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The Iwasawa theoretic Gross–Zagier theorem

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Abstract

We prove Mazur and Rubin's Λ-adic Gross–Zagier conjecture (under some restrictive hypotheses), which relates Heegner points in towers of number fields to the 2-variable p-adic L-function. The result generalizes Perrin-Riou's p-adic Gross–Zagier theorem.

Introduction

Fix forever a rational prime $p > 2$ and embeddings $\mathbb{Q}^{\mathrm{alg}} \hookrightarrow \mathbb{Q}_p^{\mathrm{alg}}$ and $\mathbb{Q}^{\mathrm{alg}} \hookrightarrow \mathbb{C}$. Fix also a normalized cuspidal newform $f \in S_2(\Gamma_0(N), \mathbb{C})$ and an imaginary quadratic field $K/\mathbb{Q}$ of discriminant D and quadratic character ϵ satisfying the Heegner hypothesis that all primes dividing N are split in K. Assume that $(p, DN) = 1$ and that f is *ordinary* at p in the sense that the Fourier coefficient $a_p(f) \in \mathbb{Q}^{\mathrm{alg}}$ has p-adic absolute value 1 at the fixed embedding $\mathbb{Q}^{\mathrm{alg}} \hookrightarrow \mathbb{Q}_p^{\mathrm{alg}}$. We let $\mathcal{B}_0$ be a number field which is large enough to contain all Fourier coefficients of f, denote by $\mathcal{A}_0$ the integer ring of $\mathcal{B}_0$, and denote by $\mathcal{A}$ and $\mathcal{B}$ the closures of $\mathcal{A}_0$ and $\mathcal{B}_0$ in $\mathbb{Q}_p^{\mathrm{alg}}$, respectively. Let H_s be the ring class field of K of conductor p^s and let H_∞ be the union over all s of H_s. We write $\Gamma = 1 + p\mathbb{Z}_p$, and let $\gamma_0 \in \Gamma$ be a topological generator. Using methods of Hida [Hid85], Perrin-Riou [PR87a, PR88] attached to f a 'two-variable' p-adic L-function

$$\mathcal{L}_f \in \mathcal{A}[[\mathrm{Gal}(H_\infty/K) \times \Gamma]] \otimes_{\mathcal{A}} \mathcal{B}$$

which interpolates the special values of twists of the complex L-function of f at $s = 1$. The p-adic L-function may be expanded as a power series in $\gamma_0 - 1$

$$\mathcal{L}_f = \mathcal{L}_{f,0} + \mathcal{L}_{f,1} \cdot (\gamma_0 - 1) + \cdots, \tag{1}$$

with each $\mathcal{L}_{f,k} \in \mathcal{A}[[\mathrm{Gal}(H_\infty/K)]] \otimes_{\mathcal{A}} \mathcal{B}$. The Heegner hypothesis forces the constant term $\mathcal{L}_{f,0}$ to vanish, and the goal of this paper is to relate the linear term $\mathcal{L}_{f,1}$ to the p-adic height pairings of Heegner points in the f-component of the Jacobian $J_0(N)$.

For every nonnegative integer s the Heegner hypothesis guarantees the existence of a Heegner point $h_s \in X_0(N)(\mathbb{C})$ of conductor p^s; that is, a cyclic N-isogeny of elliptic curves $h_s : E_s \to E'_s$ over $\mathbb{C}$ such that both E_s and E'_s have complex multiplication by *exactly* $\mathcal{O}_s = \mathbb{Z} + p^s\mathcal{O}_K$, the order of conductor p^s in K. The family $\{h_s\}$ may be chosen so that for every $s > 1$ there is a commutative diagram

$$\begin{array}{ccc} E_s & \xrightarrow{h_s} & E'_s \\ \downarrow & & \downarrow \\ E_{s-1} & \xrightarrow{h_{s-1}} & E'_{s-1} \end{array}$$

in which the vertical arrows are p-isogenies. The elliptic curve E_{s-1} (respectively E'_{s-1}) is then

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necessarily the quotient of E_s (respectively E'_s) by its $p\mathcal{O}_{s-1}$-torsion. By the theory of complex multiplication (for example [Cor02, Proposition 1.2]) the curves E_s and E'_s, as well as the isogeny connecting them, can be defined over H_s, and so define a point $h_s \in X_0(N)(H_s)$. One then has the Euler system relations (§ 1.2)

$$T_{p^r}(h_s) = \mathrm{Norm}_{H_{s+r}/H_s}(h_{s+r}) + T_{p^{r-1}}(h_{s-1})$$

if $r, s > 0$, and

$$T_p(h_0) = \begin{cases} u \cdot \mathrm{Norm}_{H_1/H_0}(h_1) + (\sigma_p + \sigma_p^*)h_0 & \text{if } \epsilon(p) = 1 \\ u \cdot \mathrm{Norm}_{H_1/H_0}(h_1) & \text{if } \epsilon(p) = -1 \end{cases}$$

as divisors on $X_0(N)$, where T_{p^r} is the usual Hecke correspondence, $2u = |\mathcal{O}_K^\times|$, and σ_p, $\sigma_p^* \in \mathrm{Gal}(H_0/K)$ are the Frobenius automorphisms of the two primes above p in the case $\epsilon(p) = 1$. Abusing notation, we also denote by h_s the image of h_s in $J_0(N)$ under the usual embedding $X_0(N) \to J_0(N)$ taking the cusp ∞ to the origin.

Let $\mathbf{T}$ be the $\mathbb{Q}$-algebra generated by the action of the Hecke operators T_ℓ with $(\ell, N) = 1$ on $J_0(N)$. The semi-simplicity of $\mathbf{T}$ gives a decomposition of $\mathbf{T} \otimes \mathcal{B}$-modules

$$J_0(N)(H_s) \otimes_{\mathbb{Z}} \mathcal{B} \cong \bigoplus_\beta J(H_s)_\beta$$

where β ranges over $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{alg}}/\mathcal{B})$-orbits of algebra homomorphisms $\beta : \mathbf{T} \to \mathbb{Q}_p^{\mathrm{alg}}$. Each summand is stable under the action of $\mathrm{Gal}(H_s/\mathbb{Q})$, and if $\beta(\mathbf{T}) \subset \mathcal{B}$ then $\mathbf{T}$ acts on $J(H_s)_\beta$ through the character β. The fixed newform f determines one such homomorphism, and we define $h_{s,f}$ to be the projection of h_s onto the associated factor $J(H_s)_f$. Let $\alpha \in \mathcal{A}^\times$ be the unit root of $X^2 - a_p(f)X + p$. As in [BD96], define the *regularized Heegner point* $z_s \in J(H_s)_f$ for $s > 0$ by

$$z_s = \frac{1}{\alpha^s} h_{s,f} - \frac{1}{\alpha^{s+1}} h_{s-1,f}.$$

In the case $s = 0$ we define

$$z_0 = u^{-1} \cdot \begin{cases} \left(1 - \dfrac{\sigma_p}{\alpha}\right)\left(1 - \dfrac{\sigma_p^*}{\alpha}\right) h_{0,f} & \text{if } \epsilon(p) = 1 \\ \left(1 - \dfrac{1}{\alpha^2}\right) h_{0,f} & \text{if } \epsilon(p) = -1. \end{cases}$$

It follows from the Euler system relations that the points z_s are compatible under the norm (trace) maps on $J(H_s)_f$.

The case $s = 0$ of the following theorem is due to Perrin-Riou [PR87a], and has been generalized to higher weight modular forms by Nekovář [Nek95].

Theorem A. *Assume that D is odd and $\neq -3$, and that $\epsilon(p) = 1$. For any character $\eta : \mathrm{Gal}(H_s/K) \to \mathbb{Q}_p^{\mathrm{alg},\times}$*

$$\eta(\kappa_s) \log_p(\gamma_0) \cdot \mathcal{L}_{f,1}(\eta) = \sum_{\sigma \in \mathrm{Gal}(H_s/K)} \eta(\sigma) \langle z_s^\vee, z_s^\sigma \rangle$$

where $\kappa_s \in \mathrm{Gal}(H_s/K)$ is the Artin symbol of $\mathfrak{d}_s = (\sqrt{D}\mathcal{O}_K) \cap \mathcal{O}_s$,

$$\langle\ ,\ \rangle = \langle\ ,\ \rangle_{J_0(N),H_s} : J_0(N)^\vee(H_s) \times J_0(N)(H_s) \to \mathbb{Q}_p$$

is the p-adic height pairing (9) extended $\mathcal{B}$ bilinearly, and $z_s^\vee$ is the image of z_s under the canonical principal polarization of $J_0(N)$ (extended $\mathcal{B}$-linearly on Mordell–Weil groups)

$$J_0(N)(H_s) \otimes \mathcal{B} \cong J_0(N)^\vee(H_s) \otimes \mathcal{B}.$$

Both sides of the stated equality are independent of the choice of γ_0.

Remark 0.0.1. The p-adic height pairing $\langle\ ,\ \rangle_{J_0(N),H_s}$ referred to in the theorem is not uniquely determined (see Proposition 3.2.1 and Remark 3.2.2). We emphasize that Theorem A holds for *any* choice of p-adic height pairing $\langle\ ,\ \rangle_{J_0(N),H_s}$ as in (9).

Remark 0.0.2. Nekovář [Nek95] claims that there is a sign error in the statement of [PR87a, Théorème 1.3], but there is no small amount of confusion over Perrin-Riou's normalization of the height pairing. This is primarily due to the change of sign in Remark 3.3.1, which is our reason for maintaining the distinction between $J_0(N)$ and $J_0(N)^\vee$, and between the pairings (9) and (10). It is also possible that [PR87a] uses a different convention for the reciprocity law of class field theory; see § 3.3.

Remark 0.0.3. Theorem A should hold without the stated hypotheses on D and $\epsilon(p)$. We note that the hypothesis $D \neq -3$ is not assumed in [PR87a].

Now suppose that f has rational Fourier coefficients, $\mathcal{B}_0 = \mathbb{Q}$, and E belongs to the isogeny class of (ordinary!) elliptic curves associated to f. Fix a modular parametrization $X_0(N) \xrightarrow{\phi} E$, and let

$$\phi_* : J_0(N) \to E, \quad \phi^* : E^\vee \to J_0(N)^\vee$$

be the Albanese and Picard maps. Extending ϕ_* and ϕ^* to $\mathbb{Q}_p$-linear maps on Mordell–Weil groups, let $y_s = \phi_*(z_s) \in E(H_s) \otimes \mathbb{Z}_p$ and let $y_s^\vee$ be the unique point of $E^\vee(H_s) \otimes \mathbb{Q}_p$ with $\phi^*(y_s^\vee) = z_s^\vee$. The canonical polarization $E \cong E^\vee$ identifies y_s with $\deg(\phi) \cdot y_s^\vee$. The points y_s and $y_s^\vee$ are norm-compatible as s varies (since the z_s are). Define the Heegner L-function $\mathcal{L}_{\mathrm{Heeg}} \in \mathbb{Z}_p[[\mathrm{Gal}(H_\infty/K)]] \otimes \mathbb{Q}_p$ by

$$\mathcal{L}_{\mathrm{Heeg}} = \varprojlim \sum_{\sigma \in \mathrm{Gal}(H_s/K)} \langle y_s^\vee, y_s^\sigma \rangle_{E,H_s} \cdot \sigma$$

where the pairing is the p-adic height pairing of (9) extended $\mathbb{Q}_p$-linearly (and *not* the height pairing of (10); as E is both a curve and an abelian variety, we have reached a notational singularity). Unlike the height pairing of Theorem A, the pairing $\langle\ ,\ \rangle_{E,H_s}$ is canonical. This follows from the ordinarity of E at p and the uniqueness claims of Proposition 3.2.1. *A priori*, $\mathcal{L}_{\mathrm{Heeg}}$ lives in the larger space $\varprojlim \mathbb{Q}_p[[\mathrm{Gal}(H_s/K)]]$, but it is known that the denominators in the height pairing are bounded as s varies (this follows from the construction of [PR87a], although it is not explicitly stated there; note also Proposition 0.0.4 below).

Theorem B. *Under the hypotheses (and notation) of Theorem A,*

$$\kappa \cdot \log_p(\gamma_0) \cdot \mathcal{L}_{f,1} = \mathcal{L}_{\mathrm{Heeg}}$$

in $\mathbb{Z}_p[[\mathrm{Gal}(H_\infty/K)]] \otimes \mathbb{Q}_p$, *where* $\kappa = \varprojlim \kappa_s \in \mathrm{Gal}(H_\infty/K)$.

Theorem B is a (very slightly) strengthened form of a conjecture of Mazur and Rubin [MR02, Conjecture 9]. To make the connection between our theorem and the conjecture of Mazur and Rubin more explicit, first note that the construction of the p-adic height $\langle\ ,\ \rangle_{E,H_s}$ depends on the auxiliary choice of the idèle class character $\rho_{H_s} : \mathbf{A}_{H_s}^\times / H_s^\times \to \Gamma \xrightarrow{\log_p} \mathbb{Z}_p$ defined at the start of § 3.3. Define $\Gamma_{\mathbb{Q}_p} = \Gamma \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and extend $\log_p$ to a $\mathbb{Q}_p$-linear isomorphism $\Gamma_{\mathbb{Q}_p} \cong \mathbb{Q}_p$. Define a pairing

$$\langle\ ,\ \rangle_{E,H_s}^\Gamma : E^\vee(H_s) \times E(H_s) \to \Gamma_{\mathbb{Q}_p}$$

by $\langle\ ,\ \rangle_{E,H_s} = \log_p \circ \langle\ ,\ \rangle_{E,H_s}^\Gamma$ and set

$$\mathcal{L}_{\mathrm{Heeg}}^\Gamma = \varprojlim \sum_{\sigma \in \mathrm{Gal}(H_s/K)} \langle y_s, y_s^\sigma \rangle_{E,H_s}^\Gamma \cdot \sigma \in \mathbb{Z}_p[[\mathrm{Gal}(H_\infty/K)]] \otimes \Gamma_{\mathbb{Q}_p},$$

where we have now identified $E \cong E^\vee$ in the canonical way, so that

$$(1 \otimes \log_p)(\mathcal{L}^\Gamma_{\mathrm{Heeg}}) = \deg(\phi) \cdot \mathcal{L}_{\mathrm{Heeg}}.$$

Let I be the kernel of the projection

$$\mathbb{Z}_p[[\mathrm{Gal}(H_\infty/K) \times \Gamma]] \otimes \mathbb{Q}_p \ \to \ \mathbb{Z}_p[[\mathrm{Gal}(H_\infty/K)]] \otimes \mathbb{Q}_p$$

and let $w : \mathbb{Z}_p[[\mathrm{Gal}(H_\infty/K)]] \otimes \Gamma_{\mathbb{Q}_p} \ \to \ I/I^2$ be the isomorphism defined by $w(\lambda \otimes \gamma) = \lambda(\gamma - 1)$. Thus $w(\mathcal{L}^\Gamma_{\mathrm{Heeg}}) = \deg(\phi) \log_p(\gamma_0)^{-1} \mathcal{L}_{\mathrm{Heeg}} \cdot (\gamma_0 - 1)$. As $\mathcal{L}_{f,0} = 0$, the p-adic L-function $\mathcal{L}_f$ is contained in I, and Theorem B may be rewritten as

$$\kappa \cdot \mathcal{L}_f = \kappa \cdot \mathcal{L}_{f,1} \cdot (\gamma_0 - 1) = \frac{1}{\log_p(\gamma_0)} \mathcal{L}_{\mathrm{Heeg}} \cdot (\gamma_0 - 1) = \frac{1}{\deg(\phi)} w(\mathcal{L}^\Gamma_{\mathrm{Heeg}})$$

in I/I^2.

Now assume the hypotheses of Theorem A, and also that $\mathrm{Gal}(K^{\mathrm{alg}}/K)$ surjects onto the $\mathbb{Z}_p$-module automorphisms of $T_p(E)$ and that p does not divide the class number of K. Let $K_\infty \subset H_\infty$ be the anticyclotomic $\mathbb{Z}_p$-extension of K, and set $K_s = K_\infty \cap H_{s+1}$, so that $[K_s : K] = p^s$. Define $\Lambda_{\mathrm{anti}} = \mathbb{Z}_p[[\mathrm{Gal}(K_\infty/K)]] \otimes \mathbb{Q}_p$, and

$$\mathcal{S}(K_s, E) = \varprojlim_k \mathrm{Sel}_{p^k}(K_s, E), \quad \mathcal{S}_\infty = (\varprojlim_s \mathcal{S}(K_s, E)) \otimes \mathbb{Q}_p$$
$$X = \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{Sel}_{p^\infty}(K_\infty, E), \mathbb{Q}_p/\mathbb{Z}_p) \otimes \mathbb{Q}_p.$$

Let $\tilde{y}_\infty \in \mathcal{S}_\infty$ be the inverse limit of $\tilde{y}_s = \mathrm{Norm}_{H_{s+1}/K_s}(y_{s+1}) \in \mathcal{S}(K_s, E)$, and define the *Heegner submodule* $\mathcal{H} \subset \mathcal{S}_\infty$ to be the Λ_{anti}-submodule generated by $\tilde{y}_\infty$. It follows from work of Cornut and Vatsal [Cor02, Vat02] that $\mathcal{H}$ is a free Λ_{anti}-module of rank one. It is known by work of Bertolini and the author [Ber95, How04] that X is a finitely-generated rank-one Λ_{anti}-module, $\mathcal{S}_\infty$ is free of rank one, and

$$\mathrm{char}(X_{\mathrm{tors}}) \text{ divides } \mathrm{char}(\mathcal{S}_\infty/\mathcal{H}) \cdot \mathrm{char}(\mathcal{S}_\infty/\mathcal{H})^\iota \tag{2}$$

where X_{tors} denotes the Λ_{anti}-torsion submodule of X, and $\lambda \mapsto \lambda^\iota$ is the involution of Λ_{anti} which is inversion on group-like elements. Perrin-Riou [PR87b, Conjecture B] has conjectured that the divisibility (2) is an equality.

Proposition 0.0.4 (Perrin-Riou [PR87b, PR91, PR92]). *There is a p-adic height pairing*

$$\mathfrak{h}_s : \mathcal{S}(K_s, E) \times \mathcal{S}(K_s, E) \ \to \ c^{-1}\mathbb{Z}_p$$

whose restriction to the image of the Kummer map $E(K_s) \otimes \mathbb{Z}_p \ \to \ \mathcal{S}(K_s, E)$ agrees with the pairing $\langle \ , \ \rangle_{E,K_s}$ of (9) after identifying $E \cong E^\vee$ in the canonical way, where $c \in \mathbb{Z}_p$ is independent of s.

There is a Λ_{anti}-adic height pairing $\mathfrak{h}_\infty : \mathcal{S}_\infty \times \mathcal{S}_\infty \ \to \ \Lambda_{\mathrm{anti}}$ defined by

$$\mathfrak{h}_\infty(\varprojlim a_s, \varprojlim b_s) = \varprojlim \sum_{\sigma \in \mathrm{Gal}(K_s/K)} \mathfrak{h}_s(a_s, b_s^\sigma) \cdot \sigma,$$

and we define the Λ_{anti}-adic regulator $\mathcal{R}$ to be the image of this map. If

$$e : \mathbb{Z}_p[[\mathrm{Gal}(H_\infty/K)]] \otimes \mathbb{Q}_p \ \to \ \Lambda_{\mathrm{anti}}$$

is the natural projection, then the norm compatibility of the height pairing (see Remark 3.2.2; in this case the compatibility is *automatic* by the uniqueness claim of Proposition 3.2.1 and the fact that E is ordinary at p) gives

$$e(\mathcal{L}_{\mathrm{Heeg}})\Lambda_{\mathrm{anti}} = \mathfrak{h}_\infty(\tilde{y}_\infty, \tilde{y}_\infty)\Lambda_{\mathrm{anti}} = \mathrm{char}(\mathcal{S}_\infty/\mathcal{H}) \cdot \mathrm{char}(\mathcal{S}_\infty/\mathcal{H})^\iota \cdot \mathcal{R}.$$

If we assume that $\mathcal{R} \neq 0$, then Theorem B allows us to rewrite the divisibility (2) as

$$\mathrm{char}(X_{\mathrm{tors}}) \text{ divides } \frac{e(\mathcal{L}_{f,1})\Lambda_{\mathrm{anti}}}{\mathcal{R}}, \tag{3}$$

which now has the look and feel of a Λ_{anti}-adic form of the Birch and Swinnerton-Dyer conjecture and no longer makes any mention of Heegner points. It was conjectured by Mazur and Rubin [MR02, Conjecture 6] that $\mathcal{R} = \Lambda_{\mathrm{anti}}$, but those authors have since retracted that conjecture.

Note that the hypothesis on the action of Galois on the p-adic Tate module excludes the case where E has complex multiplication. Results similar to (3) in the so-called exceptional case where E has complex multiplication by K can be found in [AH03].

0.1 Plan of the proof

Enlarging $\mathcal{B}_0$ if needed, we may assume that $\mathcal{A}_0$ contains the Fourier coefficients of all normalized newforms of level dividing N, so that all algebra maps $\mathbf{T} \to \mathbb{Q}^{\mathrm{alg}}$ take values in $\mathcal{B}_0$. Fix $s > 0$ and define, for each integer $0 \leqslant i \leqslant s$, degree 0 divisors on $X_0(N)_{/H_s}$

$$c_i = (h_i) - (0), \quad d_i = (h_i) - (\infty).$$

For any pair $0 \leqslant i, j \leqslant s$ and any $\sigma \in \mathrm{Gal}(H_s/K)$ we define a p-adic modular form

$$F_\sigma^{i,j} = \sum_\beta \langle c_i, d_{j,\beta}^\sigma \rangle f_\beta \ \in S_2(\Gamma_0(N), \mathcal{B}_0) \otimes_{\mathcal{B}_0} \mathcal{B}$$

where the sum is over algebra homomorphisms $\beta : \mathbf{T} \to \mathcal{B}_0$, f_β is the associated normalized primitive (i.e. new of some level dividing N) eigenform, $\langle\ ,\ \rangle = \langle\ ,\ \rangle_{X_0(N),H_s}$ is the p-adic height pairing (10) on degree zero divisors of $X_0(N)_{/H_s}$ (viewed as a pairing on $J_0(N)(H_s)$ and extended $\mathcal{B}$-linearly; by Remark 3.3.1 this is *minus* the pairing of Theorem A) and the β subscript on d_j indicates projection to the component $J(H_s)_\beta$. Define a p-adic cusp form

$$F_\sigma = U^2 F_\sigma^{s,s} - U F_\sigma^{s,s-1} - U F_\sigma^{s-1,s} + F_\sigma^{s-1,s-1} \in S_2(\Gamma_0(Np), \mathcal{B}_0) \otimes_{\mathcal{B}_0} \mathcal{B}$$

where U is the Atkin–Lehner U_p defined by $U(\sum a_m q^m) = \sum a_{mp} q^m$. For $(m, N) = 1$, the mth Fourier coefficient of F_σ is given by the formula (see Proposition 7.0.6)

$$a_m(F_\sigma) = \langle c_s, T_{mp^2}(d_s^\sigma) \rangle - \langle c_s, T_{mp}(d_{s-1}^\sigma) \rangle - \langle c_{s-1}, T_{mp}(d_s^\sigma) \rangle + \langle c_{s-1}, T_m(d_{s-1}^\sigma) \rangle. \tag{4}$$

The pairs of divisors occurring in this expression will not be relatively prime for many values of m, but if we define divisors

$$\mathbf{h}_{s,r} = \mathrm{Norm}_{H_{s+r}/H_s}(h_{s+r}), \quad \mathbf{d}_{s,r} = \mathrm{Norm}_{H_{s+r}/H_s}(d_{s+r})$$

on $X_0(N)$ and write $m = m_0 p^r$ with $(m_0, p) = 1$, then the Euler system relation allow us to rewrite (4) as

$$a_m(F_\sigma) = \langle c_s, T_{m_0}(\mathbf{d}_{s,r+2}^\sigma) \rangle - \langle c_{s-1}, T_{m_0}(\mathbf{d}_{s,r+1}^\sigma) \rangle. \tag{5}$$

The pairs of divisors occurring here are relatively prime: the geometric points of $T_{m_0}(\mathbf{h}_{s,r})$ represent elliptic curves with complex multiplication (CM) by an order $\mathcal{O}$ for which $\mathrm{ord}_p(\mathrm{cond}(\mathcal{O})) = r + s$. Working with these divisors allows us to avoid the 'intersection theory with tangent vectors' used by Gross–Zagier to deal with divisors having common support.

