8

Representing CPT

Précis. *There are many representations of time reversal symmetry, including PT, CT, and CPT, but only the standard time reversal operator T is associated with an arrow of time itself.*

In an excited phone call, John Wheeler is reported to have told his then-graduate student Richard Feynman at Princeton that he had discovered why all electrons have the same charge and mass: "Because they are all the same electron!"[1] Wheeler proceeded to suggest that a single electron worldline might be wriggling forwards and backwards in time in a great knot, and that when moving backwards we would experience it like its antiparticle, a positron. From the perspective of a future-directed observer like ourselves, the backwards-turning points would appear as electron–positron annihilation; the forwards-turning points would appear as electron–positron creation; and, the appearance of distinct particles at a given instant would be explained by the spacelike surface that is the instant cutting through the knot, as in Figure 8.1.

Feynman reported his response:

> I did not take the idea that all the electrons were the same one from him as seriously as I took the observation that positrons could simply be represented as electrons going from the future to the past in a back section of their world lines. That, I stole! (Feynman 1972, p.163)

[1] As reported by Feynman (1972, p.163). Wheeler appears to have been inspired by Stueckelberg (1942), who had earlier arrived at a similar perspective.

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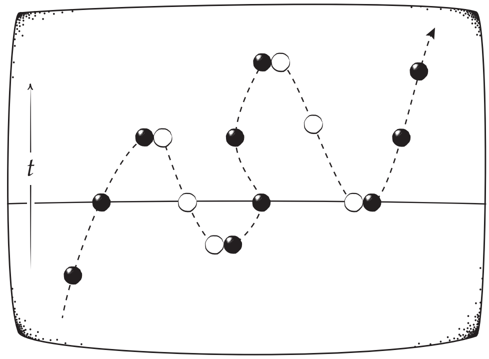

Figure 8.1 Wheeler's knot. On the horizontal spacelike surface there is an electron, a positron, another electron, and an electron–positron creation event.

The idea was fruitful for Feynman: understanding time reversal as including matter–antimatter exchange led him to the absorber theory of Wheeler and Feynman (1945) as well as to the Feynman (1949) theory of positrons.

By inspecting Figure 8.1, one can immediately see that on Feynman's view, inverting the direction of time automatically converts each instantaneous electron state into a positron, and vice versa. Of course, one would like to make this precise in the language of quantum field theory, and many have done so.[2] The proposal has also been defended by philosophers Greaves (2008, 2010) and Arntzenius and Greaves (2009). Writing 'C' to refer to matter–antimatter exchange and 'T' to refer to time reversal, they write,

> [T]he operation that ought to be called 'time reversal' – in the sense that it bears the right relation to spatiotemporal structure to deserve that name – is the operation that is usually called TC. (Arntzenius and Greaves 2009, p.584)

Unfortunately, as we have seen in Section 7.3, CT violation in electroweak theory means that there is no representation of CT symmetry in the Standard Model. However, there is a representation of CPT symmetry, which is to say, symmetry under the combination of three transformations: matter–antimatter exchange C, time reversal T, and parity P. This led to another piece of physics lore, that the operation that ought to be called 'time reversal' is the CPT operator, as expressed by Wallace:[3]

[2] Early versions appear in Stueckelberg (1942) and Watanabe (1951), and in the original Lüders (1954, p.4) construction of a CPT operator, who refers to CT as time reversal "of the second kind". This phrase is dropped from most later formulations of the CPT theorem, except for that of Bell (1955).
[3] D. Wallace (2011). "The logic of the past hypothesis", Unpublished manuscript, http://philsci-archive.pitt.edu/8894/

[I]n quantum field theory, it is the transformation called CPT, and not the one usually called T, that deserves the name. (Wallace 2011, p.4)[3]

How are we to evaluate these proposals, that time reversal is 'really' CT or CPT? Setting aside labelling conventions, the substantial proposal seems to be that CT or CPT is more appropriate for evaluating whether time has an arrow than T is. The thesis of this chapter is that they are not: the operator that ought to be called 'time reversal', in the sense that it is relevant to the question of whether time itself has an asymmetry, is the standard time reversal operator T.

