_2) \in C$. (The other conditions of the definition are necessarily satisfied.)

Now let C and D be prime bi-filters of $A \times B$. Then $(a_1, b_1) \leq (a_2, b_2)$ and $(a_1, b_1) \in C \cap D$ implies $(a_2, b_2) \in C \cap D$. Also $C \cap D$ is a proper subset. Let $(a_1, b_1) \in C$ and $(a_2, b_2) \in D$. Then $(a_1 + a_2, b_1 + b_2) \in C \cap D$, so that $C \cap D$ is non-empty. Thus $C \cap D$ is a prime bi-filter.

Now if $C, D \in \mathcal{P}$, then $C \cup D$ is non-empty. Let $(a_1, b_1) \in C \cup D$, say $(a_1, b_1) \in C$, and let $(a_1, b_1) \leq (a_2, b_2)$. Then $(a_2, b_2) \in C$, so that $(a_2, b_2) \in C \cup D$. Now C and D are proper subsets, so let $(a_1, b_1) \notin C$, $(a_2, b_2) \notin D$. If $(a_1a_2, b_1b_2) \in C \cup D$ then either $(a_1a_2, b_1b_2) \in C$ so that $(a_1, b_1) \in C$, or $(a_1a_2, b_1b_2) \in D$ so that $(a_2, b_2) \in D$. Thus $(a_1a_2, b_1b_2) \notin C \cup D$ and so $C \cup D$ is a proper subset. Hence $C \cup D$ is a prime bi-filter.

Thus $\mathcal{P}$ is a lattice, in fact a lattice of subsets of $A \times B$ under union and intersection, and so $\mathcal{P}$ is a ring of sets and is distributive. It follows by our isomorphism theorem that $S(A \otimes B)$ is a distributive lattice. (Of course, the lattice operations in $S(A \otimes B)$ are not union and intersection.)

We now study the structure of $A \otimes B$ when A and B are arbitrary distributive lattices. We shall obtain a representation of $A \otimes B$ as a ring of subsets of $\mathcal{P}$. First we recall the well known representation theorem of Stone (5) which asserts that for a distributive lattice L the map $x \to \{P \in S(L) | x \in P\}$ is an isomorphism from L to a ring of subsets of $S(L)$. For a tensor product $A \otimes B$, this map is

$$\Sigma_i \Pi_j (a_{ij} \otimes b_{ij}) \to \left\{P \in S(A \otimes B) \,\middle|\, \sum_i \prod_j (a_{ij} \otimes b_{ij}) \in P\right\}$$

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and in particular

$$a \otimes b \to \{P \in S(A \otimes B) | a \otimes b \in P\}.$$

Note that since the elements $a \otimes b (a \in A, b \in B)$ generate $A \otimes B$, the corresponding subsets $\{P \in S(A \otimes B) | a \otimes b \in P\}$ generate the ring of subsets of $S(A \otimes B)$ that is isomorphic to $A \otimes B$. Now the isomorphism $P \to C_p$ from $S(A \otimes B)$ to $\mathcal{P}$ given in the proof of Theorem 3.2 has the property that $a \otimes b \in P$ if and only if $(a, b) \in C_P$. Thus the subset of $\mathcal{P}$ corresponding to $\{P \in S(A \otimes B) | a \otimes b \in P\}$ under this isomorphism is just $\{C \in \mathcal{P} | (a, b) \in C\}$. Similarly we have $\Sigma_1^n (a_i \otimes b_i) \in P$ if and only if $(a_i, b_i) \in C_P$ for some $i = 1, \ldots, n$ and so the subset $\{P \in S(A \otimes B) | \Sigma_1^n (a_i \otimes b_i) \in P\}$ corresponds with $\{C \in \mathcal{P} | (a_i, b_i) \in C$ for some $i = 1, \ldots, n\}$. Finally, $\Sigma_{i=1}^n \Pi_{j=1}^m (a_{ij} \otimes b_{ij}) \in P$ if and only if there is an $i = 1, \ldots, n$ such that for all $j = 1, \ldots, m$, $(a_{ij}, b_{ij}) \in C_P$. Hence $\{P \in S(A \otimes B) | \Sigma_{i=1}^n \Pi_{j=1}^m (a_{ij} \otimes b_{ij}) \in P\}$ corresponds with $\{C \in \mathcal{P} |$ there is an $i = 1, \ldots, n$ such that for all $j = 1, \ldots, m$, $(a_{ij}, b_{ij}) \in C\}$.