In § 2 we recall some p-adic analytic results of Hida and Perrin-Riou. In particular, we recall the construction of a p-adic modular form $G_\sigma \in M_2(\Gamma_0(Np^\infty), \mathcal{A})$ (a space defined at the beginning of § 2) for each $\sigma \in \mathrm{Gal}(H_s/K)$, with the property that

$$\log_p(\gamma_0) \cdot \mathcal{L}_{f,1}(\eta) = \sum_{\sigma \in \mathrm{Gal}(H_s/K)} \eta(\sigma) L_f(G_\sigma)$$

for every character η of $\mathrm{Gal}(H_s/K)$. Here L_f is a linear functional

$$L_f : M_2(\Gamma_0(Np^\infty), \mathcal{A}) \to \mathcal{B}$$

which plays the Hida-theoretic role of taking the Petersson inner product with f.

Perrin-Riou gives an explicit formula for the Fourier coefficient $a_m(G_\sigma)$ when p divides m (Proposition 2.0.5), and in §§ 4, 5, and 6 we adapt the methods of Gross–Zagier and Perrin-Riou to compute (to the extent necessary) the Fourier coefficients of F_σ. More precisely, each Fourier coefficient has a decomposition over the finite places of H_s, $a_m(F_\sigma) = \sum_v a_m(F_\sigma)_v$, arising from the decomposition of the p-adic heights in (5) into local p-adic Néron symbols on $X_0(N)_{/H_{s,v}}$. For v lying above a rational prime $\neq p$ which splits in K, $a_m(F_\sigma)_v = 0$ (Proposition 4.0.8). For v above a nonsplit rational prime $\ell \neq p$ we derive an explicit formula (Proposition 5.4.1) for $\sum_{v|\ell} a_m(F_\sigma)_v$ similar to formulas of Gross–Zagier. For $v \mid p$ we can offer no explicit formula for $a_m(F_\sigma)_v$, instead we show that the contribution of $a_m(F_\sigma)_p$ to $a_m(F_\sigma)$ is killed by the operator L_f (Proposition 6.2.2). This is where we must impose the condition $\epsilon(p) = 1$, although Proposition 6.2.2 should also hold when $\epsilon(p) = -1$. Comparing these calculations with the Fourier coefficients of G_σ, we conclude that

$$L_f(U^{2s}(1 - U^2)G_{\sigma\kappa}) = L_f(F_\sigma),$$

and Theorems A and B follow easily (see § 7 for the details).

0.2 Notation and conventions

The data K, p, N, D, f, $\mathcal{A}_0$, and $\{h_s\}$ are fixed throughout. We continue to assume, as in § 0.1, that $\mathcal{A}_0$ contains the Fourier coefficients of all normalized primitive forms of level N. We typically do *not* assume that D is odd or $\neq -3, -4$, or that $\epsilon(p) = 1$, unless explicitly stated otherwise. The parity assumption on D is needed only for the results of Perrin-Riou cited in § 2. The condition $\epsilon(p) = 1$ and $D \neq -3, -4$ is used in the calculation of local Néron symbols above p in § 6.

If M is any $\mathbb{Z}$-module of finite type and r is a rational prime we set $M_r = M \otimes_{\mathbb{Z}} \mathbb{Z}_r$. For any integer n, any order $\mathcal{O} \subset K$, and any proper fractional $\mathcal{O}$-ideal $\mathfrak{a}$, we denote by $r_{\mathfrak{a}}(n)$ the number of proper, integral $\mathcal{O}$-ideals of norm n whose class in $\mathrm{Pic}(\mathcal{O})$ agrees with that of $\mathfrak{a}$. The order $\mathcal{O}$ will usually be clear from the context. If there is any ambiguity we will write $r_{\mathfrak{a}\mathcal{O}}(n)$. Since complex conjugation acts by inversion on $\mathrm{Pic}(\mathcal{O})$, $r_{\mathfrak{a}}(n) = r_{\mathfrak{a}^{-1}}(n)$. We define $R_{\mathfrak{a}}(n)$ to be the number of proper, integral $\mathcal{O}$-ideals of norm n in the $\mathcal{O}$-genus of $\mathfrak{a}$; that is, such that the image in $\mathrm{Pic}(\mathcal{O})/\mathrm{Pic}(\mathcal{O})^2$ agrees with the image of $\mathfrak{a}$. For any integer k we set

$$\delta(k) = 2^{\#\{\text{prime divisors of } (k,D)\}}.$$

The reciprocity map of class field theory is always normalized in the arithmetic fashion.

1. Preliminaries on elliptic curves

1.1 CM points, Heegner diagrams, and Serre's construction

Let S be an $\mathcal{O}_K$-scheme and let $\mathcal{O} = \mathcal{O}[c] \subset \mathcal{O}_K$ be the order of conductor c. Assume $(c, N) = 1$. An elliptic curve $E \to S$ is said to have CM by $\mathcal{O}$ if there is an embedding $\mathcal{O} \hookrightarrow \mathrm{End}_S(E)$. We always assume that such an embedding is normalized, in the sense that the action of $\mathcal{O}$ on the pull-back of the tangent sheaf of E by the identity section agrees with the action given by viewing the structure sheaf of S as a sheaf of $\mathcal{O}$-algebras. We say that $\mathcal{O}$ is the CM-order of E, or that E has CM by exactly $\mathcal{O}$, if this action does not extend to any larger order. A *Heegner diagram* of conductor c over S, h, is an $\mathcal{O}$-linear cyclic N-isogeny of elliptic curves $h : E \to E'$ over S, such that E and E' both have CM by exactly $\mathcal{O}$. An isogeny of Heegner diagrams means an isogeny of the underlying

$\Gamma_0(N)$-structure; i.e. a commutative diagram

$$\begin{array}{ccc} E_0 & \xrightarrow{h_0} & E_0' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ E_1 & \xrightarrow{h_1} & E_1' \end{array}$$

in which the vertical arrows are isogenies of elliptic curves over S, and the map f takes the scheme-theoretic kernel of h_0 isomorphically to the scheme-theoretic kernel of h_1. The degree of such an isogeny is defined to be the degree of f, which is also the degree of f'. Any Heegner diagram h over S gives rise to an S-valued point of $X_0(N)_{/\mathbb{Z}}$, which we also denote by h. Since $X_0(N)$ is not a fine moduli space, Heegner diagrams which are not isomorphic over S may give rise to the same S-valued point on $X_0(N)$.

If E is an elliptic curve over S with CM by $\mathcal{O}$ and $\mathfrak{a}$ is a proper fractional $\mathcal{O}$-ideal, a theorem of Serre [Con04, Theorem 7.2] guarantees that the functor from S-schemes to $\mathcal{O}$-modules $T \mapsto E(T) \otimes_{\mathcal{O}} \mathfrak{a}$ is represented by an elliptic curve which we denote by $E \otimes_{\mathcal{O}} \mathfrak{a}$. Define $E^{\mathfrak{a}} = E \otimes_{\mathcal{O}} \mathfrak{a}^{-1}$. As in [Con04, Corollary 7.11], this construction extends to Heegner diagrams, and so to any Heegner diagram $h : E \to E'$ of conductor c over S and any $\mathfrak{a}$ as above, we obtain a new Heegner diagram

$$h^{\mathfrak{a}} : E^{\mathfrak{a}} \to E'^{\mathfrak{a}}.$$

If $S = \mathrm{Spec}(\mathbb{C})$ and E is an elliptic curve over S with CM by exactly $\mathcal{O}$, then $E(\mathbb{C}) \cong \mathbb{C}/\mathfrak{b}$ for some proper fractional $\mathcal{O}$-ideal $\mathfrak{b}$, and we have an analytic isomorphism $E^{\mathfrak{a}}(\mathbb{C}) \cong \mathbb{C}/\mathfrak{a}^{-1}\mathfrak{b}$. By the Main Theorem of Complex Multiplication, the right-hand side is analytically isomorphic to $E^{\sigma}(\mathbb{C})$ for any $\sigma \in \mathrm{Aut}(\mathbb{C}/K)$ whose restriction to $H[c]$ (the ring class field of conductor c) agrees with $\mathfrak{a}$ under the Artin map $\mathrm{Pic}(\mathcal{O}) \cong \mathrm{Gal}(H[c]/K)$. In particular, E has a model over $H[c]$, E^{σ} and $E^{\mathfrak{a}}$ are isomorphic over $\mathbb{C}$, and $\mathrm{Gal}(H[c]/K)$ acts transitively on the $\mathbb{C}$-isomorphism classes of elliptic curves over $H[c]$ with CM by exactly $\mathcal{O}$. Similarly all Heegner diagrams over $\mathbb{C}$ of conductor c have models over the ring class field of conductor c. If h is a Heegner diagram of conductor c defined over $H[c]$, we define the *orientation* of h to be the annihilator in $\mathcal{O}$ of the kernel of $h : E(\mathbb{C}) \to E'(\mathbb{C})$. It is an ideal $\mathcal{N}$ of $\mathcal{O}$ such that $\mathcal{O}/\mathcal{N} \cong \mathbb{Z}/N\mathbb{Z}$. Then $\mathrm{Gal}(H[c]/K)$ acts transitively on the $\mathbb{C}$-isomorphism classes of conductor c Heegner points with a given orientation.

1.2 Hecke action on CM points

Let $\mathfrak{L}$ denote the set of lattices in K, modulo multiplication by $K^{\times}$. The $K^{\times}$-class of a lattice L will be denoted $[L]$. For any $[L] \in \mathfrak{L}$ we define the conductor of $[L]$ to be the conductor of the left order of L; that is, the conductor of the order $\mathcal{O}(L) = \{\alpha \in K \mid \alpha L \subset L\}$. Every lattice of conductor c is represented uniquely (up to $K^{\times}$ action) by an element of $\mathrm{Pic}(\mathcal{O})$, where $\mathcal{O} \subset K$ is the order of conductor c.

We have the usual action of Hecke operators $\{T_m\}$ on formal sums of classes in $\mathfrak{L}$, which we wish to make explicit. The following lemma is an elementary exercise.

Lemma 1.2.1. *Suppose we are given orders $\mathcal{O}$ and $\mathcal{O}'$ of K of conductors c and d, respectively, and a proper fractional $\mathcal{O}$-ideal $\mathfrak{c}$ (respectively $\mathcal{O}'$-ideal $\mathfrak{d}$). If $c \mid d$ then the multiplicity of $[\mathfrak{c}]$ in the formal sum $T_m[\mathfrak{d}]$ is equal to $r_{\mathfrak{c}\mathfrak{d}^{-1}\mathcal{O}}(mc/d)$. If instead $d \mid c$, then the multiplicity of $[\mathfrak{c}]$ in $T_m[\mathfrak{d}]$ is given by $|\mathcal{O}'^{\times}||\mathcal{O}^{\times}|^{-1} r_{\mathfrak{c}\mathfrak{d}^{-1}\mathcal{O}'}(md/c)$.*

Lemma 1.2.2 (Euler system relations). *With notation as in the introduction and $2u = |\mathcal{O}_K^{\times}|$,*

$$T_{p^r}(h_s) = \mathrm{Norm}_{H_{s+r}/H_s}(h_{s+r}) + T_{p^{r-1}}(h_{s-1})$$

if $r, s > 0$, *and*

$$T_p(h_0) = \begin{cases} u \cdot \mathrm{Norm}_{H_1/H_0}(h_1) + (\sigma_p + \sigma_p^*)h_0 & \text{if } \epsilon(p) = 1 \\ u \cdot \mathrm{Norm}_{H_1/H_0}(h_1) & \text{if } \epsilon(p) = -1. \end{cases}$$

Proof. We give a brief sketch of the proof of the first relation. Let $\mathfrak{d}$ be a proper $\mathcal{O}_{s+r}$-ideal such that $\mathbb{C}/\mathfrak{d} \cong E_{s+r}(\mathbb{C})$, and for any $0 \leqslant t \leqslant s+r$, set $\mathfrak{d}_t = \mathfrak{d}\mathcal{O}_t$, so that $E_t(\mathbb{C}) \cong \mathbb{C}/\mathfrak{d}_t$. By the theory of complex multiplication, the complex elliptic curves underlying the $\Gamma_0(N)$-structures appearing in the divisor $\mathrm{Norm}_{H_{s+r}/H_s}(h_{s+r})$ are exactly the complex tori of the form $\mathbb{C}/\mathfrak{d}'$ where $\mathfrak{d}'$ is a proper $\mathcal{O}_{s+t}$-ideal satisfying $\mathfrak{d}'\mathcal{O}_s = \mathfrak{d}_s$. Using Lemma 1.2.1, such a $\mathfrak{d}'$ occurs exactly once in the formal sum $T_{p^r}[\mathfrak{d}_s]$, and does not occur in $T_{p^{r-1}}[\mathfrak{d}_{s-1}]$. As the formal sum of lattices $T_{p^r}[\mathfrak{d}_s] - T_{p^{r-1}}[\mathfrak{d}_{s-1}]$ has degree p^r, it must be exactly the formal sum of $[\mathfrak{d}']$ with $\mathfrak{d}'$ as above. □

1.3 The Serre–Tate theorem

We recall the Serre–Tate theory of deformations of elliptic curves. More detail can be found in [Con04, § 3] and [Gor02, ch. 6]. Let k be a field of nonzero characteristic ℓ and define $\mathcal{C}_k$ to be the category of local Artinian algebras $(R, \mathfrak{m}_R)$ with residue field k, together with a chosen isomorphism $R/\mathfrak{m}_R \cong k$, with morphisms given by local algebra maps inducing the identity on k. Given an elliptic curve $E \to \mathrm{Spec}(k)$, and some $R \in \mathcal{C}_k$, we define a *deformation* of E to R to be an elliptic curve $E_R \to \mathrm{Spec}(R)$ together with an isomorphism between the closed fiber of E_R and E. Similarly, we may define the notion of a deformation of the ℓ-divisible group of an elliptic curve over k. For $(R, \mathfrak{m}_R)$ an object of $\mathcal{C}_k$, let DEF_R denote the category of pairs (E, G) where E is an elliptic curve over k and G is a deformation to R of the ℓ-divisible group of E. A morphism from (E, G) to (E', G') is a pair (f, ϕ) where $f : E \to E'$ is a morphism of elliptic curves over $\mathrm{Spec}(k)$ and $\phi : G \to G'$ is a map of ℓ-divisible groups such that the base change of ϕ to the closed fiber is the map on ℓ-divisible groups over $\mathrm{Spec}(k)$ induced by f.

Theorem 1.3.1 (Serre–Tate). *For any object* $(R, \mathfrak{m}_R)$ *of* $\mathcal{C}_k$, *the functor from elliptic curves over* R *to* DEF_R *which sends* E *to the pair* $(E \times_R k, E[\ell^\infty])$ *is an equivalence of categories, where* $E[\ell^\infty]$ *denotes the* ℓ*-divisible group of* E.

Now assume that k is algebraically closed and fix an ordinary elliptic curve E over k. We have $E[\ell^\infty] \cong \mu_{\ell^\infty} \oplus \mathbb{Q}_\ell/\mathbb{Z}_\ell$ as ℓ-divisible groups over k. For any $R \in \mathcal{C}_R$ there is a distinguished deformation of the ℓ-divisible group of E to an ℓ-divisible group over R, namely the deformation $\mu_{\ell^\infty} \oplus \mathbb{Q}_\ell/\mathbb{Z}_\ell$. Applying the Serre–Tate theorem, we obtain an elliptic curve over R called the *Serre–Tate canonical lift* of E to R.

As explained in [Con04, § 3], a theorem of Grothendieck allows one to replace 'local Artinian' by 'complete local Noetherian' in the definition of $\mathcal{C}_k$, and the discussion above holds verbatim.

2. The *p*-adic *L*-function

In this section we quickly recall the essential properties of Hida's p-adic L-function $\mathcal{L}_f$ and Perrin-Riou's calculation of its linear term. We refer the reader to [Hid85, Nek95, PR87a] for more detailed treatments. Assume that D is odd. Recall that $\mathcal{A}_0 \subset \mathbb{Q}^{\mathrm{alg}}$ is the ring of integers of a number field with closure $\mathcal{A}$ in $\mathbb{Q}_p^{\mathrm{alg}}$, $\mathcal{B}$ is the fraction field of $\mathcal{A}$, and $\alpha \in \mathcal{A}^\times$ is the unit root of $X^2 - a_p(f)X + p$.

Set

$$M_2(\Gamma_0(Np^k), \mathcal{A}) = M_2(\Gamma_0(Np^k), \mathcal{A}_0) \otimes_{\mathcal{A}_0} \mathcal{A}$$

and let $M_2(\Gamma_0(Np^\infty), \mathcal{A})$ be the completion of $\bigcup_k M_2(\Gamma_0(Np^k), \mathcal{A})$ with respect to the p-adic supremum norm on Fourier coefficients. To any $s \geqslant 0$, $\sigma \in \mathrm{Gal}(H_s/K)$, and integer C prime to Dp, Perrin-Riou [PR87a, § 2.2.3] associates a measure Φ_σ^C on $\mathbb{Z}_p^\times$ with values in the space $M_2(\Gamma_0(Np^\infty), \mathcal{A})$. These are compatible as s and σ vary in the following sense: there is a measure Φ^C

on $\mathrm{Gal}(H_\infty/K)\times\mathbb{Z}_p^\times$ with values in $M_2(\Gamma_0(Np^\infty),\mathcal{A})$ such that for any continuous characters

$$\eta:\mathrm{Gal}(H_\infty/K)\ \to\ \mathbb{Q}_p^{\mathrm{alg},\times},\quad \psi:\mathbb{Z}_p^\times\ \to\ \mathbb{Q}_p^{\mathrm{alg},\times}$$

such that η factors through $\mathrm{Gal}(H_s/K)$ we have the relation

$$\int_{\mathrm{Gal}(H_\infty/K)\times\mathbb{Z}_p^\times}\eta\psi\ d\Phi^C=\sum_{\sigma\in\mathrm{Gal}(H_s/K)}\eta(\sigma)\int_{\mathbb{Z}_p^\times}\psi\ d\Phi^C_\sigma$$

in $M_2(\Gamma_0(Np^\infty),\mathcal{A})\otimes_{\mathcal{A}}\mathbb{Q}_p^{\mathrm{alg}}$.

Use the notation $\tilde{T}_\ell$ to denote Hecke operators acting on modular forms of level $\Gamma_0(Np^\infty)$, to distinguish them from the operators on level $\Gamma_0(N)$. Define Hida's ordinary projector [Hid93, § 7.2]

$$e^{\mathrm{ord}}:M_2(\Gamma_0(Np^\infty),\mathcal{A})\ \to\ M_2(\Gamma_0(Np),\mathcal{A})$$

by $e^{\mathrm{ord}}(g)=\lim_{k\to\infty}U^{k!}(g)$, where $U=\tilde{T}_p$ is given by $U(\sum a_nq^n)=\sum a_{np}q^n$ and the limit is with respect to the supremum norm on Fourier coefficients. Define modular forms of level $\Gamma_0(Np)$ by

$$f_0(z)=f(z)-\frac{p}{\alpha}f(pz),\quad f_1(z)=f(z)-\alpha f(pz).$$

These are eigenforms for *all* Hecke operators $\tilde{T}_\ell$, and satisfy $a_\ell(f_0)=a_\ell(f)=a_\ell(f_1)$ if $\ell\neq p$, and $a_p(f_0)=\alpha$, $a_p(f_1)=p/\alpha$. The $\mathcal{B}$-algebra generated by the Hecke operators $\tilde{T}_\ell$ with $(\ell,Np)=1$ acting on $M_2(\Gamma_0(Np),\mathcal{A})\otimes_{\mathcal{A}}\mathcal{B}$ is semi-simple, and so contains an idempotent e_f such that $e_f\circ\tilde{T}_\ell=a_\ell(f)e_f$. By [Hid85, § 4] there is an idempotent e_{f_0} in the algebra generated by *all* Hecke operators $\tilde{T}_\ell$, such that $e_{f_0}\circ\tilde{T}_\ell=a_\ell(f_0)e_{f_0}$ for every ℓ. As operators on modular forms, $e_{f_0}=e_{f_0}e_f$. Define a linear functional

$$l_f:M_2(\Gamma_0(Np^\infty),\mathcal{A})\otimes_{\mathcal{A}}\mathcal{B}\ \to\ \mathcal{B}$$

by $l_f(g)=a_1(e_{f_0}e^{\mathrm{ord}}g)$, and set $L_f=(1-p/\alpha^2)(1-1/\alpha^2)l_f$ (this is denoted $\tilde{L}_{f_0}$ in [PR87a]).

Lemma 2.0.2. *The linear functional $L_f:M_2(\Gamma_0(Np^\infty),\mathcal{A})\otimes_{\mathcal{A}}\mathcal{B}\ \to\ \mathcal{B}$ satisfies:*

(a) $L_f=L_f\circ e^{\mathrm{ord}}$;
(b) $L_f(f)=1-1/\alpha^2$;
(c) *if $g\in M_2(\Gamma_0(Np^\infty),\mathcal{A})$ is such that $a_m(g)=0$ for all $(m,N)=1$, then $L_f(g)=0$;*
(d) *for any positive integer m, $L_f\circ\tilde{T}_m=a_m(f_0)L_f$; in particular, $L_f\circ U=\alpha L_f$.*

Proof. The first claim is trivial, since $e^{\mathrm{ord}}\circ e^{\mathrm{ord}}=e^{\mathrm{ord}}$. The second follows from $l_f(f_0)=1$, $l_f(f_1)=0$. If g satisfies $a_m(g)=0$ for all $(m,N)=1$, then so does $e_fe^{\mathrm{ord}}g$, so we may assume that g has level $\Gamma_0(Np)$ and that $\tilde{T}_\ell g=a_\ell(f)g$ for $(\ell,Np)=1$. By Atkin–Lehner theory, g is a linear combination of f_0 and f_1. Since $a_1(g)=0$, g must be a scalar multiple of f_0-f_1. But $a_p(f_0-f_1)\neq 0$, so this scalar must be 0. The final claim follows from $e_{f_0}\circ\tilde{T}_m=a_m(f_0)e_{f_0}$. □

Remark 2.0.3. Contrary to the proof of [Nek95, Proposition II.5.10], the weaker hypothesis that $a_m(g)=0$ for all $(m,Np)=1$ is not sufficient to conclude that $L_f(g)=0$. The modular form $g=f_0-f_1$ provides a counterexample.

Whenever ψ is a continuous character of Γ, we extend ψ to a character of $\mathbb{Z}_p^\times$ using the usual projection $\langle\ \rangle:\mathbb{Z}_p^\times\ \to\ \Gamma$. We now define the p-adic L-function $\mathcal{L}_f$ of the introduction (compare [PR87a, Définition 2.4], but note that Perrin-Riou's $\psi(C)=\psi(\mathrm{Frob}_{C\mathcal{O}_K})$ is our $\psi(C)^2$). For any continuous character $\eta\cdot\psi$ of $\mathrm{Gal}(H_\infty/K)\times\Gamma$, set

$$\mathcal{L}_f(\eta,\psi)=\frac{1}{1-C\epsilon(C)\psi(C)^{-2}}\cdot L_f\left(\int_{\mathrm{Gal}(H_\infty/K)\times\mathbb{Z}_p^\times}\eta\cdot\psi\ d\Phi^C\right),$$

where C is chosen so that $(1-C\epsilon(C)\langle C\rangle^{-2}) \in \mathbb{Z}_p[[\Gamma]]^\times$. The resulting $\mathcal{L}_f \in \mathcal{A}[[\mathrm{Gal}(H_\infty/K)\times\Gamma]]\otimes_{\mathcal{A}}\mathcal{B}$ does not depend on the choice of C. Any finite-order character $\eta\cdot\psi$ of $\mathrm{Gal}(H_\infty/K)\times\Gamma$ determines a character

$$\chi(\mathfrak{b}) = \eta(\mathrm{Frob}_{\mathfrak{b}})\cdot\psi(\mathbf{N}(\mathfrak{b}))$$

on ideals of $\mathcal{O}_K$ prime to p, and there is an interpolation formula [PR87a, Théorème 1.1] relating $\mathcal{L}_f(\eta,\psi)$ to $L(f,\bar{\chi},1)$, where $L(f,\bar{\chi},s)$ is the Rankin product of the L-function of f and the L-function of the theta series associated to $\bar{\chi}$.