There are two parts to my argument. In the first, I will draw on the arguments of Chapter 2 to make the case that time reversal must behave appropriately with respect to translations in time and space, in order to deserve the name. Using this behaviour to characterise what it means to reverse 'time alone', I argue in Section 8.1 that only the standard time reversal T is suitable for this purpose. In Section 8.2, I consider the defence of Feynman's proposal due to Greaves (2010), who argues that by following the meaning of classical time reversal through the quantisation procedure, one finds that it is represented by CT. At least when a rigorous approach to quantisation is adopted, I argue that classical time reversal is transformed into an operator T that does not exchange matter and antimatter.

The second part will consider whether there is a more systematic way to view the relationship between matter–antimatter exchange and spacetime symmetry. Both are related to a structure called the 'covering group' of the restricted Lorentz group. Thus, in Section 8.3, I consider the possibility of viewing this structure as a spacetime symmetry group and point out a sense in which it is empirically adequate as a spacetime symmetry group, although a bit eccentric. In Section 8.4, I argue that, even given this spacetime symmetry group, matter–antimatter exchange cannot be a spacetime symmetry on par with parity and time reversal, although it can be viewed in a closely related way, as a symmetry of representations. On this 'spacetime symmetry account' of matter–antimatter exchange, I find there is an interesting relationship between time reversal and matter–antimatter exchange – but still no plausible sense in which T is 'really' CT or CPT. In particular, there is no sense in which the arrow of time established by T violation is erased.

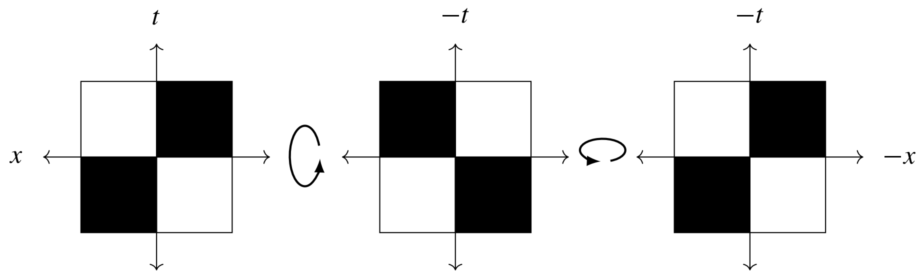

Figure 8.2 Time reversal is not a symmetry, but space-time reversal is.

8.1 Against Alternative Routes to Time Symmetry

In Chapter 7, I argued that experimental evidence of T violation from electroweak theory shows that time has an arrow. But, might there be some more liberal perspective on what 'time reversal' means, according to which time is still symmetric? Consider the 2×2 chequered grid of Figure 8.2, for which neither time reversal nor spatial reversal is a symmetry, but the combination of both is. In this section, I will argue that 'restoring' temporal symmetry in this way is, perhaps surprisingly, always possible – but that this does not undermine the fact that time is asymmetric in the sense established in Chapter 7.

Here is a toy example from Newtonian mechanics: consider a free particle, like a bead that threads a wire, which is constrained by some blocking mechanism to never move to the left, as shown in Figure 8.3. Time translations φ_t take their ordinary force-free form on the state space of positions and velocities, $\varphi_t(x,\dot{x}) = (\dot{x}t + x, \dot{x})$, but with the constraint on state space that the only possible states $(x,\dot{x})$ are those that satisfy $\dot{x} \geq 0$. This system is both T violating and P violating, because each would transform a rightward-moving trajectory to an impossible leftward-moving one. In fact, the transformations representing time reversal $T(x,\dot{x}) := (x,-\dot{x})$ and parity $P(x,\dot{x}) := (-x,-\dot{x})$ cannot even be defined on this state space, which includes only states with positive velocities. Nevertheless, one can still represent PT symmetry: if we both turn the string around *and* apply time reversal, by applying $\tilde{T}(x,\dot{x}) := (-x,\dot{x})$, then we do get a dynamical symmetry.[4]