Now combining these results with the isomorphism in Stone's theorem, we find that the map

$$\sum_{i=1}^{n} \prod_{j=1}^{m} (a_{ij} \otimes b_{ij}) \to \{C \in \mathcal{P} | \text{ there is an } i = 1, \ldots, n \text{ such that for all } j = 1, \ldots, m, (a_{ij}, b_{ij}) \in C\}$$

is an isomorphism from $A \otimes B$ to a ring of subsets of $\mathcal{P}$. Thus we have the following result.

Theorem 3.4. *Let A and B be distributive lattices. Then $A \otimes B$ is isomorphic to the ring of subsets of $\mathcal{P}$ generated by the subsets of the form $\{C \in \mathcal{P} | (a, b) \in C\}$. Under this isomorphism, the element $\Sigma_{i=1}^n \Pi_{j=1}^m (a_{ij} \otimes b_{ij})$ corresponds with the subset*

$$\{C \in \mathcal{P} | \text{ there is an } i = 1, \ldots, n \text{ such that for all } j = 1, \ldots, m, (a_{ij}, b_{ij}) \in C\}.$$

We saw in (2) that an inequality in $A \otimes B$ of the form $\Pi (a_i \otimes b_i) \leq \Sigma (c_j \otimes d_j)$ could not be characterised solely in terms of relations that hold among the elements a_i, c_j in A and b_i, d_j in B. The following corollary to Theorem 3.4 uses the notion of prime bi-filter to characterise such inequalities in a different way.

Corollary 3.5. *Let A and B be distributive lattices and let $a_i, c_j \in A$ and $b_i, d_j \in B$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$. Then $\Pi_1^n (a_i \otimes b_i) \leq \Sigma_1^m (c_j \otimes d_j)$ if and only if for all $C \in \mathcal{P}$, $\{(a_1, b_1), \ldots, (a_n, b_n)\} \subseteq C$ implies $(c_j, d_j) \in C$ for some $j = 1, \ldots, m$.*

Proof. Using the isomorphism of Theorem 3.4, we have $\Pi_1^n(a_i \otimes b_i) \leq \Sigma_1^m(c_j \otimes d_j)$ if and only if

$$\{C \in \mathcal{P} | (a_i, b_i) \in C \text{ for all } i = 1, \ldots, n\} \subseteq$$
$$\{C \in \mathcal{P} | (c_j, d_j) \in C \text{ for some } j = 1, \ldots, m\}.$$

When A and B are finite distributive lattices Theorem 3.2 provides a simple way of explicitly constructing $A \otimes B$. If A and B are finite, then $A \otimes B$ is also finite. Hence every filter of $A \otimes B$ is principal, and the prime filters are those filters determined by the non-zero join-irreducible elements of $A \otimes B$. Now if x and y are arbitrary non-zero join-irreducible elements of $A \otimes B$ and P_x and P_y are the principal filters determined by x and y then we have $x \leq y$ if and only if $P_x \supseteq P_y$. Thus the partially ordered set of prime filters of $A \otimes B$ is anti-isomorphic to the partially ordered set J of non-zero join-irreducible elements of $A \otimes B$. But by Theorem 3.2, $S(A \otimes B)$ is isomorphic to $\mathcal{P}$. Hence $\mathcal{P}$ is anti-isomorphic to J. Now any finite distributive lattice is isomorphic to the family of all hereditary subsets of its set of non-zero join-irreducible elements (**4**, p. 72). Thus $A \otimes B$ is isomorphic to the family of all hereditary subsets of J. It follows that $A \otimes B$ is isomorphic to the family of all dually hereditary subsets of $\mathcal{P}$.

We illustrate our results with the following example. Let A be 2 and let B be the three-element chain. Then by Theorem 3.3, $\mathcal{P}$ is a distributive lattice under union and intersection. We find that $\mathcal{P}$ consists of eight prime bi-filters. Regarding $\mathcal{P}$ as an abstract eight-element distributive lattice, we

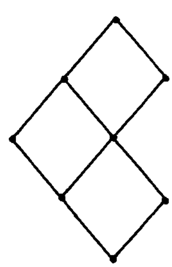

Fig. 1

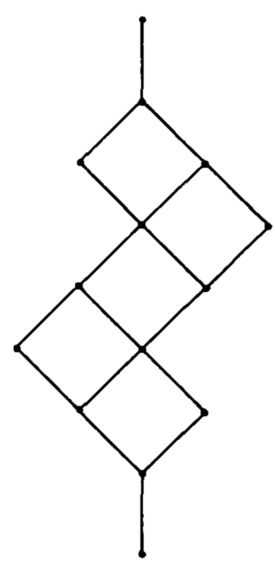

Fig. 2

determine that there are fourteen dually hereditary subsets of $\mathcal{P}$, so that $A \otimes B$ consists of fourteen elements. The diagrams of $\mathcal{P}$ and $A \otimes B$ are given in Figures 1 and 2 respectively.

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