PROPOSITION 2.0.4. *Let* $\mathbf{1}$ *denote the trivial character of* Γ. *Then* $\mathcal{L}_f(\eta,\mathbf{1}) = 0$ *for all continuous characters* η *of* $\mathrm{Gal}(H_\infty/K)$. *Furthermore, in the notation of (1),* $\mathcal{L}_{f,0} = 0$ *and*

$$\log_p(\gamma_0)\cdot\mathcal{L}_{f,1}(\eta) = \sum_{\sigma\in\mathrm{Gal}(H_s/K)}\eta(\sigma)L_f(G_\sigma)$$

for every character η *of* $\mathrm{Gal}(H_s/K)$, *where* $G_\sigma \in M_2(\Gamma_0(Np^\infty),\mathcal{A})$ *is defined by*

$$G_\sigma = \frac{1}{1-C\epsilon(C)}\cdot\int_{\mathbb{Z}_p^\times}\log_p\ d\Phi_\sigma^C.$$

Proof. Fix an integer $s>0$. For each $\sigma\in\mathrm{Gal}(H_s/K)$ define

$$\mathcal{L}^\sigma(\psi) = \frac{1}{1-C\epsilon(C)\psi(C)^{-2}}\cdot\int_{\mathbb{Z}_p^\times}\psi\ d\Phi_\sigma^C \in M_2(\Gamma_0(Np^\infty),\mathcal{A}),$$

a function on continuous characters ψ of Γ with the property that

$$\mathcal{L}_f(\eta,\psi) = \sum_{\sigma\in\mathrm{Gal}(H_s/K)}\eta(\sigma)L_f(\mathcal{L}^\sigma(\psi))$$

for any ψ and any character η of $\mathrm{Gal}(H_s/K)$. By [PR87a, Remarque 3.19] $a_m(\mathcal{L}^\sigma(\mathbf{1})) = 0$ whenever $p\mid m$, and so $U\mathcal{L}^\sigma(\mathbf{1}) = 0$. Lemma 2.0.2(d) now implies $L_f(\mathcal{L}^\sigma(\mathbf{1})) = 0$. Since s and η were arbitrary, we deduce $\mathcal{L}_f(\eta,\mathbf{1}) = 0$ for all finite order η, hence for all continuous η (since $\mathcal{L}_f(\ ,\mathbf{1}) \in \mathcal{A}[[\mathrm{Gal}(H_\infty/K)]]\otimes_{\mathcal{A}}\mathcal{B}$). This is equivalent to $\mathcal{L}_{f,0} = 0$. Finally, recall that $\langle\ \rangle$ denotes the projection $\mathbb{Z}_p^\times \to \Gamma$ and compute

$$\begin{aligned}\lim_{t\to0}\frac{\mathcal{L}_f(\eta,\langle\ \rangle^t)}{t} &= \sum_{\sigma\in\mathrm{Gal}(H_s/K)}\frac{d}{dt}\left[\frac{\eta(\sigma)}{1-C\epsilon(C)\langle C\rangle^{-2t}}\cdot L_f\left(\int_{\mathbb{Z}_p^\times}\langle x\rangle^t\,d\Phi_\sigma^C(x)\right)\right]_{t=0}\\ &= \sum_{\sigma\in\mathrm{Gal}(H_s/K)}\frac{\eta(\sigma)}{1-C\epsilon(C)}\cdot\frac{d}{dt}\left[L_f\left(\int_{\mathbb{Z}_p^\times}\langle x\rangle^t\,d\Phi_\sigma^C(x)\right)\right]_{t=0}\end{aligned}$$

where in the second equality we have used the fact, proved above, that $L_f(\int_{\mathbb{Z}_p^\times}\mathbf{1}\,d\Phi_\sigma^C) = 0$. Differentiating under the integral and using $\log_p(\gamma_0)\mathcal{L}_{f,1}(\eta) = \lim_{t\to0}(1/t)\mathcal{L}_f(\eta,\langle\ \rangle^t)$ proves the claim. □

Fix $s\geqslant 0$ and $\sigma\in\mathrm{Gal}(H_s/K)$. Choose a proper integral $\mathcal{O}_s$-ideal, $\mathfrak{a}$, such that the class of $\mathfrak{a}$ in $\mathrm{Pic}(\mathcal{O}_s)$ corresponds to σ under the Artin symbol. For any positive integer n prime to p and any positive divisor $d\mid n$, define

$$\epsilon_{\mathfrak{a}}(n,d) = \begin{cases}\left(\frac{D_1}{d}\right)\left(\frac{D_2}{-Nn/d}\right)\chi_{D_1,D_2}(\mathfrak{a}\mathcal{O}_K) & \text{if } \gcd(d,n/d,D) = 1\\ 0 & \text{otherwise}\end{cases}$$

where $D = D_1D_2$ is the factorization into fundamental discriminants with $(d,D) = |D_2|$ and χ_{D_1,D_2} is the associated genus character. That is, the quadratic character of $\mathrm{Pic}(\mathcal{O}_K)$ associated to the

extension $K(\sqrt{D_1}) = K(\sqrt{D_2})$. Set

$$\sigma'_{\mathfrak{a}}(n) = \sum_{\substack{d|n \\ d>0}} \epsilon_{\mathfrak{a}}(n,d) \log_p(n/d^2).$$

Proposition 2.0.5 (Perrin-Riou). *For any positive integer m divisible by p, the mth Fourier coefficient of G_σ is given by*

$$a_m(G_\sigma) = - \sum_{\substack{n>0 \\ (n,p)=1}} r_{\mathfrak{a}\mathfrak{d}_s}(m|D| - nN)\sigma'_{\mathfrak{a}}(n)$$

where $\mathfrak{d}_s = (\sqrt{D}\mathcal{O}_K) \cap \mathcal{O}_s$.

Proof. This is [PR87a, Proposition 3.18], where G_σ is denoted $L'_{p,\sigma,\langle\ \rangle}$. The missing minus sign in the statement of Perrin-Riou's Proposition 3.18 is a typographical error, as the proof makes clear.

In Perrin-Riou's statement $r_{\mathfrak{a}\mathfrak{d}_s}$ appears as $r_{\mathfrak{a}'}$ where $\mathfrak{a}' = \mathfrak{D}\mathfrak{a}$, and (p. 484) '$\mathfrak{D}$ est le $\mathcal{O}_s$-idéal engendré par $\sqrt{D}$'. That is, $\mathfrak{D} = \sqrt{D}\mathcal{O}_s \neq \mathfrak{d}_s$. Later, on p. 486, Perrin-Riou writes 'Lorsque $s = 0$, $\mathfrak{a}'$ et $\mathfrak{a}$ sont équivalent', although under the stated definition of $\mathfrak{D}$ they are equivalent even when $s \neq 0$, suggesting that an unannounced change of notation has occurred. The formulas of [PR87a, § 3.2.3] are correct with $\mathfrak{D}$ defined as above, while those of [PR87a, § 3.3] are correct with $\mathfrak{D}$ replaced by our $\mathfrak{d}_s$. In particular, in the proof of [PR87a, Lemme 3.17] one must interpret $\mathfrak{D}$ as our $\mathfrak{d}_s$ in order to pass from (3.7) to (3.8) ('On remplace ensuite n par $\delta_2 n \dots$'). The key point is

$$r_{\mathfrak{D}_1^{-1}\mathfrak{a}}(m\delta_1 - nN) = r_{\mathfrak{D}_1^{-1}\mathfrak{D}_2\mathfrak{a}}(m\delta - n\delta_2 N)$$

in which $\delta = |D| = \delta_1\delta_2$ and $\mathfrak{D}_i$ is the $\mathcal{O}_s$-ideal of norm δ_i (the equality is seen by using the map on $\mathcal{O}_s$-ideals $\mathfrak{b} \mapsto \mathfrak{D}_2\mathfrak{b}$ to identify the sets of ideals being counted). Using $\mathfrak{D}_1^{-1}\mathfrak{D}_2 = \mathfrak{d}_s$ in $\mathrm{Pic}(\mathcal{O}_s)$, one obtains the correct formula. Also, the first displayed equation in the proof of [PR87a, Lemme 3.17] appears to be in error; the two p-adic modular forms in the second equality differ by shifting Fourier coefficients by δ_1 (see [PR87a, (2.4) and Lemme 3.1]). This misstatement has no effect on the proof.

Perrin-Riou's $\mathfrak{a}$ is our $\mathfrak{a}^{-1}$, but both $r_{\mathfrak{a}\mathfrak{d}_s}$ and $\sigma'_{\mathfrak{a}}$ are unchanged by $\mathfrak{a} \mapsto \mathfrak{a}^{-1}$. For $\sigma'_{\mathfrak{a}}$ this is obvious; for $r_{\mathfrak{a}\mathfrak{d}_s}$ use the fact that inversion agrees with complex conjugation in $\mathrm{Pic}(\mathcal{O}_s)$, the fact that complex conjugation preserves norms, and the fact that $\mathfrak{d}_s$ has order two in $\mathrm{Pic}(\mathcal{O}_s)$. □

Lemma 2.0.6. *Suppose that n is prime to p and that there exists a proper integral $\mathcal{O}_s$-ideal $\mathfrak{b}$ in the $\mathrm{Pic}(\mathcal{O}_s)$-class of $\mathfrak{a}$ with $\mathbf{N}(\mathfrak{b}) \equiv -nN \pmod{Dp}$. Then*

$$\sigma'_{\mathfrak{a}}(n) = \sum_{\ell|n} \log_p(\ell) \cdot \begin{cases} 0 & \text{if } \epsilon(\ell) = 1 \\ \mathrm{ord}_\ell(\ell n)\delta(n)R_{\mathfrak{anc}}(n/\ell) & \text{if } \epsilon(\ell) = -1 \\ \mathrm{ord}_\ell(n)\delta(n)R_{\mathfrak{anc}}(n/\ell) & \text{if } \epsilon(\ell) = 0 \end{cases}$$

where in the second and third cases $\mathfrak{n}$ is any integral $\mathcal{O}_s$-ideal of norm N and $\mathfrak{c}$ is any proper integral $\mathcal{O}_s$-ideal with $\mathbf{N}(\mathfrak{c}) \equiv -\ell \pmod{Dp}$.

Proof. By [GZ86, Proposition IV.4.6(b)], the stated equality holds with $R_{\mathfrak{anc}}(n/\ell)$ replaced by $R_{\mathfrak{anc}\mathcal{O}_K}(n/\ell)$; that is, if we count integral $\mathcal{O}_K$-ideals of norm n/ℓ in the $\mathcal{O}_K$-genus of $\mathfrak{anc}\mathcal{O}_K$. So, we only need show that $R_{\mathfrak{anc}}(n/\ell) = R_{\mathfrak{anc}\mathcal{O}_K}(n/\ell)$ under the stated hypotheses. The map $I \mapsto I\mathcal{O}_K$ takes the collection $\mathfrak{R}_{\mathfrak{anc}}(n/\ell)$ of proper $\mathcal{O}_s$-ideals of norm n/ℓ in the $\mathcal{O}_s$-genus of $\mathfrak{anc}$ injectively to the set $\mathfrak{R}_{\mathfrak{anc}\mathcal{O}_K}(n/\ell)$ of proper $\mathcal{O}_K$-ideals of norm n/ℓ in the $\mathcal{O}_K$-genus of $\mathfrak{anc}\mathcal{O}_K$. It suffices to show that this map has an inverse. More precisely, we show that the map $J \mapsto J \cap \mathcal{O}_s$ from integral $\mathcal{O}_K$-ideals of norm prime to p to integral $\mathcal{O}_s$-ideals of norm prime to p restricts to a map $\mathcal{R}_{\mathfrak{anc}\mathcal{O}_K}(n/\ell) \to \mathcal{R}_{\mathfrak{anc}}(n/\ell)$.

Suppose that $I = J \cap \mathcal{O}_s$ is an integral $\mathcal{O}_s$-ideal of norm n/ℓ such that $J \in \mathfrak{R}_{\mathfrak{anc}\mathcal{O}_K}(n/\ell)$. Set $p^* = (-1)^{(p-1)/2}p$. Genus theory (for example, [Cox89, § 6.A] discusses the genus theory of $\mathcal{O}_K$ at length, and that of $\mathcal{O}_s$ is similar) gives a canonical isomorphism

$$\mathrm{Pic}(\mathcal{O}_s)/\mathrm{Pic}(\mathcal{O}_s)^2 \cong \mathrm{Pic}(\mathcal{O}_K)/\mathrm{Pic}(\mathcal{O}_K)^2 \times \mathrm{Gal}(K(\sqrt{p^*})/K)$$

under which the $\mathcal{O}_s$-genus of I is sent to the $\mathcal{O}_K$-genus of $J = I\mathcal{O}_K$ in the first factor, and to its Artin symbol

$$\left(\frac{I}{K(\sqrt{p^*})/K}\right) = \left(\frac{\mathbf{N}(I)}{\mathbb{Q}(\sqrt{p^*})/\mathbb{Q}}\right)$$

in the second factor. The same holds with I replaced by $\mathfrak{bnc}$, and since the $\mathcal{O}_K$-genera of J and $\mathfrak{bnc}\mathcal{O}_K$ agree by assumption, $I \in \mathfrak{R}_{\mathfrak{anc}}(n/\ell) = \mathfrak{R}_{\mathfrak{bnc}}(n/\ell)$ if and only if

$$\left(\frac{\mathbf{N}(I)}{\mathbb{Q}(\sqrt{p^*})/\mathbb{Q}}\right) = \left(\frac{\mathbf{N}(\mathfrak{bnc})}{\mathbb{Q}(\sqrt{p^*})/\mathbb{Q}}\right)$$

which occurs if and only if

$$\left(\frac{\mathbf{N}(I)}{p}\right) = \left(\frac{\mathbf{N}(\mathfrak{bnc})}{p}\right).$$

Since $\mathbf{N}(I) = n/\ell$ and $\mathbf{N}(\mathfrak{bnc}) \equiv nN^2\ell \pmod{p}$ we are done. □

Corollary 2.0.7. *Let $\kappa \in \mathrm{Gal}(H_s/K)$ be the Artin symbol of $\mathfrak{d}_s$. For any positive integer m divisible by p, the mth Fourier coefficient of $G_{\sigma\kappa}$ is given by the expression*

$$-\sum_{\substack{n>0\\(n,p)=1}}\sum_{\ell|n}\log_p(\ell)\cdot r_{\mathfrak{a}}(m|D|-nN)\cdot\begin{cases}0 & \text{if } \epsilon(\ell)=1\\ \mathrm{ord}_\ell(\ell n)\delta(n)R_{\mathfrak{anc}}(n/\ell) & \text{if } \epsilon(\ell)=-1\\ \mathrm{ord}_\ell(n)\delta(n)R_{\mathfrak{anc}}(n/\ell) & \text{if } \epsilon(\ell)=0\end{cases}$$

where in the second and third cases $\mathfrak{n}$ is any integral $\mathcal{O}_s$-ideal of norm N and $\mathfrak{c}$ is any proper integral $\mathcal{O}_s$-ideal with $\mathbf{N}(\mathfrak{c}) \equiv -\ell \pmod{Dp}$.

Proof. Combine Proposition 2.0.5 and Lemma 2.0.6, and use $\sigma'_{\mathfrak{a}} = \sigma'_{\mathfrak{a}\mathfrak{d}_s}$ (which follows from the definition of σ' and the fact that $\mathfrak{d}_s\mathcal{O}_K$ is principal) and $\kappa^2 = 1$. □

3. The p-adic height pairing

In this section we recall some known facts about p-adic Néron symbols and p-adic height pairings on abelian varieties and, when the abelian variety is the Jacobian of a curve, the connection with p-adic Néron symbols and intersection theory on the curve.

3.1 Intersection theory

Let R be a complete discrete valuation ring, $S = \mathrm{Spec}(R)$. Let $\underline{X} \to S$ be an integral, proper scheme over S with generic fiber a smooth curve X, and suppose $\underline{C}$ and $\underline{D}$ are effective Cartier divisors with no common components. Define the intersection multiplicity $i_y(\underline{C}, \underline{D})$ at a closed point y of $\underline{X}$ to be the length of the $\mathcal{O}(\underline{X})_y$-module $\mathcal{O}(\underline{X})_y/(f, g)$ where f and g are defining equations of $\underline{C}$ and $\underline{D}$ in a neighborhood of y. Define the total intersection multiplicity $i(\underline{C}, \underline{D}) = \sum_y i_y(\underline{C}, \underline{D})[k(y) : k(s)]$ where s is the closed point of S and the sum is over closed points of $\underline{X}$.

We now assume that $\underline{X}$ is regular (in particular, we do not need to distinguish between Weil divisors and Cartier divisors), and record some fundamental properties of the total intersection multiplicity. We refer the reader to [Gro85] and [La88, ch. III] for details. The total intersection multiplicity is bi-additive, and so extends to divisors with rational coefficients. We define,

for C and D degree zero divisors on X with disjoint support,

$$[C, D] = i(\underline{C} + C', \underline{D}) = i(\underline{C}, \underline{D} + D')$$

where $\underline{C}$ and $\underline{D}$ are the horizontal divisors on $\underline{X}$ whose generic fibers are C and D, respectively, and C' (respectively D') is a fibral divisor with rational coefficients chosen so that the symbol $i(\underline{C}+C',\ )$ (respectively $i(\ , \underline{D} + D')$) vanishes on all fibral divisors. Let L be the fraction field of R and let v denote the normalized valuation on L, so that $v(\varpi) = 1$ for a uniformizer π. If $C = (f)$ is a principal divisor then $[C, D] = v(f(D))$ where $D = \sum n_i(D_i)$ is a linear combination of prime divisors D_i with residue field L_i and

$$f(D) = \prod_i \mathbf{N}_{L_i/L}(f(D_i)^{n_i}). \tag{6}$$

3.2 p-adic Néron symbols, I

We now define local p-adic Néron symbols on abelian varieties. The contents of this subsection are taken from [PR87a, § 4] essentially verbatim.

Let ℓ be a rational prime and L a finite extension of $\mathbb{Q}_\ell$. Let A be an abelian variety over L and assume that either $\ell \neq p$ or that A has good reduction. Fix a nontrivial continuous additive character $\rho : L^\times \to \mathbb{Z}_p$. If $\ell = p$ we assume that ρ is ramified.

Proposition 3.2.1. *There is a $\mathbb{Q}_p$-valued Néron symbol $\langle \mathfrak{C}, d\rangle = \langle \mathfrak{C}, d\rangle_{A,\rho}$ defined whenever $\mathfrak{C}$ is an algebraically trivial divisor on A, d is a zero cycle of degree zero on A rational point-by-point over L, and the supports of $\mathfrak{C}$ and d have no common points. This symbol satisfies:*

(a) *$\langle\ ,\ \rangle$ is bilinear (whenever this makes sense) and invariant under translation by elements of $A(L)$;*

(b) *if $\mathfrak{C} = (h)$ is principal then $\langle \mathfrak{C}, d\rangle = \rho(h(d))$, where $h(d) = \prod_i f(d_i)$ is defined as in (6);*

(c) *for any endomorphism $\phi : A \to A$, $\langle \phi^*\mathfrak{C}, d\rangle = \langle \mathfrak{C}, \phi_* d\rangle$;*

(d) *for any $x_0 \in A(L)$ and any $\mathfrak{C}$ as above, the function $x \mapsto \langle \mathfrak{C}, (x) - (x_0)\rangle$ is continuous for the ℓ-adic topology on $A(L)$;*

(e) *if $\ell = p$, L' is a finite extension of L contained in the $\mathbb{Z}_p$-extension of L cut out by ρ, and $\mathfrak{C}$ is a degree zero divisor on $A_{/L'}$, then*

$$\langle \mathbf{N}_{L'/L}\mathfrak{C}, d\rangle \subset c^{-1}\rho(\mathbf{N}_{L'/L}(L'))$$

whenever this is defined, for some constant $c \in \mathbb{Z}_p$ independent of L', $\mathfrak{C}$, and d.

Furthermore, if $\ell \neq p$, or if $\ell = p$ and A has ordinary reduction, then such a symbol is unique.

Proof. In the case $\ell \neq p$, or $\ell = p$ but A has ordinary reduction, see the references after [PR87a, Théorème 4.2] for existence. In the case $\ell = p$ with nonordinary reduction, the existence is [PR87a, Théorème 4.7]. The translation invariance is not stated explicitly by Perrin-Riou, but follows from the construction as in [Blo80, Lemma 2.14]. We sketch the proof of the uniqueness. If $\langle\ ,\ \rangle'$ is another such symbol then we may define

$$G(\mathfrak{C}, x) = \langle \mathfrak{C}, (x) - (0)\rangle - \langle \mathfrak{C}, (x) - (0)\rangle'.$$

This defines a function $A^\vee(L) \times A(L) \to \mathbb{Q}_p$ which is linear in the first variable and continuous in the second. Using translation invariance and the theorem of the square [Mil86, Theorem 6.7], one can show that G is also linear in the second variable. Hence for fixed $\mathfrak{C}$, $G(\mathfrak{C},\ )$ defines a continuous linear map $A(L) \to \mathbb{Q}_p$. If $\ell \neq p$ this map must be trivial for topological reasons. If $\ell = p$ and A has ordinary reduction, then $A^\vee$ also has ordinary reduction, and [Maz72, Proposition 4.39] implies that the universal norms from the (ramified) $\mathbb{Z}_p$-extension cut out by ρ have finite index in $A^\vee(L)$. From this and the boundedness property (e), we see that G is identically zero. □

When $\ell \neq p$ the Néron symbol is compatible with base extension in the following sense. If L'/L is a finite extension, $A' = A \times_L L'$, and $\rho' = \rho \circ \mathbf{N}_{L'/L}$, then

$$\langle \mathfrak{C}, d \rangle_{A',\rho'} = \langle \mathbf{N}_{L'/L}\mathfrak{C}, d \rangle_{A,\rho} \tag{7}$$

for $\mathfrak{C}$ an algebraically trivial divisor on A' and d a point-by-point rational zero cycle of degree zero on A. This allows us to remove the hypothesis in Proposition 3.2.1 that d is rational point-by-point, by choosing an extension L'/L over which d becomes pointwise rational and defining

$$\langle \mathfrak{C}, d \rangle_{A,\rho} = [L' : L]^{-1} \langle \mathfrak{C}, d \rangle_{A',\rho'}.$$

This is independent of the choice of L' by (7). Proposition 3.2.1(b) continues to hold for this slight extension of the Néron symbol, provided that one extends the definition of $h(d)$ as in (6).

When $\ell = p$ the Néron symbol on A may not uniquely determined by the properties above, but one can choose a compatible family (in the sense that (7) holds) of Néron symbols $\langle\ ,\ \rangle_{A',\rho'}$ as L' varies over the finite extensions of L. Again, this allows one to remove the hypothesis that d is defined point-by-point. Perrin-Riou only states the existence of compatible families for subfields of the extension of L cut out by ρ, but the same argument holds for all finite extensions.

Remark 3.2.2. Although the choice of a Néron symbol on A in residue characteristic p is (sometimes) not unique, our results do not depend on the choice. Hence we fix, once and for all, a choice of Néron symbol on $J_0(N)_{H_s,v}$ for every s and every prime v of H_s above p, with the understanding that these choices are compatible as s varies in the sense of (7).

Now suppose that A is the Jacobian of a smooth, proper, geometrically connected curve X over L, and that X has an L-rational point ∞. Let $\alpha : X \to A$ be the canonical embedding $x \mapsto (x) - (\infty)$. Suppose we are given degree zero divisors C and D on X with disjoint support. Pullback by α restricts to an isomorphism $\alpha^* : \mathrm{Pic}^0(A) \to \mathrm{Pic}^0(X)$, and so there is an algebraically trivial divisor $\mathfrak{C}$ whose associated line bundle pulls back to the line bundle associated to C. Thus $C = \alpha^*\mathfrak{C} + (f)$ for some rational function f on X. The pair $(\mathfrak{C}, f)$ may be chosen so that (f) is disjoint from D and then it follows that $\mathfrak{C}$ has no points in common with $\alpha_* D$. We now define

$$\langle C, D \rangle_{X,\rho} = \langle \mathfrak{C}, \alpha_* D \rangle_{A,\rho} + \rho(f(D)), \tag{8}$$

where $f(D)$ is defined by (6). This is independent of the choice of $\mathfrak{C}$ (by Proposition 3.2.1(b)) and the choice of f (which is determined up to $L^\times$ once $\mathfrak{C}$ is chosen).