[4] To confirm this, observe that for all $(x,\dot{x})$ in the state space, we have, $\tilde{T}\varphi_t\tilde{T}^{-1}(x,\dot{x}) = \tilde{T}\varphi_t(-x,\dot{x}) = \tilde{T}(-x+\dot{x}t,\dot{x}) = (x-\dot{x}t,\dot{x}) = \varphi_{-t}(x,\dot{x})$. Note that this 'PT' transformation, $\tilde{T}$, is *not* the composition of two transformations P and T, neither of which exist. So, it is better notation to introduce a new symbol like $\tilde{T}$. The same turns out to hold of the CPT transformation, as Swanson (2019, p.107) has emphasised.

Figure 8.3 Time translations for the asymmetric bead–string system.

Of the two classes of dynamical symmetry described in Section 4.1.3, $S\varphi_t S^{-1} = \varphi_t$ and $S\varphi_t S^{-1} = \varphi_{-t}$, this PT transformation $\tilde{T}$ is an example of the latter: it is a 'time reversing' symmetry,

$$\tilde{T}\varphi_t\tilde{T}^{-1} = \varphi_{-t}. \tag{8.1}$$

Since $\tilde{T}$ reverses all the curves in space, in the sense that $x(t) \mapsto x(-t)$, one might be tempted to call it 'time reversal', especially given the absence of any other candidate. In fact, we have considered this possibility before in Section 3.2.1, where I associated this transformation with a 'folk' view of Newtonian time reversal.

One can make this concern more powerful from the perspective of the Representation View of Section 2.3. On this view, a spacetime symmetry gets its meaning on state space through the presence of a representation. As a spacetime symmetry, time reversal τ satisfies $\tau t \tau^{-1} = -t$ for all time translations t, as I argued in Chapter 2. So, since a representation φ of that group is just a homomorphism, Eq. (8.1) says that $\tilde{T}$ provides a representation of time translation reversal! Does this mean that the parity–time reversal transformation $\tilde{T}$ can be understood as referring to the 'true' time reversal operator for this system, or at least an adequate one? If the PT transformation (or some other transformation like it) always restores temporal symmetry, then one might be tempted to conclude that T violation is not enough to establish an arrow of time.

I will make the case that this argument does not work. But, before I turn to that, let me point out a sense in which this more general kind of symmetry is really ubiquitous in any quantum theory, even without appeal to the CPT theorem. If we are willing to do more than reverse 'time alone', then there are always many different ways to obtain a dynamical symmetry that reverses time. Here is the statement of that fact, followed by its interpretation.

Proposition 8.1 *Let $t \mapsto U_t$ be a strongly continuous unitary representation of $(\mathbb{R}, +)$ on a separable Hilbert space $\mathcal{H}$. Then this representation always extends to a representation of time reversal symmetry, in the sense that there exists some antiunitary $\mathcal{T}$ such that*

$$\mathcal{T}U_t\mathcal{T}^{-1} = U_{-t}. \tag{8.2}$$

Moreover, for every unitary U' such that $[U_t, U'] = 0$, the operator $\mathcal{T}' := U'\mathcal{T}$ is an antiunitary representation of time reversal symmetry in this sense.

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Proof Our first statement is a corollary of Proposition 3.4. The fact that $\mathcal{T}' := U'\mathcal{T}$ satisfies Eq. (8.2) when $[U_t, U'] = 0$ is thus immediate. ■

Here is what this means. First, recall that dynamical systems in fundamental physics are generically associated with half-bounded energy.[5] For quantum systems, it is known that every representation $\mathcal{T}$ of time reversal must be *antiunitary*, which is to say that $\mathcal{T}$ is antilinear, $\mathcal{T}(a\psi + b\phi) = a^*\mathcal{T}\psi + b^*\mathcal{T}\phi$, and that $\mathcal{T}^*\mathcal{T} = I$. This result was proved in the second part of Proposition 3.4. So, if we are willing to accept a transformation as 'time reversal symmetry' whenever it represents an automorphism that reverses time translations, $\mathcal{T}U_t\mathcal{T}^{-1} = U_{-t}$, then time reversal symmetry is always assured. This is true even in the electroweak theory, which is only T violating in the sense that the 'canonical' time reversal transformation T does not provide a representation of time reversal symmetry. In this more general sense, virtually all quantum systems are temporally symmetric.