3.3 *p*-adic Néron symbols, II

Identifying Γ with the Galois group of the unique $\mathbb{Z}_p$-extension of $\mathbb{Q}$ via the cyclotomic character, the reciprocity map of class field theory and the p-adic logarithm define an idèle class character

$$\rho_{\mathbb{Q}} : \mathbf{A}_{\mathbb{Q}}^\times / \mathbb{Q}^\times \to \Gamma \xrightarrow{\log_p} \mathbb{Z}_p.$$

Fix a finite extension $L/\mathbb{Q}$, let ρ_L be the idèle class character of L defined by $\rho_L = \rho_{\mathbb{Q}} \circ \mathbf{N}_{L/\mathbb{Q}}$. For each finite place v of L, let π_v be a uniformizer of L_v and let $\mathbf{N}(v)$ denote the absolute residue degree of v. We may decompose $\rho_L = \sum_v \rho_{L_v}$ as a sum of local characters, and then $\rho_{L_v}(\pi_v) = \log_p(\mathbf{N}(v))$ for any prime v not above p. We note that this does not agree with [PR87a, p. 501], which seems to be in error (note also the remarks of [Nek95, § II.6.4]), although perhaps this is attributable to a different normalization of class field theory. We remind the reader that we always use the *arithmetic* conventions.

Let A be an abelian variety over L with good reduction above p. Summing the local Néron symbols $\langle\ ,\ \rangle_v = \langle\ ,\ \rangle_{A_v,\rho_{L_v}}$ on the completions $A_v = A \times_L L_v$ defines a bilinear pairing on Mordell–Weil groups

$$\langle\ ,\ \rangle_{A,L} : A^\vee(L) \times A(L) \to \mathbb{Q}_p. \tag{9}$$

Indeed, given $a \in A^\vee(L)$ and $b \in A(L)$, let $\mathfrak{C}$ be an algebraically trivial divisor on A which represents a and let $d = \sum n_i(d_i)$ be a zero cycle of degree zero on A with $\sum n_i d_i = b$. These can be chosen so that $\mathfrak{C}$ and d have no points in common and we then define

$$\langle a, b\rangle_{A,L} = \sum_v \langle \mathfrak{C}, d\rangle_v$$

where the sum is over the finite places of L. A different choice of $\mathfrak{C}$ changes the pairing by

$$\sum_v \langle (h), d\rangle_v = \sum_v \rho_{L,v}(h(d)) = \rho_L(h(d)) = 0$$

for some rational function h on A. Now fix $\mathfrak{C}$ and consider the expression $\sum_v \langle \mathfrak{C}, d\rangle_v$. We have just seen that this depends only on the linear equivalence class of $\mathfrak{C}$ (which is translation invariant), and thus the translation invariance of each $\langle\ ,\ \rangle_v$ shows that $\sum_v \langle \mathfrak{C}, d\rangle_v$ is translation invariant in the second variable (with $\mathfrak{C}$ held fixed). From this one may deduce

$$\sum_v \langle \mathfrak{C}, d\rangle_v = \sum_v \langle \mathfrak{C}, (b) - (0)\rangle_v,$$

and so the left-hand side depends only on b and not on the choice of d.

Now suppose X is a proper, smooth, geometrically connected curve over L with an L-rational point, and that A is the Jacobian of X. Let $\alpha : X \to A$ be the associated canonical embedding. For each place v of L we have from § 3.2 a $\mathbb{Q}_p$-valued symbol $\langle\ ,\ \rangle_{X_v,\rho_{L_v}}$ on disjoint divisors on $X_v = X \times_L L_v$. By summing over all places, we obtain a symbol

$$\langle\ ,\ \rangle_{X,L} = \sum_v \langle\ ,\ \rangle_{X_v,\rho_{L_v}} \tag{10}$$

defined on degree zero divisors of X with disjoint support. This pairing descends to a (symmetric) pairing on linear equivalence classes (this follows from Proposition 3.3.2(a,b) below and the fact that $\rho = \sum_v \rho_{L_v}$ vanishes on $L^\times$). In particular, $\langle\ ,\ \rangle_{X,L}$ extends bilinearly to all pairs of degree zero divisors, without the assumption of disjoint support.

Remark 3.3.1. As $\langle\ ,\ \rangle_{X,L}$ is defined on linear equivalence classes, it descends to a bilinear pairing

$$\langle\ ,\ \rangle_{X,L} : A(L) \times A(L) \to \mathbb{Q}_p.$$

which agrees with the pairing $-\langle\ ,\ \rangle_{A,L}$ when one identifies $A \cong A^\vee$ via the canonical principal polarization [PR87a, § 4.3].

Proposition 3.3.2. *Let v be a prime of L above a rational prime ℓ. The local Néron symbol $\langle C, D\rangle_v = \langle C, D\rangle_{X_v,\rho_{L_v}}$, defined on degree zero divisors on X_v with disjoint support, satisfies:*

(a) *$\langle\ ,\ \rangle_v$ is symmetric and bilinear;*

(b) *if $C = (f)$ is a principal divisor, then $\langle C, D\rangle_v = \rho_{L_v}(f(D))$;*

(c) *if T is a correspondence from X to itself and T^ι is the dual correspondence, then*

$$\langle TC, D\rangle_v = \langle C, T^\iota D\rangle_v;$$

(d) *for $d_0 \in X_v(L_v) - \mathrm{supp}(C)$, the function on $X_v(L_v) - \mathrm{supp}(C)$*

$$d \mapsto \langle C, (d) - (d_0)\rangle_v$$

is continuous for the v-adic topology;

(e) *if $\ell = p$, L' is a finite extension of L_v contained in the cyclotomic $\mathbb{Z}_p$-extension of L_v, and C and D are degree zero divisors on $X_v \times_{L_v} L'$ and X_v, respectively, then*

$$\langle \mathbf{N}_{L'/L_v} C, D\rangle_v \subset c^{-1}\rho_{\mathbb{Q}_p}(\mathbf{N}_{L'/\mathbb{Q}_p}(L'))$$

whenever this is defined, for some constant $c \in \mathbb{Z}_p$ independent of C, D, and L'.

Furthermore $\langle\ ,\ \rangle_v$ takes values in a compact subset of $\mathbb{Q}_p$.

Proof. Properties (a)–(e) are direct consequences of the analogous properties of the Néron symbol on A in Proposition 3.2.1, except for the symmetry (which is stated without proof in [PR87a], but can be deduced from the construction of the pairing of Proposition 3.2.1). For the final claim one uses the finite generation of the p-primary part $A(L_v)$ as a $\mathbb{Z}_p$-module and the specified behavior on principal divisors. □

Proposition 3.3.3. *For any prime v of L with residue characteristic $\neq p$ and any degree zero divisors C and D on X_v with disjoint support,*

$$\langle C, D\rangle_v = \log_p(\mathbf{N}(v))\ [C, D]$$

where $[C, D]$ is the pairing of § 3.1 for any regular, integral, proper scheme $\underline{X}$ over the integer ring of L_v whose generic fiber is X_v.

Proof. Using the discussion of § 3.1, one can show that the right-hand side satisfies properties (a)–(d) of Proposition 3.3.2, and so it suffices to show that these determine $\langle\ ,\ \rangle_v$ uniquely. This is similar to the uniqueness argument of Proposition 3.2.1; the difference of two such symbols would define a continuous bilinear function $A(L_v) \times A(L_v) \to \mathbb{Q}_p$, which must be trivial for topological reasons. □

4. Intersections on modular curves

Fix $s > 0$ and $\sigma \in \mathrm{Gal}(H_s/K)$. Let ℓ be a rational prime, v a place of H_s above ℓ, F the completion of the maximal unramified extension of $H_{s,v}$, W the integer ring of F, and $\mathfrak{m}$ the maximal ideal of W. Set $W_n = W/\mathfrak{m}^{n+1}$. We denote by $X = X_0(N)_{/\mathbb{Z}}$ the canonical integral model of [KM85], and set $\underline{X} = X \times_{\mathbb{Z}} W$.

Definition 4.0.4. Given elliptic curves with $\Gamma_0(N)$-structure $\underline{x}$ and $\underline{y}$ over $\mathrm{Spec}(W)$, we define $\mathrm{Hom}_{W_n}(\underline{y}, \underline{x})_{\deg(m)}$ to be the set of degree m isogenies (of elliptic curves with $\Gamma_0(N)$-structure, in the sense of § 1.1)

$$\underline{y} \times_W W_n \to \underline{x} \times_W W_n.$$

Proposition 4.0.5. *Let $\underline{x}, \underline{y} \in \underline{X}(W)$ represent elliptic curves with $\Gamma_0(N)$-structure over W, and assume that these sections intersect properly and reduce to regular, noncuspidal points in the special fiber. Then*

$$i(\underline{x}, \underline{y}) = \frac{1}{2} \sum_{n \geqslant 0} |\mathrm{Hom}_{W_n}(\underline{y}, \underline{x})_{\deg(1)}|.$$

Proof. This is [GZ86, Proposition III.6.1], or [Con04, Theorem 4.1]. □

Now assume $\ell \neq p$ and fix an integer $m = m_0 p^r$ with $r > 0$ and $(m_0, Np) = 1$. Choose an embedding $H_\infty \hookrightarrow F$ extending $H_s \hookrightarrow F$. Recall the notation

$$\mathbf{h}_{s,r} = \mathrm{Norm}_{H_{s+r}/H_s}(h_{s+r}), \quad \mathbf{d}_{s,r} = \mathrm{Norm}_{H_{s+r}/H_s}(d_{s+r})$$

of the introduction. For any $t \geqslant 0$, let $\underline{h}_t$ be the Zariski closure (with the reduced subscheme structure) of $h_t \in X(F)$ in $\underline{X}$ and let $T_{m_0}(\underline{\mathbf{h}}^\sigma_{s,r})$ be the horizontal Weil divisor on $\underline{X}$ with generic fiber $T_{m_0}(\mathbf{h}^\sigma_{s,r})$. By the valuative criterion of properness, the closed subscheme $\underline{h}_{s+r}$ has the form $\mathrm{Spec}(W) \to \underline{X}$. Moreover, the section $\underline{h}_{s+r}$ arises from a Heegner diagram defined over W. Indeed, by [Cor02, Proposition 1.2] or [SeTa69, Theorems 8,9] the point $h_{s+r} \in X(H_{s+r})$ arises from a Heegner diagram over H_{s+r} with good reduction above ℓ, and so the section $\underline{h}_{s+r}$ represents the Néron model over W of this Heegner diagram. Taking the quotient of $\underline{h}_{s+r}$ by its $p\mathcal{O}_{s+r-1}$-torsion,

we obtain a Heegner diagram represented by the section $\underline{h}_{s+r-1} \in \underline{X}(W)$, and so on through all lower conductors. In particular, we now have a p-isogeny of Heegner diagrams defined over W.

$$\begin{array}{ccc} \underline{E}_s & \xrightarrow{\underline{h}_s} & \underline{E}'_s \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi'} \\ \underline{E}_{s-1} & \xrightarrow{\underline{h}_{s-1}} & \underline{E}'_{s-1} \end{array} \tag{11}$$

Although the expression for the local Néron symbol at $\ell \neq p$ in terms of intersection theory requires working on a regular model (which $\underline{X}$ is not when $\ell \mid N$) and modifying the divisors in questions by a fibral divisor, in our situation these details can be ignored.

Proposition 4.0.6. *Suppose that $\ell \neq p$ and $0 \leqslant t \leqslant s$. Then*

$$\langle c_t, T_{m_0}(\mathbf{d}^{\sigma}_{s,r})\rangle_v = \log_p(\mathbf{N}(v)) \cdot i(\underline{h}_t, T_{m_0}(\underline{\mathbf{h}}^{\sigma}_{s,r})),$$

where the pairing on the left is the local Néron symbol on $X_{/H_{s,v}}$ of Proposition 3.3.2 and i is the intersection multiplicity on $\underline{X}$ of § 3.1.

The proof is as in [GZ86, Proposition III.3.3], together with Proposition 3.3.3.

Remark 4.0.7. In order to make sense of $i(\underline{h}_t, T_{m_0}(\underline{\mathbf{h}}^{\sigma}_{s,r}))$ when $\ell \mid N$ we need to justify why the prime Weil divisors occurring in $T_{m_0}(\underline{\mathbf{h}}^{\sigma}_{s,r})$ are locally principal, so that $T_{m_0}(\underline{\mathbf{h}}^{\sigma}_{s,r})$ may be viewed as a Cartier divisor. The geometric points of $T_{m_0}(\mathbf{h}^{\sigma}_{s,r})$ all occur in the support of $T_m(h^{\sigma}_s)$. If $\ell \mid N$ then these points represent Heegner diagrams which are prime-to-ℓ isogenous to h^{σ}_s, and so are all defined over F. Arguing as in [Con04, Corollary 2.7] (Conrad's p is our ℓ), the Zariski closures of these points on $\underline{X}$ are sections to the structure map $\underline{X} \to \mathrm{Spec}(W)$ and lie in the smooth locus. In particular, the associated ideal sheaves are locally free of rank one.

Proposition 4.0.8. *Suppose $\ell \neq p$ and $\epsilon(\ell) = 1$. Then for all $0 \leqslant t \leqslant s$, $\langle c_t, T_{m_0}(\mathbf{d}^{\sigma}_{s,r})\rangle_v = 0$, where the pairing $\langle\ ,\ \rangle_v$ is as in Proposition 4.0.6.*

Proof. By Proposition 4.0.6 we must show that $i(\underline{h}_t, T_{m_0}(\underline{\mathbf{h}}^{\sigma}_{s,r})) = 0$. The claim is unchanged if we replace W by the integer ring of a finite extension of F. Doing so, we assume that the divisor $T_{m_0}(\mathbf{h}^{\sigma}_{s,r})$ is defined point-by-point over F and that the horizontal divisor $T_{m_0}(\mathbf{d}^{\sigma}_{s,r})$ on $\underline{X}$ is a sum of sections to the structure map, each of which represents a Heegner diagram over W whose conductor divides mp^s and has exact valuation $s + r > t$ at p. Let $\underline{x}$ be one such Heegner diagram, and let $\mathcal{O}$ and $\mathcal{O}'$ be the endomorphism rings of $\underline{x}$ and its closed fiber, respectively. These are orders in K, as $\underline{x}$ has ordinary reduction, and $\mathcal{O} \subset \mathcal{O}'$. By the Serre–Tate theorem, $\mathcal{O}$ is the intersection (in $K \otimes \mathbb{Q}_\ell$) of $\mathcal{O}'$ and $\mathcal{O} \otimes \mathbb{Z}_\ell$, therefore

$$\mathrm{ord}_p(\mathrm{cond}(\mathcal{O}')) = \mathrm{ord}_p(\mathrm{cond}(\mathcal{O})) = s + r > t.$$

The same argument shows that the valuation at p of the conductor of the CM order of the special fiber of $\underline{h}_t$ is t, and so the Heegner diagram $\underline{h}_t$ is distinct in the special fiber from all Heegner diagrams appearing in $T_{m_0}(\underline{\mathbf{h}}^{\sigma}_{s,r})$. By Proposition 4.0.5, $i(\underline{h}_t, T_{m_0}(\underline{\mathbf{h}}^{\sigma}_{s,r})) = 0$. □

5. Nonsplit primes away from p

In this section we examine the local Néron pairings between Heegner points at places lying above rational primes $\neq p$ which are nonsplit in K. The methods are based on those of Chapter III of [GZ86], and this portion of Gross and Zagier's work has been reworked and rewritten by Conrad [Con04] with the addition of considerably more detail.

Keep the notation of § 4, and assume $\ell \neq p$ is nonsplit in K. In particular $\ell \nmid N$. Fix a prime v of H_s (with $s > 0$, as always) above ℓ and an integral $\mathcal{O}_s$-ideal $\mathfrak{a}$ of norm prime to $D\ell p$ whose class in $\mathrm{Pic}(\mathcal{O}_s)$ represents σ under the Artin map. We denote by $\mathfrak{l}$ the unique prime of $\mathcal{O}_s$ above ℓ (we sometimes let $\mathfrak{l}$ denote the $\mathcal{O}_K$-ideal $\mathfrak{l}\mathcal{O}_K$; a mild abuse of notation). If $\epsilon(\ell) = -1$ then $\mathfrak{l} = \ell\mathcal{O}_s$ is trivial in $\mathrm{Pic}(\mathcal{O}_s)$, $\mathfrak{l}$ splits completely in H_s, and v has absolute residue degree 2. If $\epsilon(\ell) = 0$ then $\mathfrak{l}^2 = \ell\mathcal{O}_s$ and $\mathfrak{l}$ is not a principal ideal of $\mathcal{O}_s$ (if D is not prime then $\mathfrak{l}\mathcal{O}_K$ is not principal, if $D = -\ell$ is prime then $\mathfrak{l} = (\sqrt{D} \cap \mathcal{O}_s)$ is not principal since $s > 0$). Thus, when $\epsilon(\ell) = 0$, $\mathfrak{l}$ has order 2 in $\mathrm{Pic}(\mathcal{O}_s)$ and again v has residue degree 2.

5.1 Intersection via Hom sets

Proposition 5.1.1. *For any integer $m = m_0 p^r$ with $(m_0, Np) = 1$,*

$$\langle c_s, T_{m_0}(\mathbf{d}^{\sigma}_{s,r+2})\rangle_v = \log_p(\ell) \sum_{n \geqslant 0} (|\mathrm{Hom}_{W_n}(\underline{h}^{\mathfrak{a}}_s, \underline{h}_s)_{\deg(mp^2)}| - |\mathrm{Hom}_{W_n}(\underline{h}^{\mathfrak{a}}_{s-1}, \underline{h}_s)_{\deg(mp)}|)$$

$$\langle c_{s-1}, T_{m_0}(\mathbf{d}^{\sigma}_{s,r+1})\rangle_v = \log_p(\ell) \sum_{n \geqslant 0} (|\mathrm{Hom}_{W_n}(\underline{h}^{\mathfrak{a}}_s, \underline{h}_{s-1})_{\deg(mp)}| - |\mathrm{Hom}_{W_n}(\underline{h}^{\mathfrak{a}}_{s-1}, \underline{h}_{s-1})_{\deg(m)}|)$$

where $\langle\ ,\ \rangle_v$ is the local Néron symbol on $X_{/H_{s,v}}$ of Proposition 3.3.2, and the Hom *sets are those of Definition 4.0.4.*

Proof. We prove the first equality. The proof of the second involves only a change of subscripts.

First consider the easy case where $(\ell, m_0) = 1$. Then the divisor $T_{m_0}(\mathbf{h}^{\sigma}_{s,r+2})$ on $\underline{X}_{/F}$ (recall that F is the completion of the maximal unramified extension of $H_{s,v}$, W is its integer ring, and $\underline{X} = X_0(N)_{/W}$) is a sum of sections to the structure map. Hence, the same is true of the horizontal divisor $T_{m_0}(\underline{\mathbf{h}}^{\sigma}_{s,r+2})$ on $\underline{X}$, and each section represents a Heegner diagram over $\mathrm{Spec}(W)$. Namely, if we fix an extension of σ to $\mathrm{Gal}(H_{s+r+2}/K)$ and an ideal $\mathfrak{a}$ of $\mathcal{O}_{s+r+2}$ representing this extension, then

$$T_{m_0}(\underline{\mathbf{h}}^{\sigma}_{s,r+2}) = \sum_{\mathfrak{b}} \sum_{C} \underline{h}^{\mathfrak{a}\mathfrak{b}}_{s+r+2/C} \tag{12}$$

where $\mathfrak{b}$ runs over classes in $\mathrm{Pic}(\mathcal{O}_{s+r+2})$ which are trivial in $\mathrm{Pic}(\mathcal{O}_s)$, C runs over the order m_0-subgroup schemes of the Heegner diagram $\underline{h}^{\mathfrak{a}\mathfrak{b}}_{s+r+2}$ over $\mathrm{Spec}(W)$ and the subscript $/C$ means the quotient by C (which makes sense since $(m_0, N) = 1$). Since ℓ does not divide m_0, each C is étale (in fact, constant), determined uniquely by its reduction to W_n for any n, and the decomposition (12) holds over W_n. By Proposition 4.0.5

$$\begin{aligned} i(\underline{h}_s, T_{m_0}(\underline{\mathbf{h}}^{\sigma}_{s,r+2})) &= \sum_{\mathfrak{b}} \sum_{C} i(\underline{h}_s, \underline{h}^{\mathfrak{a}\mathfrak{b}}_{s+r+2/C}) \\ &= \frac{1}{2} \sum_{n} \sum_{\mathfrak{b}} \sum_{C} |\mathrm{Hom}_{W_n}(\underline{h}^{\mathfrak{a}\mathfrak{b}}_{s+r+2/C}, \underline{h}_s)_{\deg(1)}| \\ &= \frac{1}{2} \sum_{n} \sum_{\mathfrak{b}} |\mathrm{Hom}_{W_n}(\underline{h}^{\mathfrak{a}\mathfrak{b}}_{s+r+2}, \underline{h}_s)_{\deg(m_0)}|, \end{aligned}$$

and by Proposition 4.0.6 the first equality of Proposition 5.1.1 follows once we show

$$|\mathrm{Hom}_{W_n}(\underline{h}^{\mathfrak{a}}_s, \underline{h}_s)_{\deg(mp^2)}| = |\mathrm{Hom}_{W_n}(\underline{h}^{\mathfrak{a}}_{s-1}, \underline{h}_s)_{\deg(mp)}| + \sum_{\mathfrak{b}} |\mathrm{Hom}_{W_n}(\underline{h}^{\mathfrak{a}\mathfrak{b}}_{s+r+2}, \underline{h}_s)_{\deg(m_0)}|. \tag{13}$$

The p^{r+2}-torsion on $\underline{h}^{\mathfrak{a}}_s$ is constant as a group scheme, and so the kernel of any degree mp^2 isogeny $f : \underline{h}^{\mathfrak{a}}_s \to \underline{h}_s$ over W_n determines an order p^{r+2}-subgroup of $\underline{h}^{\mathfrak{a}}_s(W)$. By the Euler system relations of § 1.2, every such subgroup is either the kernel of a map which factors through $\phi^{\mathfrak{a}} : \underline{h}^{\mathfrak{a}}_s \to \underline{h}^{\mathfrak{a}}_{s-1}$, or is the kernel of the dual isogeny to $\phi^{\mathfrak{a}\mathfrak{b}} \circ \cdots \circ \phi^{\mathfrak{a}\mathfrak{b}} : \underline{h}^{\mathfrak{a}\mathfrak{b}}_{s+r+2} \to \underline{h}^{\mathfrak{a}}_s$ for some choice of $\mathfrak{b}$, and the

two cases are mutually exclusive. Thus f has one of the two forms

$$\underline{h}_s^{\mathfrak{a}} \xrightarrow{\phi^{\mathfrak{a}}} \underline{h}_{s-1}^{\mathfrak{a}} \xrightarrow{\psi} \underline{h}_s, \quad \underline{h}_s^{\mathfrak{a}} \xrightarrow{(\phi^{\mathfrak{a}\mathfrak{b}}\circ\cdots\circ\phi^{\mathfrak{a}\mathfrak{b}})^\vee} \underline{h}_{s+r+2}^{\mathfrak{a}\mathfrak{b}} \xrightarrow{\psi} \underline{h}_s$$

where ψ has degree either mp or m_0 (respectively). The equality (13) follows.

Now consider the case where ℓ divides m_0. This is considerably more involved, but nearly all of what we need is covered by the generality of [Con04, § 6] (which is based on [GZ86, III § 4–6]), to which we refer the reader for the proof of (14) below. Write $m_0 = m_1\ell^t$ with $(\ell, m_1) = 1$. As above, the divisor $T_{m_1}(\underline{\mathbf{h}}^{\sigma}_{s,r+2})$ on $\underline{X}$ is a sum of sections, each of which represents a Heegner diagram over $\mathrm{Spec}(W)$, and we denote by Z the set of such sections

$$Z = \{\underline{h}^{\mathfrak{a}\mathfrak{b}}_{s+r+2/C} \mid \mathfrak{b} \in \mathrm{Ker}(\mathrm{Pic}(\mathcal{O}_{s+r+2}) \to \mathrm{Pic}(\mathcal{O}_s))\}$$

where C runs over the order m_1 subgroup schemes of $\underline{h}^{\mathfrak{a}\mathfrak{b}}_{s+r+2}$. For each $\underline{z} \in Z$, one has the expected (but much more subtle) equality

$$\begin{aligned} i(\underline{h}_s, T_{m_0}(\underline{h}^{\mathfrak{a}}_{s+r+2})) &= \sum_{\underline{z}\in Z} i(\underline{h}_s, T_{\ell^t}(\underline{z})) \\ &= \frac{1}{2}\sum_{\underline{z}\in Z}\sum_{n\geqslant 0} |\mathrm{Hom}_{W_n}(\underline{z}, \underline{h}_s)_{\deg(\ell^t)}| \\ &= \frac{1}{2}\sum_{n}\sum_{\mathfrak{b}} |\mathrm{Hom}_{W_n}(\underline{h}^{\mathfrak{a}\mathfrak{b}}_{s+r+2}, \underline{h}_s)_{\deg(m_0)}|. \end{aligned} \tag{14}$$

With this in hand, the remainder of the proof is exactly as in the case $(\ell, m_0) = 1$. □

5.2 Inclusion-exclusion

Our goal is, for any positive integer m with $(m, N) = 1$, to express the sum over n of

$$\begin{aligned} &|\mathrm{Hom}_{W_n}(\underline{h}^{\mathfrak{a}}_s, \underline{h}_s)_{\deg(mp^2)}| - |\mathrm{Hom}_{W_n}(\underline{h}^{\mathfrak{a}}_{s-1}, \underline{h}_s)_{\deg(mp)}| \\ &\qquad - |\mathrm{Hom}_{W_n}(\underline{h}^{\mathfrak{a}}_s, \underline{h}_{s-1})_{\deg(mp)}| + |\mathrm{Hom}_{W_n}(\underline{h}^{\mathfrak{a}}_{s-1}, \underline{h}_{s-1})_{\deg(m)}| \end{aligned} \tag{15}$$

as a sum over elements in the quaternion algebra $B = \mathrm{End}_{W_0}(\underline{h}_s) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Lemma 5.2.1. *Base change to the fiber induces a degree preserving injection*

$$\mathrm{Hom}_{W_n}(\underline{h}^{\mathfrak{a}}_s, \underline{h}_s) \to \mathrm{Hom}_{W_0}(\underline{h}^{\mathfrak{a}}_s, \underline{h}_s),$$

and similarly for the other Hom *sets occurring in (15).*

Proof. This is [Con04, Lemma 2.1(2)] or [Gor02, Proposition VI.2.4(2)]. □

The isogeny ϕ induces injections

$$\begin{aligned} \mathrm{Hom}_{W_n}(\underline{h}^{\mathfrak{a}}_{s-1}, \underline{h}_s) &\xrightarrow{\circ\phi^{\mathfrak{a}}} \mathrm{Hom}_{W_n}(\underline{h}^{\mathfrak{a}}_s, \underline{h}_s) \to \mathrm{Hom}_{W_0}(\underline{h}^{\mathfrak{a}}_s, \underline{h}_s) \\ \mathrm{Hom}_{W_n}(\underline{h}^{\mathfrak{a}}_s, \underline{h}_{s-1}) &\xrightarrow{\phi^\vee\circ} \mathrm{Hom}_{W_n}(\underline{h}^{\mathfrak{a}}_s, \underline{h}_s) \to \mathrm{Hom}_{W_0}(\underline{h}^{\mathfrak{a}}_s, \underline{h}_s) \end{aligned}$$

whose images we denote by L_n and $L_n^\vee$, respectively. We also define M_n to be the image of the injective composition

$$\mathrm{Hom}_{W_n}(\underline{h}^{\mathfrak{a}}_{s-1}, \underline{h}_{s-1}) \to \mathrm{Hom}_{W_n}(\underline{h}^{\mathfrak{a}}_s, \underline{h}_s) \to \mathrm{Hom}_{W_0}(\underline{h}^{\mathfrak{a}}_s, \underline{h}_s)$$

where the first arrow is given by $f \mapsto \phi^\vee \circ f \circ \phi^{\mathfrak{a}}$. Clearly $M_n \subset L_n \cap L_n^\vee$. The scheme-theoretic kernels

$$\ker(\phi : \underline{E}_s \to \underline{E}_{s-1}), \quad \ker(\phi^{\mathfrak{a}} : \underline{E}^{\mathfrak{a}}_s \to \underline{E}^{\mathfrak{a}}_{s-1})$$

are constant group schemes of order p over W. We define

$$C = (\ker\ \phi)(W_0), \quad C^{\mathfrak{a}} = (\ker\ \phi^{\mathfrak{a}})(W_0).$$

Definition 5.2.2. We say that $f \in \mathrm{Hom}_{W_n}(\underline{h}_s^{\mathfrak{a}}, \underline{h}_s)$ is *stable* if the restriction of f to the fiber $f_0 : \underline{E}_s^{\mathfrak{a}}(W_0) \to \underline{E}_s(W_0)$ takes $C^{\mathfrak{a}}$ into C. We say that f is *unstable* otherwise, and make similar definitions for maps from $\underline{h}_s$ to itself. If $Z \subset \mathrm{Hom}_{W_n}(\underline{h}_s^{\mathfrak{a}}, \underline{h}_s)$ is any subset, we write Z^{stable} and Z^{unstable} for the subsets of stable and unstable elements of Z.

Lemma 5.2.3. *Suppose that m is any positive integer with $(m, N) = 1$. Base change to the fiber identifies the stable elements of degree mp^2 in $\mathrm{Hom}_{W_n}(\underline{h}_s^{\mathfrak{a}}, \underline{h}_s)$ with the degree mp^2 elements of $L_n \cup L_n^{\vee}$.*

Proof. Fix $f \in \mathrm{Hom}_{W_n}(\underline{h}_s^{\mathfrak{a}}, \underline{h}_s)$ of degree divisible by p and prime to N. Letting f_0 denote the restriction of f to geometric points as above, f is stable if and only if either $f_0(C^{\mathfrak{a}}) = 0$ or $f_0(C^{\mathfrak{a}}) = C$. The first condition is equivalent to $f_0 = g_0 \circ \phi^{\mathfrak{a}}$ for some $g_0 \in \mathrm{Hom}_{W_0}(\underline{E}_{s-1}^{\mathfrak{a}}, \underline{E}_s)$. Since $\phi^{\mathfrak{a}}$ has degree p it induces an isomorphism on ℓ-divisible groups over W_n, and so the map on ℓ-divisible groups induced by g_0 lifts to W_n. By the Serre–Tate theorem g_0 itself lifts to a morphism over W_n, and so $f \in L_n$. Now suppose $f_0(C^{\mathfrak{a}}) = C$. Since the degree of f is divisible by p we must have $f_0(\underline{E}_s^{\mathfrak{a}}(W_0)[p]) = C$, and so $f_0^{\vee}(C) = (f_0^{\vee} \circ f_0)(\underline{E}_s^{\mathfrak{a}}(W_0)[p]) = 0$. Hence, $f_0^{\vee} = g_0 \circ \phi$ for some $g_0 \in \mathrm{Hom}_{W_0}(\underline{E}_{s-1}, \underline{E}_s^{\mathfrak{a}})$, and so $f_0 \in L_n^{\vee}$ as above.

Conversely, if $f_0 \in L_n \cup L_n^{\vee}$ then either $f_0(C^{\mathfrak{a}}) = 0$ or $f_0^{\vee}(C) = 0$. In the second case we compute the Weil e_p-pairing

$$e_p(f_0(\underline{E}_s^{\mathfrak{a}}(W_0)[p]), C) = e_p(\underline{E}_s^{\mathfrak{a}}(W_0)[p], f_0^{\vee}(C)) = 0.$$

This implies $f_0(\underline{E}_s^{\mathfrak{a}}(W_0)[p]) \subset C$, and so, in either case, $f_0(C^{\mathfrak{a}}) \subset C$ and f is stable. □

Lemma 5.2.4. *For any positive integer m with $(m, N) = 1$, the composition*

$$\mathrm{Hom}_{W_n}(\underline{h}_s^{\mathfrak{a}}, \underline{h}_s) \to \mathrm{Hom}_{W_0}(\underline{h}_s^{\mathfrak{a}}, \underline{h}_s) \xrightarrow{p} \mathrm{Hom}_{W_0}(\underline{h}_s^{\mathfrak{a}}, \underline{h}_s)$$

taking $f \mapsto pf_0$ identifies the unstable elements of $\mathrm{Hom}_{W_n}(\underline{h}_s^{\mathfrak{a}}, \underline{h}_s)_{\deg(m)}$ with the complement of $(M_n)_{\deg(mp^2)}$ in $(L_n \cap L_n^{\vee})_{\deg(mp^2)}$ (the degree mp^2 elements of M_n and $L_n \cap L_n^{\vee}$, respectively).

Proof. First suppose that we are given some $f \in \mathrm{Hom}_{W_n}(\underline{h}_s^{\mathfrak{a}}, \underline{h}_s)$; the claim is that $pf_0 \in M_n$ if and only if f is stable. By definition $pf_0 \in M_n$ if and only if there is some $f' \in \mathrm{Hom}_{W_n}(\underline{E}_{s-1}^{\mathfrak{a}}, \underline{E}_{s-1})$ such that $pf = \phi^{\vee} \circ f' \circ \phi^{\mathfrak{a}}$, or equivalently, such that $\phi \circ f = f' \circ \phi^{\mathfrak{a}}$. Furthermore, this is equivalent to finding $f_0' \in \mathrm{Hom}_{W_0}(\underline{E}_{s-1}^{\mathfrak{a}}, \underline{E}_{s-1})$ such that $\phi \circ f_0 = f_0' \circ \phi^{\mathfrak{a}}$ holds in the fiber (since ϕ and $\phi^{\mathfrak{a}}$ induce isomorphisms on ℓ-divisible groups over W_n, the map on ℓ-divisible groups induced by f_0' lifts to W_n, and so the Serre–Tate theorem implies that f_0' itself lifts). Such an f_0' exists if and only if $(\phi \circ f_0)(C^{\mathfrak{a}}) = 0$, which is equivalent to f being stable.

Now suppose we are given a homomorphism $g_0 \in L_n \cap L_n^{\vee}$ of degree divisible by p^2, with $g_0 \notin M_n$. There is some $y \in \mathrm{Hom}_{W_n}(\underline{h}_s^{\mathfrak{a}}, \underline{h}_{s-1})$ such that g_0 is the restriction of $g = \phi^{\vee} \circ y$ to the fiber. Let y_0 denote the restriction of y to the fiber. If $y_0(C^{\mathfrak{a}}) = 0$ we could write $y_0 = y_0' \circ \phi^{\mathfrak{a}}$ for some $y_0' \in \mathrm{Hom}_{W_0}(\underline{E}_{s-1}^{\mathfrak{a}}, \underline{E}_{s-1})$. As above, the map on ℓ-divisible groups induced by such a y_0' would lift to W_n, and so by the Serre–Tate theorem y_0' itself would lift to some $y' \in \mathrm{Hom}_{W_n}(\underline{E}_{s-1}^{\mathfrak{a}}, \underline{E}_{s-1})$ with g_0 equal to the restriction of $\phi^{\vee} \circ y' \circ \phi^{\mathfrak{a}}$ to the fiber. This contradicts $g_0 \notin M_n$, so $y_0(C^{\mathfrak{a}}) \neq 0$. Since p divides the degree of y_0 we must have $y_0(\underline{E}_s^{\mathfrak{a}}(W_0)[p]) = y_0(C^{\mathfrak{a}})$. Now $g_0 \in L_n$ implies

$$0 = g_0(C^{\mathfrak{a}}) = (\phi_0^{\vee} \circ y_0)(C^{\mathfrak{a}}) = g_0(\underline{E}_s^{\mathfrak{a}}(W_0)[p]),$$

so $g_0 = pf_0$ for some $f_0 \in \mathrm{Hom}_{W_0}(\underline{E}_s^{\mathfrak{a}}, \underline{E}_s)$. As above, the Serre–Tate theorem guarantees that f_0 lifts to a morphism f over W_n. □

Corollary 5.2.5. *The expression (15) is equal to*

$$|\mathrm{Hom}_{W_n}(\underline{h}_s^{\mathfrak{a}}, \underline{h}_s)^{\mathrm{unstable}}_{\deg(mp^2)}| - |\mathrm{Hom}_{W_n}(\underline{h}_s^{\mathfrak{a}}, \underline{h}_s)^{\mathrm{unstable}}_{\deg(m)}|.$$

Proof. By the definitions of M_n, L_n and $L_n^\vee$,

$$\begin{aligned}|\mathrm{Hom}_{W_n}(\underline{h}_{s-1}^{\mathfrak{a}}, \underline{h}_{s-1})_{\deg(m)}| &= |(M_n)_{\deg(mp^2)}|\\ |\mathrm{Hom}_{W_n}(\underline{h}_{s-1}^{\mathfrak{a}}, \underline{h}_s)_{\deg(mp)}| &= |(L_n)_{\deg(mp^2)}|\\ |\mathrm{Hom}_{W_n}(\underline{h}_s^{\mathfrak{a}}, \underline{h}_{s-1})_{\deg(mp)}| &= |(L_n^\vee)_{\deg(mp^2)}|.\end{aligned}$$

Consequently, the expression (15) is equal to

$$|\mathrm{Hom}_{W_n}(\underline{h}_s^{\mathfrak{a}}, \underline{h}_s)^{\mathrm{unstable}}_{\deg(mp^2)}| + |\mathrm{Hom}_{W_n}(\underline{h}_s^{\mathfrak{a}}, \underline{h}_s)^{\mathrm{stable}}_{\deg(mp^2)}| - |(L_n)_{\deg(mp^2)}| - |(L_n^\vee)_{\deg(mp^2)}| + |(M_n)_{\deg(mp^2)}|.$$

By Lemma 5.2.3 this is

$$|\mathrm{Hom}_{W_n}(\underline{h}_s^{\mathfrak{a}}, \underline{h}_s)^{\mathrm{unstable}}_{\deg(mp^2)}| + |(L_n \cup L_n^\vee)_{\deg(mp^2)}| - |(L_n)_{\deg(mp^2)}| - |(L_n^\vee)_{\deg(mp^2)}| + |(M_n)_{\deg(mp^2)}|$$

which we write as

$$\begin{aligned}&|\mathrm{Hom}_{W_n}(\underline{h}_s^{\mathfrak{a}}, \underline{h}_s)^{\mathrm{unstable}}_{\deg(mp^2)}| - |(L_n \cap L_n^\vee)_{\deg(mp^2)}| + |(M_n)_{\deg(mp^2)}|\\ &= |\mathrm{Hom}_{W_n}(\underline{h}_s^{\mathfrak{a}}, \underline{h}_s)^{\mathrm{unstable}}_{\deg(mp^2)}| - |\mathrm{Hom}_{W_n}(\underline{h}_s^{\mathfrak{a}}, \underline{h}_s)^{\mathrm{unstable}}_{\deg(m)}|\end{aligned}$$

using Lemma 5.2.4. □

Set $R = \mathrm{Hom}_{W_0}(\underline{h}_s, \underline{h}_s)$ and $B = R \otimes_{\mathbb{Z}} \mathbb{Q}$. Thus, B is a rational quaternion algebra ramified exactly at ℓ and ∞, and $R \subset B$ is a level-N Eichler order [Con04, Lemma 7.1]. The reduction map

$$\mathrm{Hom}_W(\underline{h}_s, \underline{h}_s) \;\to\; \mathrm{Hom}_{W_0}(\underline{h}_s, \underline{h}_s)$$

induces an embedding $\iota : K \to B$ which, by the Serre–Tate theorem, is *optimal* for the pair $(\mathcal{O}_s, R)$ in the sense that $\iota(K) \cap R = \iota(\mathcal{O}_s)$. We henceforth regard K as a subfield of B, suppressing ι from the notation. There is a canonical decomposition

$$B = B^+ \oplus B^- = K \oplus Kj$$

where $j \in B$ is a trace zero element with the property $jxj^{-1} = \bar{x}$ for all $x \in K$. This characterizes j up to multiplication by $\mathbb{Q}^\times$. The reduced norm is additive with respect to this decomposition, i.e. $\mathbf{N}(b^+ + b^-) = \mathbf{N}(b^+) + \mathbf{N}(b^-)$. We wish to determine which $b \in R = \mathrm{Hom}_{W_0}(\underline{h}_s, \underline{h}_s)$ are unstable.

Lemma 5.2.6. *An endomorphism $b \in R$ is unstable if and only if*

$$\mathrm{ord}_p \mathbf{N}(b^+) = \mathrm{ord}_p \mathbf{N}(b^-) = -2s,$$

where $b^\pm$ is the projection of b to the summand $B^\pm$.

Proof. We are free to assume that j is chosen in R. Let T denote the p-adic Tate module of $\underline{E}_s(W_0)[p^\infty]$ and set $V = T \otimes \mathbb{Q}_p$. The split quaternion algebra $B_p = B \otimes \mathbb{Q}_p$ acts on V, and the stabilizer of $T \subset V$ is exactly $R_p = R \otimes \mathbb{Z}_p$ (since the order R is locally maximal away from N). Under the identification of V/T with $\underline{E}_s(W_0)[p^\infty]$, the subgroup $\mathcal{O}_{s-1,p}T/T$ is identified with C, and so the unstable elements of R are exactly those which do not stabilize the lattice $T' = \mathcal{O}_{s-1,p}T \supset T$. As an $\mathcal{O}_{s,p}$-module, T is free of rank one (proof: T is isomorphic as an $\mathcal{O}_{s,p}$-module to some fractional $\mathcal{O}_{s,p}$-ideal; by the optimality of $K \to B$ with respect to $(\mathcal{O}_s, R)$, this ideal is proper, and all proper ideals of $\mathcal{O}_{s,p}$ are principal). Fix a generator $t \in T$, and let $X \in \mathcal{O}_{s,p}$ be such that $jt = Xt$. This implies, in particular, that $\mathbf{N}(X) = \mathbf{N}(j)$. As a $\mathbb{Z}_p$-module, T is generated by t and $p^s\sqrt{D}t$, and so $\alpha + \beta j \in B$ (with $\alpha, \beta \in K$) stabilizes T if and only if the elements

$$(\alpha + \beta j)t = (\alpha + \beta X)t, \quad (\alpha + \beta j)p^s\sqrt{D}t = (\alpha - \beta X)p^s\sqrt{D}t$$

are in T. From this we deduce that

$$R_p = \{\alpha + \beta j \in B_p \mid \alpha, \beta X \in (p^s\sqrt{D})^{-1}\mathcal{O}_{s,p}, \alpha + \beta X \in \mathcal{O}_{s,p}\}.$$

Applying similar reasoning to the lattice T', we find that the order of B_p leaving both T and T' stable is

$$R_p^{\text{stable}} = \{\alpha + \beta j \in B_p \mid \alpha, \beta X \in (p^{s-1}\sqrt{D})^{-1}\mathcal{O}_{s-1,p}, \alpha + \beta X \in \mathcal{O}_{s,p}\}.$$

Given $b = \alpha + \beta j \in R_p$, set $\alpha' = p^s\sqrt{D}\alpha$ and $\beta' = p^s\sqrt{D}\beta$. It is easily seen that the set of elements of $\mathcal{O}_{s,p}$ of norm divisible by p is equal to the unique maximal ideal $p\mathcal{O}_{s-1,p} \subset \mathcal{O}_{s,p}$. Since $\alpha' \equiv -\beta' X \pmod{p^s\sqrt{D}\mathcal{O}_{s,p}}$, α' is a unit if and only if $\beta' X$ is a unit. Both elements are units if and only if $\text{ord}_p\mathbf{N}(\alpha) = \text{ord}_p\mathbf{N}(\beta X) = -2s$, and both are nonunits if and only if $\alpha + \beta j \in R_p^{\text{stable}}$. □

Proposition 5.2.7. *For any nonnegative integers m, n with $(m, N) = 1$, there is a bijection between $\text{Hom}_{W_n}(\underline{h}_s^{\mathfrak{a}}, \underline{h}_s)^{\text{unstable}}_{\deg(m)}$ and the set of all $b \in R \cdot \mathfrak{a}$ such that:*

(a) $\mathbf{N}(b) = m\mathbf{N}(\mathfrak{a})$;

(b) $\text{ord}_p\mathbf{N}(b^+) = \text{ord}_p\mathbf{N}(b^-) = -2s$;

(c) *and*

$$\text{ord}_\ell(D\mathbf{N}(b^-)) \geqslant \begin{cases} 2n+1 & \text{if } \epsilon(\ell) = -1 \\ n+1 & \text{if } \epsilon(\ell) = 0. \end{cases}$$

Proof. By [GZ86, Proposition III.7.3] or [Con04, Theorem 7.12 and (7-3)] there is an isomorphism of left $\mathcal{O}_s$-modules

$$\text{Hom}_{W_n}(\underline{h}_s^{\mathfrak{a}}, \underline{h}_s) \cong \text{Hom}_{W_n}(\underline{h}_s, \underline{h}_s) \otimes_{\mathcal{O}_s} \mathfrak{a}$$

whose image (viewed as a lattice in $R\mathfrak{a}$) is exactly those elements satisfying property (c), under which the degree m isogenies correspond to those satisfying property (a). We must show that this bijection takes the stable elements onto those $b = b^+ + b^-$ for which property (b) fails. The isomorphism in question is defined as follows. The map

$$\text{End}_{W_n}(\underline{E}_s) \otimes_{\mathcal{O}_s} \mathfrak{a} \xrightarrow{\xi_n} \text{Hom}_{W_n}(\text{Hom}_{\mathcal{O}_s}(\mathfrak{a}, \underline{E}_s), \underline{E}_s) \cong \text{Hom}_{W_n}(\underline{E}_s^{\mathfrak{a}}, \underline{E}_s)$$

defined by $\xi_n(f \otimes x)(\phi) = f(\phi(x))$ is an isomorphism of $\mathcal{O}_s$-modules by Lemma 7.13 of [Con04], and taking level N structure into account we obtain an injection of left $\mathcal{O}_s$-modules

$$\text{Hom}_{W_n}(\underline{h}_s^{\mathfrak{a}}, \underline{h}_s) \cong \text{Hom}_{W_n}(\underline{h}_s, \underline{h}_s) \otimes_{\mathcal{O}_s} \mathfrak{a} \hookrightarrow R\mathfrak{a}.$$

This injection identifies

$$\text{Hom}_{W_n}(\underline{h}_s^{\mathfrak{a}}, \underline{h}_s)^{\text{stable}} \cong \text{Hom}_{W_n}(\underline{h}_s, \underline{h}_s)^{\text{stable}} \otimes_{\mathcal{O}_s} \mathfrak{a}$$

inside of $R\mathfrak{a}$ (this is easily checked everywhere locally using the fact that $\mathfrak{a}$ is proper, hence locally principal). Localizing at p and using $(\mathbf{N}(\mathfrak{a}), p) = 1$, the claim follows from Lemma 5.2.6. □

For any order S of B, define

$$D_s^{\mathfrak{a}}(S, m) = \left\{ b \in S \cdot \mathfrak{a} \;\middle|\; \begin{array}{c} \mathbf{N}(b) = m\mathbf{N}(\mathfrak{a}) \\ \text{ord}_p\mathbf{N}(b^+) = \text{ord}_p\mathbf{N}(b^-) = -2s \end{array} \right\}$$

$$\Delta_s^{\mathfrak{a}}(S, m) = \sum_{b \in D_s^{\mathfrak{a}}(S,m)} \begin{cases} \frac{1}{2}(1 + \text{ord}_\ell\mathbf{N}(b^-)) & \text{if } \epsilon(\ell) = -1 \\ \text{ord}_\ell(D\mathbf{N}(b^-)) & \text{if } \epsilon(\ell) = 0. \end{cases} \tag{16}$$

Corollary 5.2.8. *For $(m, N) = 1$,*

$$\sum_{n \geqslant 0} |\text{Hom}_{W_n}(\underline{h}_s^{\mathfrak{a}}, \underline{h}_s)^{\text{unstable}}_{\deg(m)}| = \Delta_s^{\mathfrak{a}}(R, m).$$

Proof. When $\epsilon(\ell) = 0$ this is immediate from the proposition above. When $\epsilon(\ell) = -1$ it is similarly clear, provided one knows that $\mathrm{ord}_\ell \mathbf{N}(b^-)$ is always odd; but (as we see in the next section) we are free to choose j in such a way that $\mathrm{ord}_\ell \mathbf{N}(j) = 1$, so writing $b^- = \beta j$ with $\beta \in K$, $\mathrm{ord}_\ell(\mathbf{N}(b^-)) = 1 + \mathrm{ord}_\ell \mathbf{N}(\beta)$ is odd. □

5.3 Quaternionic sums

We continue to let B be the rational quaternion algebra of discriminant ℓ and assume we have a fixed embedding $K \hookrightarrow B$. As noted before, this embedding induces a splitting $B = B^+ + B^- = K \oplus Kj$. Let $\mathcal{S}$ denote the (finite) set of $K^\times$-conjugacy classes of $\mathcal{O}_s$-optimal, level N Eichler orders in B; that is, level N Eichler orders S such that $S \cap K = \mathcal{O}_s$, modulo the conjugation action of $K^\times$. For such an S, the value of $\Delta_s^{\mathfrak{a}}(S, m)$ (defined in (16)) depends only on the class of S in $\mathcal{S}$. Define

$$\Delta_s^{\mathfrak{a}}(m) = \sum_{S \in \mathcal{S}} \Delta_s^{\mathfrak{a}}(S, m).$$

The remainder of this subsection is devoted to the proof of the following proposition. The statement holds without parity restrictions on D, but *we will assume throughout that D is odd*, referring the reader to [Man04] for a description of the needed changes to the proof in the case where D is even. The method of proof follows the calculations performed in [GZ86, § III.9] (and described in great detail in [Man04]). The main difference (apart from working in higher conductor) is that we have 'removed the Euler factor at p' by adding the condition $\mathrm{ord}_p \mathbf{N}(b^+) = \mathrm{ord}_p \mathbf{N}(b^-) = -2s$ to the set $D_s^{\mathfrak{a}}(S, m)$ over which the summation $\Delta_s^{\mathfrak{a}}(S, m)$ occurs.

Proposition 5.3.1. *There is a proper integral $\mathcal{O}_s$-ideal $\mathfrak{q}$ such that for every positive integer m*

$$\Delta_s^{\mathfrak{a}}(m) = \sum_{\substack{n>0 \\ \ell \mid n, (n,p)=1}} \delta(n) r_{\mathfrak{a}}(mp^{2s}|D| - nN) \cdot \begin{cases} \mathrm{ord}_\ell(\ell n) R_{\mathfrak{aqn}}(n/\ell) & \text{if } \epsilon(\ell) = -1 \\ \mathrm{ord}_\ell(n) R_{\mathfrak{aqn}\mathfrak{l}}(n/\ell) & \text{if } \epsilon(\ell) = 0 \end{cases}$$

where $\mathfrak{n}$ is any integral $\mathcal{O}_s$-ideal with $\mathcal{O}_s/\mathfrak{n} \cong \mathbb{Z}/N\mathbb{Z}$. When $\epsilon(\ell) = -1$, we may take $\mathbf{N}(\mathfrak{q}) \equiv -\ell \pmod{Dp}$, and when $\epsilon(\ell) = 0$ we may take $\mathbf{N}(\mathfrak{q}\mathfrak{l}) \equiv -\ell \pmod{Dp}$.

If $\hat{K}^\times$ denotes the group of finite idèles of K and $\hat{\mathcal{O}}_s^\times \subset \hat{K}^\times$ is the group of units in the profinite completion of $\mathcal{O}_s$, then there is an action of the ring class group $\hat{K}^\times / K^\times \hat{\mathcal{O}}_s^\times \cong \mathrm{Pic}(\mathcal{O}_s)$ on $\mathcal{S}$: if $x = (x_r) \in \hat{K}^\times$ and $S \in \mathcal{S}$ then S^x is defined by the relation $(S^x)_r = x_r S_r x_r^{-1} \subset B_r$ for every rational prime r. In terms of $\mathcal{O}_s$-ideals the action is again by conjugation: $S^{\mathfrak{b}} = \mathfrak{b} S \mathfrak{b}^{-1}$.

Lemma 5.3.2. *The action of $\mathrm{Pic}(\mathcal{O}_s)$ on $\mathcal{S}$ is transitive, and the stabilizer of any element is the subgroup generated by the class of $\mathfrak{l}$ (so has order 1 if $\epsilon(\ell) = -1$ and order 2 if $\epsilon(\ell) = 0$).*

Proof. Let S and S' be $\mathcal{O}_s$-optimal level N Eichler orders. To prove the transitivity of the action of $\mathrm{Pic}(\mathcal{O}_s)$ on $\mathcal{S}$, we must show that S_r and S'_r are conjugate by elements of $K_r^\times$ for every prime r. The proof of [Man04, Theorem A.15] shows that this is the case if either $\mathcal{O}_{s,r}$ is maximal (which occurs for all $r \neq p$) or if S_r and S'_r are maximal (which occurs for all $(r, N) = 1$). To compute the kernel of the action, fix $S \in \mathcal{S}$ and let $x = (x_r)$ be a finite idèle of K. If $S = S^x$ in $\mathcal{S}$ then there is some $y \in K^\times$ such that $x_r y_r^{-1}$ is contained in $N(S_r)$, the normalizer of S_r in $B_r^\times$, for every prime r.

If $(r, N\ell) = 1$ then $N(S_r) = \mathbb{Q}_r^\times S_r^\times$, and so

$$x_r y_r^{-1} \in (\mathbb{Q}_r^\times S_r^\times) \cap K_r^\times = \mathbb{Q}_r^\times \mathcal{O}_{s,r}^\times.$$

If $r \mid N$ then $\mathbb{Q}_r^\times S_r^\times$ has index 2 in $N(S_r)$. Fix an isomorphism $\psi : B_r \cong M_2(\mathbb{Q}_r)$ in such a way that $\psi(K_r) \cong \mathbb{Q}_r \oplus \mathbb{Q}_r$ is the quadratic subalgebra of diagonal matrices, and let $S'_r \subset M_2(\mathbb{Q}_r)$ be the usual Eichler order of integral matrices whose lower left entry is divisible by $N_r = r^{\mathrm{ord}_r(N)}$. As S_r and $\psi^{-1}(S'_r)$ are both $\mathcal{O}_{s,r}$-optimal, by the discussion above there is a $z \in K_r^\times$ such

that $zS_rz^{-1} = \psi^{-1}(S'_r)$. Thus, replacing ψ by a $\psi(K^\times)$-conjugate we may also assume that $\psi(S_r) = S'_r$. Having made such a choice, we now suppress ψ from the notation. The nontrival coset of $\mathbb{Q}_r^\times S_r^\times$ in $N(S_r)$ is represented by the matrix

$$\alpha = \begin{pmatrix} 0 & 1 \\ N_r & 0 \end{pmatrix},$$

and one now checks directly that

$$x_r y_r^{-1} \in N(S_r) \cap K_r^\times = (\mathbb{Q}_r^\times S_r^\times \sqcup \alpha \mathbb{Q}_r^\times S_r^\times) \cap K_r^\times = \mathbb{Q}_r^\times \mathcal{O}_{s,r}^\times.$$

When $r = \ell$, B_ℓ has a unique maximal order, hence $N(S_\ell) \cap K_\ell^\times = K_\ell^\times$. We have shown that a finite idèle (x_r) acts trivially on $\mathcal{S}$ if and only if $(x_r) \in \hat{\mathbb{Q}}^\times \hat{\mathcal{O}}_s^\times K_\ell^\times K^\times = \hat{\mathcal{O}}_s^\times K_\ell^\times K^\times$. □

Let $\mathcal{W}_0$ denote the set of prime divisors of Dp if $\epsilon(\ell) = -1$, and the set of prime divisors $\neq \ell$ of Dp if $\epsilon(\ell) = 0$. Let $\mathcal{W}$ be the free abelian group (written multiplicatively) of exponent 2 on the elements of $\mathcal{W}_0$, and define a homomorphism

$$\mathcal{W} \ \to \ \mathrm{Pic}(\mathcal{O}_s)[2]$$

by sending $w \mapsto (\sqrt{D})_w$, the finite idèle of K which is 1 away from w and equal to the image of $\sqrt{D}$ under $K^\times \ \to \ K_r^\times$ at each $r \mid w$. This map allows us to view $\mathcal{S}$ as a $\mathcal{W}$-module. By genus theory, the map $\mathcal{W} \ \to \ \mathrm{Pic}(\mathcal{O}_s)[2]$ is surjective. The kernel has order 2 if $\epsilon(\ell) = -1$, and has order 1 if $\epsilon(\ell) = 0$.

As in [GZ86, pp. 265–266], we now choose a particular model for the quaternion algebra B. Detailed proofs of the following assertions can be found in [Man04]. If $\epsilon(\ell) = -1$ then choose a prime q such that $\left(\frac{-\ell q}{r}\right) = 1$ for all primes $r \mid D$. For such a q the quaternion algebra B is isomorphic to the quaternion algebra $\left(\frac{D,-\ell q}{\mathbb{Q}}\right)$ (meaning the quaternion algebra $B = \mathbb{Q} \oplus \mathbb{Q}i \oplus \mathbb{Q}j \oplus \mathbb{Q}ij$ with $i^2 = D$, $j^2 = -\ell q$, $ij = -ji$) and q is split in K. We may, and do, further impose the condition $q \equiv -\ell \pmod{Dp}$. If $\epsilon(\ell) = 0$ then choose a prime $q \neq \ell$ such that $\left(\frac{-q}{r}\right) = 1$ for all primes $r \mid (D/\ell)$, and with $\left(\frac{-q}{\ell}\right) = -1$. For such a q the quaternion algebra B is isomorphic to the quaternion algebra $\left(\frac{D,-q}{\mathbb{Q}}\right)$, and again such a q is split in K. We further impose the condition $\ell q \equiv -\ell \pmod{Dp}$. We henceforth fix a q as above and identify

$$B \cong \begin{cases} \left(\dfrac{D,-\ell q}{\mathbb{Q}}\right) & \text{if } \epsilon(\ell) = -1 \\ \left(\dfrac{D,-q}{\mathbb{Q}}\right) & \text{if } \epsilon(\ell) = 0. \end{cases}$$

In either case we regard K as a subfield of B via $\sqrt{D} \mapsto i$, so that conjugation by j acts as complex conjugation on K. Let $\mathfrak{D}_s = p^s\sqrt{D}\mathcal{O}_s$ denote the different of the order $\mathcal{O}_s$. Fix an integral $\mathcal{O}_s$-ideal $\mathfrak{n}$ such that $\mathcal{O}_s/\mathfrak{n} \cong \mathbb{Z}/N\mathbb{Z}$, and let $\mathfrak{q}$ be an integral $\mathcal{O}_s$-ideal of norm q.

Lemma 5.3.3. *If $\epsilon(\ell) = -1$ there is a collection $\{X_r \in \mathbb{Z}_r^\times \mid r \in \mathcal{W}_0\}$ such that*

$$R = \{\alpha + \beta j \mid \alpha \in \mathfrak{D}_s^{-1},\ \beta \in \mathfrak{D}_s^{-1}\mathfrak{n}\mathfrak{q}^{-1},\ \alpha - X_r\beta \in \mathcal{O}_{s,r}\ \forall r \in \mathcal{W}_0\}$$

is an $\mathcal{O}_s$-optimal level N Eichler order, and such that $X_r^2 = -\ell q$. If $\epsilon(\ell) = 0$ there is a collection $\{X_r \in \mathbb{Z}_r^\times \mid r \in \mathcal{W}_0\}$ such that

$$R = \{\alpha + \beta j \mid \alpha \in \mathfrak{D}_s^{-1}\mathfrak{l},\ \beta \in \mathfrak{D}_s^{-1}\mathfrak{l}\mathfrak{n}\mathfrak{q}^{-1},\ \alpha - X_r\beta \in \mathcal{O}_{s,r}\ \forall r \in \mathcal{W}_0\}$$

has the above property, and $X_r^2 = -q$.

Proof. Suppose $\epsilon(\ell) = -1$. The order $S = \mathcal{O}_s + \mathfrak{q}^{-1}j \subset B$ has reduced discriminant $p^{2s}D\ell$, and for a prime r not dividing pND, $R_r = S_r$. Thus, the lattice R_r is a maximal order at such primes.

If $r \mid N$ then $R_r = \mathcal{O}_{s,r} + \mathfrak{n}_r j$ is an Eichler order of level $r^{\mathrm{ord}_r N}$, so it remains to consider R_r for $r \mid Dp$. We have assumed $q \equiv -\ell \pmod{Dp}$, so that by Hensel's lemma $j^2 = -\ell q$ has a square root $X_r \in \mathbb{Z}_r^\times$ for each $r \mid Dp$. If we set $t_r = X_r - j$ then one readily computes $jt_r = -X_r t_r$, so that $B_r \cdot t_r = K_r \cdot t_r$ is a two-dimensional $\mathbb{Q}_r$-vector space on which B_r acts by left multiplication. Exactly as in the proof of Lemma 5.2.6, the (necessarily maximal) order leaving $\mathcal{O}_{s,r} \cdot t_r$ stable is

$$R_r = \{\alpha + \beta j \in B_r \mid \alpha, \beta X_r \in \mathfrak{D}_{s,r}^{-1}, \alpha - \beta X_r \in \mathcal{O}_{s,r}\}.$$

This shows that R is a level N Eichler order, and the $\mathcal{O}_s$-optimality is immediate from the explicit description. The case $\epsilon(\ell) = 0$ is entirely similar. □

Fix a family $\{X_r\}$ and an order R as in the lemma. It is verified by direct calculation that for any $w \in \mathcal{W}$, R^w has the same explicit form as R, but with X_r replaced by

$$X_r^w = \begin{cases} -X_r & \text{if } r \mid w \\ X_r & \text{otherwise.} \end{cases}$$

Lemma 5.3.4. *If $\mathfrak{g}$ is any integral $\mathcal{O}_s$-ideal of norm prime to Dp then*

$$\sum_{w \in \mathcal{W}} \sum_{b \in D_s^{\mathfrak{a}}(R^{w\mathfrak{g}}, m)} (1 + \mathrm{ord}_\ell \mathbf{N}(b^-)) = \sum_{\substack{n > 0 \\ \ell \mid n, (n,p)=1}} \delta(n) r_{\mathfrak{a}}(mp^{2s}|D| - nN) \cdot \begin{cases} 4 \cdot r_{\mathfrak{a}\mathfrak{q}\bar{\mathfrak{n}}\bar{\mathfrak{g}}^2}(n/\ell)\ \mathrm{ord}_\ell(\ell n) & \text{if } \epsilon(\ell) = -1 \\ 2 \cdot r_{\mathfrak{a}\mathfrak{q}\bar{\mathfrak{n}}\bar{\mathfrak{g}}^2\mathfrak{l}}(n/\ell)\ \mathrm{ord}_\ell(n) & \text{if } \epsilon(\ell) = 0. \end{cases} \tag{17}$$

Proof. Suppose that $\epsilon(\ell) = -1$. The lattice $R^{w\mathfrak{g}}\mathfrak{a}$ is given explicitly by

$$R^{w\mathfrak{g}}\mathfrak{a} = \{\alpha + \beta j \mid \alpha \in \mathfrak{D}_s^{-1}\mathfrak{a},\ \beta \in \mathfrak{D}_s^{-1}\mathfrak{n}\mathfrak{q}^{-1}\mathfrak{g}\bar{\mathfrak{g}}^{-1}\bar{\mathfrak{a}},\ \alpha - X_r^w \beta \in \mathcal{O}_{s,r} \forall r \mid Dp\}.$$

Denote by $\mathfrak{C}$ the set of all pairs $(\mathfrak{c}^+, \mathfrak{c}^-)$ of proper, integral $\mathcal{O}_s$-ideals such that:

(a) $\mathbf{N}(\mathfrak{c}^+) + \ell N \mathbf{N}(\mathfrak{c}^-) = mp^{2s}|D|$;
(b) $\mathfrak{c}^+$ and $\mathfrak{c}^-$ are prime to p;
(c) $\mathfrak{c}^+$ lies in the $\mathrm{Pic}(\mathcal{O}_s)$-class of $\bar{\mathfrak{a}}$;
(d) $\mathfrak{c}^-$ lies in the $\mathrm{Pic}(\mathcal{O}_s)$-class of $\mathfrak{a}\bar{\mathfrak{n}}\mathfrak{q}\bar{\mathfrak{g}}^2$;

and for each $w \in \mathcal{W}$ let $F^w : D_s^{\mathfrak{a}}(R^{w\mathfrak{g}}, m) \to \mathfrak{C}$ be the function defined by sending $b = \alpha + \beta j$ to the pair

$$\mathfrak{c}^+ = \alpha \mathfrak{D}_s \mathfrak{a}^{-1}, \quad \mathfrak{c}^- = \beta \mathfrak{D}_s \mathfrak{q}\mathfrak{n}^{-1}\mathfrak{g}^{-1}\bar{\mathfrak{g}}\bar{\mathfrak{a}}^{-1}. \tag{18}$$

If $D_s^{\mathfrak{a}}(R^{w\mathfrak{g}}, m)$ contained both $b = \alpha + \beta j$ and $\alpha - \beta j$ then we would have $b^+ = \alpha \in \mathcal{O}_{s,p}$, contradicting $\mathrm{ord}_p \mathbf{N}(b^+) = -2s$. This implies that F^w is two-to-one.

The claim is that every element of $\mathfrak{C}$ is in the image of F^w for exactly $2\delta(\mathbf{N}(\mathfrak{c}^-))$ choices of w, so that

$$\sum_{w \in \mathcal{W}} \sum_{b \in D_s^{\mathfrak{a}}(R^{w\mathfrak{g}}, m)} (1 + \mathrm{ord}_\ell \mathbf{N}(b^-)) = 4 \sum_{(\mathfrak{c}^+, \mathfrak{c}^-) \in \mathfrak{C}} (2 + \mathrm{ord}_\ell \mathbf{N}(\mathfrak{c}^-)) \cdot \delta(\mathbf{N}(\mathfrak{c}^-)). \tag{19}$$

To verify this, fix $(\mathfrak{c}^+, \mathfrak{c}^-) \in \mathfrak{C}$ and choose generators

$$\alpha \mathcal{O}_s = \mathfrak{c}^+ \mathfrak{D}_s^{-1} \mathfrak{a}, \quad \beta \mathcal{O}_s = \mathfrak{c}^- \mathfrak{D}_s^{-1} \mathfrak{q}^{-1} \mathfrak{n}\mathfrak{g}\bar{\mathfrak{g}}^{-1}\bar{\mathfrak{a}}.$$

Then $b = \alpha + \beta j$ lies in $D_s^{\mathfrak{a}}(R^{w\mathfrak{g}}, m)$ if and only if $\alpha - X_r^w \beta \in \mathcal{O}_{s,r}$ for every prime divisor r of Dp, or equivalently, if $\alpha' \equiv X_r^w \beta' \pmod{\mathfrak{D}_{s,r}}$ for every r, where $\alpha' = p^s \sqrt{D}\, \alpha$, $\beta' = p^s \sqrt{D}\, \beta \in \mathcal{O}_s$. The action of complex conjugation on $\mathcal{O}_s / \mathfrak{D}_s$ is trivial and so we have

$$\alpha'^2 \equiv \mathbf{N}(\alpha') = \mathbf{N}(\mathfrak{a})\mathbf{N}(\mathfrak{c}^+) \equiv -\ell N \mathbf{N}(\mathfrak{c}^-)\mathbf{N}(\mathfrak{a}) = -\ell q \mathbf{N}(\beta') \equiv X_r^2 \beta'^2$$

modulo $\mathfrak{D}_{s,r}$. When $r \neq p$, $\mathcal{O}_{s,r}/\mathfrak{D}_{s,r}$ is a field, and so $\alpha' \equiv \pm X_r \beta'$. The congruence holds for both signs if and only if $\alpha' \equiv 0$, which holds if and only if $r \mid \mathbf{N}(\mathfrak{c}^-)$. When $r = p$, $\alpha' \in \mathcal{O}_{s,r}^\times$ and the unit group of the ring $\mathbb{Z}/p^{2s}\mathbb{Z} \cong \mathcal{O}_{s,r}/\mathfrak{D}_{s,r}$ has no 2-torsion apart from ± 1. Hence, $\alpha' \equiv \pm X_r \beta'$ for a unique choice of sign. We have shown that $\alpha + \beta j$ is contained in $D_s^{\mathfrak{a}}(R^{w\mathfrak{g}}, m)$ for exactly $\delta(\mathbf{N}(\mathfrak{c}^-))$ choices of w. The element $\alpha - \beta j$ lies in $D_s^{\mathfrak{a}}(R^{w\mathfrak{g}}, m)$ for another $\delta(\mathbf{N}(\mathfrak{c}^-))$ choices of w, all distinct from the first set of choices. This proves (19). The right-hand side of (19) agrees with the right-hand sum in the statement of the lemma by setting $n = \ell \mathbf{N}(\mathfrak{c}^-)$.

The case where $\epsilon(\ell) = 0$ is very similar: the set $\mathfrak{C}$ is instead taken to be the collection of pairs of proper, integral $\mathcal{O}_s$-ideals $(\mathfrak{c}^+, \mathfrak{c}^-)$ such that:

(a) $\mathbf{N}(\mathfrak{c}^+) + N\mathbf{N}(\mathfrak{c}^-) = mp^{2s}|D|$;

(b) $\mathfrak{c}^+$ and $\mathfrak{c}^-$ are prime to p and divisible by $\mathfrak{l}$;

(c) $\mathfrak{c}^+$ lies in the $\mathrm{Pic}(\mathcal{O}_s)$-class of $\bar{\mathfrak{a}}$;

(d) $\mathfrak{c}^-$ lies in the $\mathrm{Pic}(\mathcal{O}_s)$-class of $\mathfrak{a}\bar{\mathfrak{n}}\mathfrak{q}\bar{\mathfrak{g}}^2$.

The function from $D_s^w(R^{w\mathfrak{g}}, m)$ to $\mathfrak{C}$ is then exactly as in (18), and the expression on the left-hand side of (17) is equal to

$$4 \sum_{(\mathfrak{c}^+,\mathfrak{c}^-)\in\mathfrak{C}} \mathrm{ord}_\ell \mathbf{N}(\mathfrak{c}^-) \cdot 2^{\#\{r\in\mathcal{W}_0 | r \text{ divides } \mathbf{N}(\mathfrak{c}^-)\}} = 2 \sum_{\substack{n>0 \\ \ell|n,(n,p)=1}} r_{\mathfrak{a}}(mp^{2s}|D| - nN) r_{\mathfrak{a}\mathfrak{q}\bar{\mathfrak{n}}\bar{\mathfrak{g}}^2}(n) \delta(n) \mathrm{ord}_\ell(n)$$

by taking $n = \mathbf{N}(\mathfrak{c}^-)$. This is equivalent to the stated equality. □

Proof of Proposition 5.3.1. Fix a set $\mathfrak{G} = \{\mathfrak{g}\}$ of proper integral $\mathcal{O}_s$-ideals of norm prime to Dp such that $\{\mathfrak{g}^2 \mid \mathfrak{g} \in \mathfrak{G}\}$ represents $\mathrm{Pic}(\mathcal{O}_s)^2$. As $\mathfrak{g}$ varies over $\mathfrak{G}$ and w varies over $\mathcal{W}$, $w\mathfrak{g}$ varies over $\mathrm{Pic}(\mathcal{O}_s)$ hitting each ideal class once if $\epsilon(\ell) = 0$ and twice if $\epsilon(\ell) = -1$. By Lemmas 5.3.2 and 5.3.4 (recall also that we are assuming D odd) we have

$$\begin{aligned}
\Delta_s^{\mathfrak{a}}(m) &= \frac{1}{2} \sum_{w\in\mathcal{W}} \sum_{\mathfrak{g}\in\mathfrak{G}} \Delta_s^{\mathfrak{a}}(R^{w\mathfrak{g}}, m) \\
&= \frac{1}{2(1-\epsilon(\ell))} \sum_{\mathfrak{g}\in\mathfrak{G}} \sum_{w\in\mathcal{W}} \sum_{b\in D_s^{\mathfrak{a}}(R^{w\mathfrak{g}},m)} \left(1 + \mathrm{ord}_\ell \mathbf{N}(b^-)\right) \\
&= \sum_{\substack{n>0 \\ \ell|n,(n,p)=1}} \delta(n) r_{\mathfrak{a}}(mp^{2s}|D| - nN) \cdot \begin{cases} \mathrm{ord}_\ell(\ell n) R_{\mathfrak{a}\mathfrak{q}\mathfrak{n}}(n/\ell) & \text{if } \epsilon(\ell) = -1 \\ \mathrm{ord}_\ell(n) R_{\mathfrak{a}\mathfrak{q}\mathfrak{n}\mathfrak{l}}(n/\ell) & \text{if } \epsilon(\ell) = 0. \end{cases}
\end{aligned}$$

□

5.4 The ℓ-contribution to the height

Fix $m = m_0 p^r$ with $(m_0, Np) = 1$. Let $\mathfrak{b}$ be a proper integral $\mathcal{O}_s$-ideal, and denote by $\tau \in \mathrm{Gal}(H_s/K)$ the Artin symbol of $\mathfrak{b}$. We consider the quantity

$$\langle c_s^\tau, T_{m_0}(\mathbf{d}_{s,r+2}^{\sigma\tau}) \rangle_v - \langle c_{s-1}^\tau, T_{m_0}(\mathbf{d}_{s,r+1}^{\sigma\tau}) \rangle_v$$

where the pairing is the local Néron symbol on $X_{/H_{s,v}}$ of Proposition 3.3.2. By replacing h_i with h_i^τ in Proposition 5.1.1, this is equal to

$$\log_p(\ell) \sum_{n\geqslant 0} (|\mathrm{Hom}_{W_n}(\underline{h}_s^{\mathfrak{a}\mathfrak{b}}, \underline{h}_s^{\mathfrak{b}})_{\deg(mp^2)}| - |\mathrm{Hom}_{W_n}(\underline{h}_{s-1}^{\mathfrak{a}\mathfrak{b}}, \underline{h}_s^{\mathfrak{b}})_{\deg(mp)}| \\ - |\mathrm{Hom}_{W_n}(\underline{h}_s^{\mathfrak{a}\mathfrak{b}}, \underline{h}_{s-1}^{\mathfrak{b}})_{\deg(mp)}| + |\mathrm{Hom}_{W_n}(\underline{h}_{s-1}^{\mathfrak{a}\mathfrak{b}}, \underline{h}_{s-1}^{\mathfrak{b}})_{\deg(m)}|),$$

which is equal, by Corollary 5.2.5, to

$$\log_p(\ell)\sum_{n\geqslant 0}(|\mathrm{Hom}_{W_n}(\underline{h}_s^{\mathfrak{a}\mathfrak{b}},\underline{h}_s^{\mathfrak{b}})^{\mathrm{unstable}}_{\deg(mp^2)}| - |\mathrm{Hom}_{W_n}(\underline{h}_s^{\mathfrak{a}\mathfrak{b}},\underline{h}_s^{\mathfrak{b}})^{\mathrm{unstable}}_{\deg(m)}|).$$

By Corollary 5.2.8, this last expression is equal to

$$\log_p(\ell)(\Delta_s^{\mathfrak{a}}(R^{\mathfrak{b}^{-1}},mp^2) - \Delta_s^{\mathfrak{a}}(R^{\mathfrak{b}^{-1}},m)),$$

where we have used [Con04, (7-8)] to identify $\mathrm{End}_{W_0}(\underline{h}_s^{\mathfrak{b}})$ with $\mathfrak{b}^{-1}\cdot\mathrm{End}_{W_0}(\underline{h}_s)\cdot\mathfrak{b}$ inside of $B = \mathrm{Hom}_{W_0}(\underline{h}_s,\underline{h}_s)\otimes\mathbb{Q}$.

Proposition 5.4.1. *For any positive integer $m = m_0p^r$ with $(m_0,Np) = 1$ and any ℓ nonsplit in K,*

$$\sum_w(\langle c_s, T_{m_0}(\mathbf{d}^{\sigma}_{s,r+2})\rangle_w - \langle c_{s-1}, T_{m_0}(\mathbf{d}^{\sigma}_{s,r+1})\rangle_w) = \log_p(\ell)(\Delta_s^{\mathfrak{a}}(mp^2) - \Delta_s^{\mathfrak{a}}(m))$$

where the sum is over all primes w of H_s above ℓ and $\Delta_s^{\mathfrak{a}}(m)$ is the quantity defined in § 5.3 (and computed explicitly in Proposition 5.3.1), and the pairing is the local Néron symbol on $X_{/H_{s,w}}$ of Proposition 3.3.2.

Proof. Let $\mathrm{Pic}^{\ell}(\mathcal{O}_s)$ denote the quotient of $\mathrm{Pic}(\mathcal{O}_s)$ by the subgroup generated by the class of the unique prime of K above ℓ. Then $\mathrm{Pic}^{\ell}(\mathcal{O}_s)$ acts simply transitively on the set $\mathcal{S}$ by Lemma 5.3.2, and also acts simply transitively on the primes of H_s above ℓ. If we let $\mathfrak{b}$ vary over a set of representatives of $\mathrm{Pic}^{\ell}(\mathcal{O}_s)$ and use the relation $\langle x^{\tau},y^{\tau}\rangle_v = \langle x,y\rangle_{\tau^{-1}(v)}$ for $\tau\in\mathrm{Gal}(H_s/K)$, then the claim follows from the discussion above. □

6. Néron symbols above p

In this section we use the methods of Perrin-Riou [PR87a, § 5.3] to analyze the p-adic Néron symbol on $X_0(N)$ at primes above p.

Fix $s>0$, $\sigma\in\mathrm{Gal}(H_s/K)$, and assume that $\epsilon(p) = 1$ and $D\neq -3,-4$. As always, we let $\mathfrak{a}\subset\mathcal{O}_s$ be a proper ideal whose Artin symbol is σ. For any positive integer m, we let T_m be the usual Hecke correspondence on $X_0(N)$ (taking the Atkin–Lehner U_ℓ at primes dividing N). For any correspondence T from a curve to itself, we let T^{ι} denote the transpose correspondence. Thus $T_m = T_m^{\iota}$ for $(m,N) = 1$. If $\mathfrak{p}$ is one of the two primes of K above p, we let δ be the order of $\mathfrak{p}$ in the ideal class group of K.

6.1 Some modular forms

Fix a place v of H_s above p.

Lemma 6.1.1. *Let R be the integer ring of $H_{s,v}$ and let $\underline{\mathbf{h}}^{\sigma}_{s,r}$ be the horizontal divisor of $X_0(N)_{/R}$ with generic fiber $\mathbf{h}^{\sigma}_{s,r}$. For any divisor $\underline{C}$ on $X_0(N)_{/R}$, there is a constant $c = c(\underline{C})$ such that the intersection multiplicity $i(\underline{C},\underline{\mathbf{h}}^{\sigma}_{s,r})$ of § 3.1 depends only on $r \pmod{\delta}$ when $r>c$.*

Proof. It suffices to prove this when $\underline{C}$ is effective. The extension H_∞/H_0 is totally ramified at v, and we let w denote the unique place of H_∞ above v. Let $F(r)$ be the completion of the maximal unramified extension of $H_{s+r,w}$ with integer ring $W(r)$, and let $W(r)_k$ be the quotient of $W(r)$ by the $(k+1)$th power of the maximal ideal. Let $\hat{\mathbb{Q}}_p^{\mathrm{unr}}$ denote the completion of the maximal unramified extension of $\mathbb{Q}_p$. The extension $H_{s+r,w}/H_{0,w}$ is totally ramified of degree $p^{r+s-1}(p-1)$, and $H_{0,w}\subset\hat{\mathbb{Q}}_p^{\mathrm{unr}}$. From this one easily deduces that $F(r)$ is the compositum of $\hat{\mathbb{Q}}_p^{\mathrm{unr}}$ and $H_{s+r,w}$ (so is abelian over $\mathbb{Q}_p$), and that $F(r)/\hat{\mathbb{Q}}_p^{\mathrm{unr}}$ is totally ramified of degree $p^{r+s-1}(p-1)$. By class field theory $F(r) = \hat{\mathbb{Q}}_p^{\mathrm{unr}}(\mu_{p^{s+r}})$. Decompose $\underline{C} = \sum_{k=0}^{e_r}\underline{y}(k)$ as a sum of prime divisors on $X_0(N)_{/W(r)}$.

For r greater than or equal to some r_0 the sequence e_r is constant and $\underline{h}^{\sigma}_{s,r}$ has no components in common with $\underline{C}$. Abbreviate $e = e_{r_0}$ and take $c = r_0 + \delta$.

Fix $r_1 > c$, $r = r_1 + i\delta$ with $i \geqslant 0$, and an extension of σ to $\mathrm{Gal}(H_\infty/K)$. By [Con04, Lemma 2.4] or [SeTa69, Theorems 8, 9(1)] the point $h^{\sigma}_{s+r} \in X_0(N)(F(r))$ represents a Heegner diagram over $F(r)$ having good reduction, and so its Zariski closure $\underline{h}^{\sigma}_{s+r}$ in $X_0(N)_{/W(r)}$ is a section to the structure map representing a Heegner diagram over $W(r)$. As in § 4, the choice of Heegner diagram $\underline{h}^{\sigma}_{s+r}$ determines a family of isogenous Heegner diagrams over $W(r)$,

$$\underline{h}^{\sigma}_{s+r} \ \to\ \underline{h}^{\sigma}_{s+r-1} \ \to\ \cdots .$$

The generic geometric kernel of the map $\underline{h}^{\sigma}_{s+r} \to \underline{h}^{\sigma}_{s+r-1}$ is stable under the action of the absolute Galois group of $F(r)$, and the Euler system relations of § 1.2 tell us that no other order p subgroup of $\underline{h}^{\sigma}_{s+r}(F(r)^{\mathrm{alg}})$ has this property. Indeed, the remaining p quotients by order p subgroups are permuted simply transitively by $\mathrm{Gal}(F(r+1)/F(r))$. It follows that this kernel must be the kernel in $\underline{h}^{\sigma}_{s+r}[p]$ of reduction to $W(r)_0$ (recall $\epsilon(p) = 1$, so $\underline{h}^{\sigma}_{s+r}$ has ordinary reduction) and the map $\underline{h}^{\sigma}_{s+r} \to \underline{h}^{\sigma}_{s+r-1}$ reduces to the absolute Frobenius in the closed fiber. The action of $\mathcal{O}_{s+r}$ on the closed fiber of $\underline{h}^{\sigma}_{s+r}$ extends to an action of the maximal order (we have just shown that the closed fiber of $\underline{h}^{\sigma}_{s+r}$ is isomorphic to a Galois conjugate of the closed fiber of $\underline{h}^{\sigma}_{0}$), and if $\mathfrak{p}$ denotes the prime of K below v, then the action of any generator of the principal ideal $\mathfrak{p}^{\delta}$ is a degree p^{δ} purely inseparable endomorphism, whose kernel must therefore be the kernel of the δ^{th}-iterate of Frobenius. This shows that the Heegner diagrams $\underline{h}^{\sigma}_{s+r}$ and $\underline{h}^{\sigma}_{s+r-\delta}$ are isomorphic over $\mathrm{Spec}(W(r)_0)$, and that the closed fiber of $\underline{h}^{\sigma}_{s+r}$ is the base change to $W(r)$ of the closed fiber of the Zariski closure of $h^{\sigma}_{s+r_1}$ on $X_0(N)_{/W(r_1)}$.

We claim that the Heegner diagram $\underline{h}^{\sigma}_{s+r-\delta}$ is distinct from $\underline{h}^{\sigma}_{s+r}$ over $W(r)_1$, so that Proposition 4.0.5 gives the intersection formula

$$i(\underline{h}^{\sigma}_{s+r}, \underline{h}^{\sigma}_{s+r-\delta}) = \tfrac{1}{2}|\mathcal{O}_K^{\times}| = 1 \tag{20}$$

on $X_0(N)_{/W(r)}$. Indeed, if these Heegner diagrams are isomorphic over $W(r)_1$, then the reduction of such an isomorphism to $W(r)_0$ allows us to view $\underline{h}^{\sigma}_{s+r-\delta}$ and $\underline{h}^{\sigma}_{s+r}$ over $W(r)_1$ as isomorphic *deformations* of the common closed fiber, which we denote by g. Let $T = \varprojlim g(W(r)_0)[p^k] \cong \mathbb{Z}_p$. The theory of Serre–Tate coordinates (for example [Gor02, ch. 3, Theorem 4.2]) associates to these Heegner diagrams over $W(r)$ (viewed as deformations of g) two bilinear maps

$$q_{s+r-\delta}, q_{s+r} : T \otimes T \ \to\ 1 + \mathfrak{m}_{W(r)}.$$

The first surjects onto $\mu_{p^{s+r-\delta}}$, and the second onto $\mu_{p^{s+r}}$. Since we assume the Heegner diagrams over $W(r)_1$ are isomorphic as deformations of g, the bilinear maps $q_{s+r-\delta}, q_{s+r}$ are congruent modulo $1 + \mathfrak{m}^2_{W(r)}$. This is a contradiction, as $\mu_{p^{s+r-\delta}}$ is contained in $1 + \mathfrak{m}^2_{W(r)}$ while $\mu_{p^{s+r}}$ is not (use the fact, noted above, that $F(r) = \hat{\mathbb{Q}}_p^{\mathrm{unr}}(\mu_{p^{r+s}})$ to replace $\mathfrak{m}_{W(r)}$ with the maximal ideal of $\mathbb{Z}_p[\mu_{p^{r+s}}]$).

Each prime divisor $\underline{y}(k)$ occurring in the support of $\underline{C}$ either does not meet the common closed point of $\underline{h}^{\sigma}_{s+r-\delta}$, $\underline{h}^{\sigma}_{s+r}$, in which case $i(\underline{y}(k), \underline{h}^{\sigma}_{s+r}) = 0$, or it does, in which case $\underline{y}(k)$ intersects both $\underline{h}^{\sigma}_{s+r-\delta}$ and $\underline{h}^{\sigma}_{s+r}$. Assume we are in the latter case. The divisors $\underline{y}(k)$ and $\underline{h}^{\sigma}_{s+r-\delta}$ on $X_0(N)_{/W(r)}$ both arise as the base change of divisors defined over $W(r-\delta)$. Since base change through a finite extension multiplies intersections by the ramification degree, $i(\underline{y}(k), \underline{h}^{\sigma}_{s+r-\delta}) > 1$. If also $i(\underline{y}(k), \underline{h}^{\sigma}_{s+r}) > 1$, then $i(\underline{h}^{\sigma}_{s+r}, \underline{h}^{\sigma}_{s+r-\delta}) > 1$, contradicting (20). Thus $i(\underline{y}(k), \underline{h}^{\sigma}_{s+r}) = 1$. We have shown that

$$i(\underline{C}, \underline{\mathbf{h}}^{\sigma}_{s,r})_R = i(\underline{C}, \underline{h}^{\sigma}_{s+r})_{W(r)} = \sum_{k=0}^{e} i(\underline{y}(k), \underline{h}^{\sigma}_{s+r})_{W(r)}$$

(the subscripts denoting the bases over which the intersections are computed) is equal to the number of $\underline{y}(k)$, $0 \leqslant k \leqslant e$, which contain the closed point of $\underline{h}^{\sigma}_{s+r}$. By the discussion earlier this is equal to

the number of $\underline{y}(k)$ on $X_0(N)_{/W(r_1)}$ which contain the closed point of the Zariski closure of $h^\sigma_{s+r_1}$ on $X_0(N)_{/W(r_1)}$, which is equal to $i(\underline{C}, \underline{\mathbf{h}}^\sigma_{s,r_1})_R$ by taking $r = r_1$ in the preceding argument. □

Let us say that a divisor C on $X_0(N)_{/H_{s,v}}$ has *good support* if its support contains no cusps except possibly for the cusp 0. Note that the set of such divisors is stable under the action of T^i_m for any m. This follows easily from the fact that the main Atkin–Lehner involution w on $X_0(N)$ satisfies $wT_m w = T^\iota_m$ and $w \cdot \infty = 0$, and that $T_m \cdot \infty$ is supported at ∞. For C of degree zero with good support we define a formal q-expansion

$$\phi(C)_v = \sum_{m=m_0p^r} \langle C, T_{m_0}\mathbf{d}^\sigma_{s,r}\rangle_v q^m \tag{21}$$

where $\langle\ ,\ \rangle_v$ is the p-adic Néron symbol on $X_0(N)_{/H_{s,v}}$ of Proposition 3.3.2, and where for any integer $m > 0$ we write $m = m_0p^r$ with $(m_0, p) = 1$. Let U denote the shift operator on formal q-expansions $U(\sum a_m q^m) = \sum a_{mp} q^m$. The q-expansion $\phi(C)_v$ is only defined if C has support prime to $T_{m_0}(\mathbf{d}^\sigma_{s,r})$ for every $m = m_0p^r$, but for any C with good support and degree 0 the q-expansion $U^k\phi(C)_v$ is defined for $k \gg 0$. Indeed, the geometric points in the support of $T_{m_0}(\mathbf{d}^\sigma_{s,r+k})$ each represent either the cusp ∞ or a CM elliptic curve such that the valuation at p of the conductor of the CM order is exactly $s + r + k$.

We can use the Lemma 6.1.1 to compute p-adic Néron symbols at v in the only case where they are known to be related to intersection pairings: the case where one divisor is principal.

Corollary 6.1.2. *Suppose C is the divisor of a rational function on $X_0(N)_{/H_{s,v}}$, and that C has good support. Then for each integer $m > 0$*

$$\lim_{k\to\infty} a_m(U^k(U^\delta - 1)\phi(C)_v) = 0.$$

Proof. Write $m = m_0p^r$ with $(m_0, p) = 1$. The divisor $T^\iota_{m_0}(C)$ is again principal with good support, and we fix a rational function f with $(f) = T^\iota_{m_0}(C)$. Writing v for the normalized valuation on $H_{s,v}$, the intersection theory of § 3.1 gives

$$v(f(\mathbf{d}^\sigma_{s,r+k+\delta})) = [(f), \mathbf{d}^\sigma_{s,r+k+\delta}] = i(\underline{(f)}, \underline{\mathbf{h}}^\sigma_{s,r+k+\delta})) - p^{r+k+\delta} \cdot i(\underline{(f)}, \underline{\infty}))$$

where the underlining of divisors indicates passing to horizontal divisors on $X_0(N)_{/R}$, R the integer ring of $H_{s,v}$. Similarly

$$v(f(\mathbf{d}^\sigma_{s,r+k})) = [(f), \mathbf{d}^\sigma_{s,r+k}] = i(\underline{(f)}, \underline{\mathbf{h}}^\sigma_{s,r+k})) - p^{r+k} \cdot i(\underline{(f)}, \underline{\infty})).$$

From this and Lemma 6.1.1 we deduce

$$\begin{aligned} v\left(\frac{f(\mathbf{h}^\sigma_{s,r+k+\delta})}{f(\mathbf{h}^\sigma_{s,r+k})}\right) &= v\left(\frac{f(\mathbf{d}^\sigma_{s,r+k+\delta})}{f(\mathbf{d}^\sigma_{s,r+k})}\right) + (p^\delta - 1)p^{r+k} \cdot v(f(\infty)) \\ &= (p^\delta - 1)p^{r+k} \cdot [v(f(\infty)) - i(\underline{(f)}, \underline{\infty})] \end{aligned}$$

for k large. Multiplying f by an element of $H^\times_{s,v}$ does not change (f), and so we may assume that $v(f(\infty)) = i(\underline{(f)}, \underline{\infty})$. Then $f(\mathbf{h}^\sigma_{s,r+k+\delta})/f(\mathbf{h}^\sigma_{s,r+k})$ is a unit in $H_{s,v}$ for k large. It is also the norm of some $u_k \in H_{s+r+k,v}$, the completion of H_{s+r+k} at the unique prime above v. Using Proposition 3.3.2(b)

$$\begin{aligned} a_m(U^k(U^\delta - 1)\phi(C)_v) &= \langle C, T_{m_0}\mathbf{d}^\sigma_{s,r+k+\delta}\rangle_v - \langle C, T_{m_0}\mathbf{d}^\sigma_{s,r+k}\rangle_v \\ &= \rho_{H_{s,v}}(f(\mathbf{d}^\sigma_{s,r+k+\delta})) - \rho_{H_{s,v}}(f(\mathbf{d}^\sigma_{s,r+k})) \\ &= \rho_{H_{s,v}}\left(\frac{f(\mathbf{h}^\sigma_{s,r+k+\delta})}{f(\mathbf{h}^\sigma_{s,r+k})}\right) - (p^\delta - 1)p^{r+k}\rho_{H_{s,v}}(f(\infty)) \\ &= \rho_{\mathbb{Q}_p}(\mathrm{Norm}_{H_{s+r+k,v}/\mathbb{Q}_p}(u_k)) - (p^\delta - 1)p^{r+k}\rho_{H_{s,v}}(f(\infty)). \end{aligned}$$

Since p is split, the field $H_{s+r+k,v}$ is abelian over $\mathbb{Q}_p$, the unit norms from $H_{s+r+k,v}$ to $\mathbb{Q}_p$ converge to 1 as $k \to \infty$, and so the final expression converges to 0. □

Given any point $P \in J_0(N)(H_{s,v})$ we may choose a degree zero divisor C on $X_0(N)_{/H_{s,v}}$ having good support which represents P. Corollary 6.1.2 implies that for any sequence of integers $b = (b_k)$ with $b_k \to \infty$, the q-expansion with $\mathbb{Q}_p$-coefficients

$$\Phi_b(P)_v \overset{\text{def}}{=} \lim_{k\to\infty} U^{b_k}(U^\delta - 1)\phi(C)_v,$$

if the limit exists (in the sense of coefficient-by-coefficient convergence; there is no assumption of uniformity) depends only on P and not on the choice of C.

Definition 6.1.3. A sequence of integers $b = (b_k)$ is *admissible* if $b_k \to \infty$ and if the limit (coefficient-by-coefficient) defining $\Phi_b(P)_v$ exists for every $P \in J_0(N)(H_{s,v})$.

Lemma 6.1.4. *Any sequence of integers tending to ∞ admits an admissible subsequence.*

Proof. Fix a sequence $b = (b_k)$ of integers tending to ∞. Let C be a degree zero divisor on $X_0(N)_{H_{s,v}}$ with good support, and consider the first Fourier coefficient

$$a_1(U^{b_k}(U^\delta - 1)\phi(C)_v) = \langle C, \mathbf{d}^\sigma_{s,b_k+\delta} - \mathbf{d}^\sigma_{s,b_k}\rangle_v.$$

By the final claim of Proposition 3.3.2 the sequence on the right-hand side takes values in a compact subset of $\mathbb{Q}_p$, and so we may choose a convergent subsequence. By Corollary 6.1.2 and the finite dimensionality of $J_0(N)(H_{s,v}) \otimes \mathbb{Q}_p$, we may repeat this process, eventually replacing b by a subsequence (still denoted b, abusively) such that

$$\lim_{k\to\infty} a_1(U^{b_k}(U^\delta - 1)\phi(C)_v)$$

exists for every degree zero divisor with good support. By the same argument we may assume that the limit $\lim_{k\to\infty} a_p(U^{b_k}(U^\delta - 1)\phi(C)_v)$ also exists for all such divisors. Now fix $m = m_0 p^r$ with $(m_0, p) = 1$. From the definition of ϕ we have

$$a_m(U^{b_k}(U^\delta - 1)\phi(C)_v) = a_{p^r}(U^{b_k}(U^\delta - 1)\phi(T^\iota_{m_0} C)_v) \tag{22}$$

(for k large enough that both sides are defined). If $r = 0$ or 1, then the limit as $k \to \infty$ exists by the above choice of b. For $r > 1$ we use the Euler system relations of § 1.2 to see that

$$\begin{aligned}\mathbf{d}^\sigma_{s,r+b_k} &= \mathrm{Norm}_{H_{s+b_k+1}/H_s}\mathbf{d}^\sigma_{s+b_k+1,r-1}\\ &= \mathrm{Norm}_{H_{s+b_k+1}/H_s}(T_{p^{r-1}}d^\sigma_{s+b_k+1} - T_{p^{r-2}}d^\sigma_{s+b_k})\\ &= T_{p^{r-1}}\mathbf{d}^\sigma_{s,b_k+1} - pT_{p^{r-2}}\mathbf{d}^\sigma_{s,b_k}\end{aligned}$$

which, together with the same formula with b_k replaced by $b_k + \delta$, implies that the right-hand side of (22) equals (for $k \gg 0$)

$$a_p(U^{b_k}(U^\delta - 1)\phi(T^\iota_{m_0 p^{r-1}} C)_v) - p \cdot a_1(U^{b_k}(U^\delta - 1)\phi(T^\iota_{m_0 p^{r-2}} C)_v),$$

and this limit exists as $k \to \infty$. □

Fix an admissible sequence b. Note that the above proof shows that

$$a_{mp}(\Phi_b(P)_v) = \begin{cases} a_p(\Phi_b(T^\iota_m P)_v) & \text{if } (m,p) = 1\\ a_p(\Phi_b(T^\iota_m P)_v) - pa_1(\Phi_b(T^\iota_{m/p} P)_v) & \text{otherwise.}\end{cases} \tag{23}$$

Let $\mathbf{T}^{\text{full}}$ denote the $\mathbb{Q}_p$-algebra generated by the Hecke operators T_m for all $m > 0$ acting on $J_0(N)$. For any $P \in J_0(N)(H_{s,v})$ and any $i > 0$, the linear functional on $\mathbf{T}^{\text{full}}$ defined by $T \mapsto a_i(\Phi_b(T^\iota P)_v)$

determines a p-adic modular form

$$h_i(P) = \sum a_i(\Phi_b(T_m^\iota P)_v) \cdot q^m \in S_2(\Gamma_0(N), \mathbb{Q}) \otimes \mathbb{Q}_p$$

of level $\Gamma_0(N)$ (as does any linear functional on $\mathbf{T}^{\text{full}}$; this follows from [Hid93, § 5.3 Theorem 1] and the identification of $\mathbf{T}^{\text{full}}$ with the Hecke algebra acting on weight two cusp forms). The relation (23) can be written as

$$U \cdot \Phi_b(P)_v = h_p(P) - pV \cdot h_1(P)$$

where $V(\sum a_n q^n) = \sum a_n q^{pn}$. As V takes modular forms of level $\Gamma_0(N)$ to modular forms of level $\Gamma_0(Np)$, we may define

$$\Psi_b(P)_v = U \cdot \Phi_b(P)_v \in M_2(\Gamma_0(Np), \mathcal{A}) \otimes_{\mathcal{A}} \mathcal{B}$$

for any $P \in J_0(N)(H_{s,v})$.

6.2 Annihilation of E_σ

Recall Hida's ordinary projector $e^{\text{ord}} = \lim_{k\to\infty} U^{k!}$ from § 2. Fix an admissible (in the sense of Definition 6.1.3, and for all primes above p simultaneously) subsequence $b = (b_k)$ of $k!$ and define, for any $P \in J_0(N)(H_s)$, a p-adic modular form $\Psi_b(P) = \sum_{v|p} \Psi_b(P)_v$ where the sum is over primes v of H_s above p. Similarly, define $\phi(C) = \sum_v \phi(C)_v$ (whenever $\phi(C)_v$ is defined for all v above p).

In the next section, we shall see that there is a modular form

$$E_\sigma \in M_2(\Gamma_0(Np^\infty), \mathcal{A}) \otimes \mathcal{B}$$

with the following property: if $\langle\ ,\ \rangle_p$ denotes the sum of the local p-adic Néron symbols on $X_0(N)_{/H_{s,v}}$ at the primes of H_s above p, then for any $m = m_0 p^r$ with $(m_0, Np) = 1$ the mth Fourier coefficient of E_σ is given by the expression

$$\begin{aligned} a_m(E_\sigma) &= \langle c_s, T_{m_0}(\mathbf{d}^\sigma_{s,r+2})\rangle_p - \langle c_{s-1}, T_{m_0}(\mathbf{d}^\sigma_{s,r+1})\rangle_p \\ &= a_{mp^2}(\phi(c_s)) - a_{mp}(\phi(c_{s-1})), \end{aligned}$$

where, as in § 0.1, $c_i = (h_i) - (0)$. From this we immediately deduce the following.

Lemma 6.2.1. *There is a modular form $g \in M_2(\Gamma_0(Np), \mathcal{A}) \otimes \mathcal{B}$ such that $a_m(g) = 0$ whenever $(m, N) = 1$, and*

$$(U^\delta - 1)e^{\text{ord}} E_\sigma = U\Psi_b(c_s) - \Psi_b(c_{s-1}) + g.$$

Proof. Compare both sides coefficient by coefficient. □

The significance of Lemma 6.2.1 is the following: while E_σ depends *a priori* on the divisors c_s and c_{s-1}, the p-adic modular forms $\Psi_b(c_s)$ and $\Psi_b(c_{s-1})$ depend only on the images in $J_0(N)(H_s)$. This plays a crucial role in the proof of the following proposition.

Proposition 6.2.2. *Let f be the modular form fixed in the introduction. The p-adic modular form E_σ is annihilated by the linear functional L_f of Lemma 2.0.2.*

Proof. By Lemmas 2.0.2(c) and (d) and 6.2.1

$$(\alpha^\delta - 1)L_f(E_\sigma) = L_f((U^\delta - 1)e^{\text{ord}} E_\sigma) = \alpha L_f(\Psi_b(c_s)) - L_f(\Psi_b(c_{s-1})),$$

and so it suffices to show that $L_f(\Psi_b(P)_v) = 0$ for every $P \in J_0(N)(H_s)$ and every prime v of H_s above p. Fix one such prime and let $\mathbf{T}$ be the $\mathbb{Q}$-algebra generated by all T_ℓ with $(\ell, N) = 1$ acting on $J_0(N)$. Recall from the introduction the decomposition

$$J_0(N)(H_s) \otimes \mathcal{B} \cong \bigoplus_\beta J(H_s)_\beta$$

where the sum is over all algebra homomorphisms $\beta : \mathbf{T} \to \mathbb{Q}_p^{\mathrm{alg}}$ (and recall that all such maps take values in $\mathcal{B}$ by hypothesis) and $\mathbf{T}$ acts on $J(H_s)_\beta$ through the character β. Let β_f be the homomorphism associated to the fixed newform f.

Suppose that $P \in J(H_s)_\beta$ for some character β, and extend $\Psi_b(\ )_v$ $\mathcal{B}$-linearly to $J_0(N)(H_s) \otimes \mathcal{B}$. We treat the cases $\beta \neq \beta_f$ and $\beta = \beta_f$ separately.

Lemma 6.2.3. *If $\beta \neq \beta_f$, then $L_f(\Psi_b(P)_v) = 0$.*

Proof. Use the notation $\tilde{T}_m$ for Hecke operators in level $\Gamma_0(Np)$. For any m prime to Np we have

$$a_m(f)L_f(\Psi_b(P)_v) = L_f(\tilde{T}_m\Psi_b(P)_v) = L_f(\Psi_b(T_mP)_v) = \beta(T_m)L_f(\Psi_b(P)_v)$$

(the first equality is by Lemma 2.0.2, the second is a straightforward calculation, and the third is obvious). Thus, if $L_f(\Psi_b(P)_v) \neq 0$, then $\beta_f(T_m) = \beta(T_m)$ for all $(m, Np) = 1$. The Atkin–Lehner strong multiplicity one theorem [AL70, Lemma 24] thus implies that $\beta_f = \beta$, a contradiction. □

Lemma 6.2.4. *If $\beta = \beta_f$, then $L_f(\Psi_b(P)_v) = 0$.*

Proof. We follow the lead of [PR87a, Example 4.12]. Let R be the integer ring of $H_{s,v}$, $\mathfrak{m}$ the maximal ideal of R, and $\mathbf{F} = R/\mathfrak{m}$. Let G_n be the p^n-torsion of the Néron model of $J_0(N)$ over R, a finite group scheme over R. Let G_n^0 and G_n^{et} be the connected component and maximal étale quotient of G_n, respectively, and let $G_n^{0,\mathrm{et}}$ (respectively $G_n^{0,0}$) be the maximal subgroup scheme of G_n^0 with étale dual (respectively quotient with connected dual).

By the theory of Dieudonné modules the Frobenius and Verschiebung morphisms on $(G_n^{0,0})_{/\mathbf{F}}$ are nilpotent, and so by the Eichler–Shimura congruence the same is true of the Hecke operator T_p. This is equivalent to $T_p^i(I) \subset \mathfrak{m}I$ for some i, where A is the Hopf algebra over R associated to the affine group scheme $G_n^{0,0}$, I is the kernel of the augmentation map $A \to R$, and T_p is now viewed as an R-algebra map $A \to A$. For any Artinian quotient $R/\mathfrak{m}^kR$ of R and any R-algebra map $\tau : A \to R/\mathfrak{m}^kR$,

$$(\tau \circ T_p^{ik})(I) \subset \tau(\mathfrak{m}^k I) = 0.$$

Back in the world of group schemes, this says that T_p acts as a nilpotent operator on $G_n^{0,0}(R/\mathfrak{m}^k)$ for any k and any n. From this it follows easily that T_p acts as a topologically nilpotent operator on R-valued points of the formal group scheme $\hat{G}^{0,0}$ associated to the p-divisible group $\varinjlim G_n^{0,0}$.

Let $\hat{G}^0$ and $\hat{G}^{0,\mathrm{et}}$ be the formal group schemes associated to G_n^0 and $G_n^{0,\mathrm{et}}$, respectively. As $\hat{G}^0(R) \subset J_0(N)(H_{s,v})$ with finite index, we may identify

$$\hat{G}^0(R) \otimes \mathcal{B} \cong J_0(N)(H_{s,v}) \otimes \mathcal{B}.$$

As $\beta_f(T_p) = a_p(f) \in \mathcal{A}^\times$ is a unit, any element of $\hat{G}^0(R) \otimes \mathcal{B}$ on which $\mathbf{T}$ acts through β_f must come from the subspace $\hat{G}^{0,\mathrm{et}}(R) \otimes_{\mathbb{Z}_p} \mathcal{B}$. We are thus reduced to the case $P \in \hat{G}^{0,\mathrm{et}}(R)$. By [Sch87, Theorem 1(i)] (together with the proof of [Sch87, Theorem 2]), the universal norms in $\hat{G}^{0,\mathrm{et}}(R)$ from any ramified $\mathbb{Z}_p$-extension of $H_{s,v}$ have finite index. We are thus further reduced to the case where $P \in J_0(N)(H_{s,v})$ is a universal norm from L_∞, the cyclotomic $\mathbb{Z}_p$-extension of $H_{s,v}$. Let $L_n \subset L_\infty$ be the extension of $H_{s,v}$ with $[L_n : H_{s,v}] = p^n$, and write $P = \mathbf{N}_{L_n/L_0}Q_n$ for some $Q_n \in J_0(N)(L_n)$. Lift Q_n to a degree zero divisor on $X_0(N)_{/L_n}$ with support prime to the cusps. Then for $m = m_0p^r$ with $(m_0, p) = 1$,

$$\begin{aligned} a_m(\Psi_b(P)_v) &= \lim_{k\to\infty} a_m(U^{b_k+1}(U^\delta - 1)\phi(\mathbf{N}_{L_n/L_0}Q_n)_v) \\ &= \lim_{k\to\infty} \langle \mathbf{N}_{L_n/L_0}Q_n, T_{m_0}\mathbf{d}^\sigma_{s,b_k+1+\delta+r} - T_{m_0}\mathbf{d}^\sigma_{s,b_k+1+r}\rangle_{X_0(N),H_{s,v}}. \end{aligned}$$

Using Proposition 3.3.2(e), we at last deduce $\Psi_b(P)_v = 0$. □

This completes the proof of Proposition 6.2.2. □

Remark 6.2.5. The reader is invited to reconsider the case $\beta = \beta_f$ under the additional hypothesis that f is ordinary at *every* place of $\mathbb{Q}^{\mathrm{alg}}$ above p. Then the abelian variety (up to isogeny) A_f attached to f by Eichler–Shimura theory is ordinary at p, and a theorem of Mazur [Maz72, Proposition 4.39] tells us that the universal norm subgroup of $A_f(H_{s,v})$ from a ramified $\mathbb{Z}_p$-extension has finite index.

7. Completion of the proofs

Assume that D is odd and $\neq -3$, and that $\epsilon(p) = 1$. Fix $s > 0$ and $\sigma \in \mathrm{Gal}(H_s/K)$. Let $\mathfrak{a}$ be a proper integral $\mathcal{O}_s$-ideal of norm prime to p whose class in $\mathrm{Pic}(\mathcal{O}_s)$ represents σ. Recall from § 0.1 the p-adic modular form F_σ defined by

$$F_\sigma = U^2 F_\sigma^{s,s} - U F_\sigma^{s,s-1} - U F_\sigma^{s-1,s} + F_\sigma^{s-1,s-1} \in M_2(\Gamma_0(Np), \mathcal{A}) \otimes_{\mathcal{A}} \mathcal{B}.$$

Proposition 7.0.6. *For every $m = m_0 p^r$ with $(m_0, Np) = 1$,*

$$\begin{aligned} a_m(F_\sigma) &= \langle c_s, T_{mp^2}(d_s^\sigma)\rangle - \langle c_s, T_{mp}(d_{s-1}^\sigma)\rangle + \langle c_{s-1}, T_m(d_{s-1}^\sigma)\rangle - \langle c_{s-1}, T_{mp}(d_s^\sigma)\rangle \\ &= \langle c_s, T_{m_0}(\mathbf{d}_{s,r+2}^\sigma)\rangle - \langle c_{s-1}, T_{m_0}(\mathbf{d}_{s,r+1}^\sigma)\rangle. \end{aligned} \tag{24}$$

where $\langle\ ,\ \rangle = \langle\ ,\ \rangle_{X_0(N),H_s}$ is the global pairing of (10) viewed as a pairing on $J_0(N)(H_s)$, and c_s, d_s, $\mathbf{c}_{s,r}$, and $\mathbf{d}_{s,r}$ are as in § 0.1. Furthermore, extending the height pairing $\mathcal{B}$-bilinearly to $J_0(N)(H_s) \otimes \mathcal{B}$,

$$L_f(F_\sigma) = (\alpha^2 - 1)\alpha^{2s}\langle z_s, z_s^\sigma\rangle$$

where L_f is the linear functional on $M_2(\Gamma_0(Np^\infty), \mathcal{A})$ of Lemma 2.0.2 and z_s is the regularized Heegner point appearing in Theorem A.

Proof. Recall, for $i, j \leqslant s$ and any m, that

$$a_m(F_\sigma^{i,j}) = \sum_\beta \langle c_i, d_{j,\beta}^\sigma\rangle a_m(f_\beta)$$

where the sum is over algebra homomorphisms $\beta : \mathbf{T} \to \mathbb{Q}^{\mathrm{alg}}$, f_β is the associated primitive eigenform, and $d_{j,\beta}^\sigma$ is the projection of $d_j^\sigma \in J_0(N)(H_s)$ to $J(H_s)_\beta$. Thus, if $(m, N) = 1$,

$$a_m(F_\sigma^{i,j}) = \sum_\beta \langle c_i, \beta(T_m) d_{j,\beta}^\sigma\rangle = \sum_\beta \langle c_i, T_m d_{j,\beta}^\sigma\rangle = \langle c_i, T_m d_j^\sigma\rangle.$$

The first claim follows easily from this and the Euler system relations of § 1.2.

For the second claim,

$$L_f(F_\sigma) = \alpha^2 L_f(F_\sigma^{s,s}) - \alpha L_f(F_\sigma^{s,s-1}) - \alpha L_f(F_\sigma^{s-1,s}) + L_f(F_\sigma^{s-1,s-1})$$

by the final claim of Lemma 2.0.2. It follows from the same lemma that $L_f(f_\beta) = 0$ unless $f_\beta = f$ (as in the proof of Lemma 6.2.3), while $L_f(f) = 1 - \alpha^{-2}$. Therefore,

$$L_f(F_\sigma^{i,j}) = (1 - \alpha^{-2})\langle c_i, d_{j,f}^\sigma\rangle = (1 - \alpha^{-2})\langle d_{i,f}, d_{j,f}^\sigma\rangle$$

where the subscript f indicates projection to the component $J(H_s)_{\beta_f}$ of the algebra homomorphism $\beta_f : \mathbf{T} \to \mathbb{Q}^{\mathrm{alg}}$ associated to f, and the second equality uses the fact that $c_i - d_i = (\infty) - (0)$ is torsion in $J_0(N)(H_s)$ and that summands $J(H_s)_\beta$ are orthogonal for distinct β (an easy consequence of Proposition 3.3.2(c)). This gives

$$\begin{aligned} L_f(F_\sigma) &= (1 - \alpha^{-2})[\alpha^2\langle d_{s,f}, d_{s,f}^\sigma\rangle - \alpha\langle d_{s,f}, d_{s-1,f}^\sigma\rangle - \alpha\langle d_{s-1,f}, d_{s,f}^\sigma\rangle + \langle d_{s-1,f}, d_{s-1,f}^\sigma\rangle] \\ &= (1 - \alpha^{-2})\langle \alpha d_{s,f} - d_{s-1,f}, \alpha d_{s,f}^\sigma - d_{s-1,f}^\sigma\rangle \\ &= (\alpha^2 - 1)\langle \alpha^s z_s, \alpha^s z_s^\sigma\rangle \end{aligned}$$

as z_s was defined to be $\alpha^{-s}(d_{s,f} - \alpha^{-1}d_{s-1,f})$ (in the introduction we abusively confused h_i with $d_i = (h_i) - (\infty)$). □

As explained in § 0.1, in each of the pairings of (24) the divisors have disjoint supports, and so we may decompose $a_m(F_\sigma) = \sum_v a_m(F_\sigma)_v$ as a sum of local Néron symbols on $X_v = X_0(N) \times_{\mathbb{Q}} H_{s,v}$ by defining

$$a_m(F_\sigma)_v = \langle c_s, T_{m_0}(\mathbf{d}^\sigma_{s,r+2})\rangle_v - \langle c_{s-1}, T_{m_0}(\mathbf{d}^\sigma_{s,r+1})\rangle_v$$

where for each prime v of H_s, $\langle\ ,\ \rangle_v = \langle\ ,\ \rangle_{X_v,\rho_{H_{s,v}}}$ is the local Néron symbol of Proposition 3.3.2. We also define, for a rational prime ℓ, $a_m(F_\sigma)_\ell = \sum_{v|\ell} a_m(F_\sigma)_v$.

Proposition 7.0.7. *Suppose* $(m, N) = 1$. *Then*

$$\sum_{\ell \neq p} a_m(F_\sigma)_\ell = a_{mp^{2s}}(G_{\sigma\kappa}) - a_{mp^{2s+2}}(G_{\sigma\kappa}),$$

where G_σ *is the p-adic modular form of Proposition 2.0.4.*

Proof. For any $\ell \neq p$, Proposition 4.0.8 shows that $a_m(F_\sigma)_\ell = 0$ when $\epsilon(\ell) = 1$, while Propositions 5.3.1 and 5.4.1 give an explicit formula for $a_m(F_\sigma)_\ell$ when $\epsilon(\ell) \neq 1$. Corollary 2.0.7 gives an explicit formula for the right-hand side. □

Proof of Theorem A. If we define a p-adic modular form $E_\sigma \in M_2(\Gamma_0(Np^\infty), \mathcal{A}) \otimes_{\mathcal{A}} \mathcal{B}$ by

$$E_\sigma = F_\sigma - U^{2s}(1 - U^2)G_{\sigma\kappa},$$

then for every $m = m_0p^r$ with $(m_0, Np) = 1$ Proposition 7.0.7 implies

$$a_m(E_\sigma) = \langle c_s, T_{m_0}(\mathbf{d}^\sigma_{s,r+2})\rangle_p - \langle c_{s-1}, T_{m_0}(\mathbf{d}^\sigma_{s,r+1})\rangle_p.$$

Proposition 6.2.2 now implies $L_f(E_\sigma) = 0$, and so

$$L_f(F_\sigma) = L_f(U^{2s}(1 - U^2)G_{\sigma\kappa}).$$

Applying Lemma 2.0.2(d) and Proposition 7.0.6

$$(\alpha^2 - 1)\alpha^{2s}\langle z_s, z_s^\sigma\rangle_{X_0(N),H_s} = \alpha^{2s}(1 - \alpha^2)L_f(G_{\sigma\kappa}).$$

Summing over σ and applying Proposition 2.0.4,

$$\sum_\sigma \eta(\sigma)\langle z_s, z_s^\sigma\rangle_{X_0(N),H_s} = -\sum_\sigma \eta(\sigma)L_f(G_{\sigma\kappa}) = -\log_p(\gamma_0)\eta(\kappa)\cdot\mathcal{L}_{f,1}(\eta)$$

for any character η of $\mathrm{Gal}(H_s/K)$. We now view z_s as an element of $J_0(N)(H_s) \otimes \mathcal{B}$, let $z_s^\vee$ be the image of z_s in $J_0(N)(H_s)^\vee \otimes \mathcal{B}$ under the canonical polarization, and switch to the height pairing $\langle\ ,\ \rangle_{J_0(N),H_s}$ of (9). Recalling Remark 3.3.1,

$$\sum_\sigma \eta(\sigma)\langle z_s^\vee, z_s^\sigma\rangle_{J_0(N),H_s} = \log_p(\gamma_0)\eta(\kappa)\cdot\mathcal{L}_{f,1}(\eta).$$

This completes the proof of Theorem A when $s > 0$. If η is a character of $\mathrm{Gal}(H_0/K)$, then we may view η as a character of $\mathrm{Gal}(H_s/K)$ for some $s > 0$, and this does not change the value of $\mathcal{L}_{f,1}(\eta)$. As the z_s and $z_s^\vee$ are norm compatible

$$\begin{aligned}\sum_{\sigma\in\mathrm{Gal}(H_s/K)} \eta(\sigma)\langle z_s^\vee, z_s^\sigma\rangle_{J_0(N),H_s} &= \sum_{\sigma\in\mathrm{Gal}(H_0/K)} \eta(\sigma)\langle z_s^\vee, z_0^\sigma\rangle_{J_0(N),H_s} \\ &= \sum_{\sigma\in\mathrm{Gal}(H_0/K)} \eta(\sigma)\langle z_0^\vee, z_0^\sigma\rangle_{J_0(N),H_0},\end{aligned}$$

so Theorem A also holds when $s = 0$. □

Proof of Theorem B. If we show that

$$\langle y_s^\vee, y_s^\sigma \rangle_{E,H_s} = \langle z_s, z_s^\sigma \rangle_{J_0(N),H_s} \tag{25}$$

for any s then we are done, as Theorem A shows that the two sides of the equality of Theorem B agree on all finite-order characters. Implicit in this statement is that (25) holds for any choice of height pairing $\langle\ ,\ \rangle_{J_0(N),H_s}$ as in (9) (recall that the definition of (9) depends on the possibly noncanonical choice of the local symbol $\langle\ ,\ \rangle_{J_0(N)_v,\rho_{H_{s,v}}}$ of Proposition 3.2.1 for each place v above p, and that there is a unique choice of local symbol $\langle\ ,\ \rangle_{E_v,\rho_{H_{s,v}}}$ at every place v). Fix a prime v of H_s and define a $\mathbb{Q}_p$-valued symbol $\langle c, d\rangle$ on pairs of degree zero divisors on $E_v = E \times_{\mathbb{Q}} H_{s,v}$ with disjoint support (and d rational over $H_{s,v}$ point-by-point) by

$$\langle c, d\rangle = \frac{1}{n}\langle \phi^* c, \delta\rangle_{J_0(N)_v,\rho_{H_{s,v}}}$$

where δ is a zero cycle on $J_0(N)_v$ such that $n \cdot d = \phi_*\delta$ for some n (using the fact that $\phi_* : J_0(N)(H_{s,v}) \to E(H_{s,v})$ has finite cokernel). It can be shown that the symbol $\langle\ ,\ \rangle$ satisfies the properties of Proposition 3.2.1, and so must be the *unique* symbol $\langle\ ,\ \rangle_{E_v,\rho_{H_{s,v}}}$. From this one easily deduces the compatibility of the global symbols (9)

$$\langle c, \phi_* d\rangle_{E,H_s} = \langle \phi^* c, d\rangle_{J_0(N),H_s}$$

for $c \in E(H_s)$ and $d \in J_0(N)(H_s)$. The equality (25) is then obvious from the definition of y_s and $y_s^\vee$. □

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