I find this result fascinating. But, I do not think it provides an argument that time is symmetric after all. Although a representation of time reversal symmetry must reverse time translations, it is also usually part of a larger spacetime symmetry group or gauge group, and this fact constrains it in other ways. For example, in both the Galilei and Lorentz groups, time reversal preserves spatial translations and also reverses velocity boosts. In this context, the meaning of time reversal is constrained by more than just the time translations: it is constrained by its also having appropriate relations to the other symmetry transformations.

For example, returning to the bead on a wire, the parity–time reversal transformation $\tilde{T}$ does not preserve spatial translations; this was shown in the discussion of Newtonian time reversal in Section 3.2.4. Nor does it reverse velocity boosts. Neither of these features are plausible properties of a transformation that reverses 'time alone'. So, once the larger group of spacetime symmetries is taken into consideration, it becomes implausible to identify the parity–time reversal transformation $\tilde{T}$ or the general transformation $\mathcal{T}$ in Proposition 3.4 with time reversal. If anything, the latter can only be viewed as establishing 'UT symmetry', where T is time reversal and U is some unitary symmetry.[6]

One can say more: if an irreducible representation of time reversal exists, then it is quite generally unique. This was pointed out in Section 3.3.3 and in particular in Proposition 3.2. For example, consider the case of the Standard

[5] For a discussion of this, see Section 3.1.3.

[6] This is always possible because both T and $\mathcal{T}$ are antiunitary: as a result, $U = \mathcal{T}T^{-1}$ is unitary, which implies that $UT = \mathcal{T}$.

Model, which has a representation of all the continuous symmetries of the Poincaré group. Then time reversal must be defined so as to have the correct relations to all those symmetries. And, in an irreducible representation of the Poincaré group, those considerations constrain time reversal so strongly as to make it unique, by Schur's lemma. So, the 'canonical' time reversal operator T is not arbitrary, despite the many available operators given by Proposition 8.1: it is the only representation of time reversal available that behaves appropriately with respect to the entire group of spacetime symmetries. So, the fact that canonical time reversal symmetry is violated in electroweak theory means that no other good options are available to represent temporal symmetry.

Matter–antimatter exchange, or charge conjugation, is no exception to this: in particular, if it is not a spacetime symmetry, then it is not relevant to time asymmetry as I have defined it. That said, some have argued that matter–antimatter exchange actually *can* be viewed as a spacetime symmetry, with the operator CT behaving the way time reversal should behave, even with respect to the other symmetries. In the next section I will address one such argument, which appeals to the nature of quantisation. In Sections 8.3 and 8.4, I will then propose a different way to try to connect time reversal and charge conjugation, using the universal covering of the Poincaré group. In both cases, I will argue that the standard time reversal operator T prevails and that the argument for time asymmetry developed in Chapter 7 still stands.

8.2 On Feynman's View

One precise way to interpret Feynman's proposal is as the claim that CT is the 'correct' time reversal operator. This way of thinking appears to have been smuggled into his famous diagrams, which represent contributions to a scattering amplitude in a perturbative approximation. If the CT transformation exists, then reflecting a Feynman diagram about the vertical 'time' axis produces a CT transformation, in the sense that it both reverses time and exchanges matter and antimatter, as in Figure 8.4.[7]

Even if we follow Feynman and refer to CT as 'time reversal', this does not erase the arrow of time: electroweak theory is CT violating as well, as a consequence of Wu's parity-violating experiment. One can raise further doubts as well: what reason is there to think that Feynman diagrams capture

[7] This was pointed out by Ramakrishnan (1967) and by Feynman (1985, Chapter 3) himself.

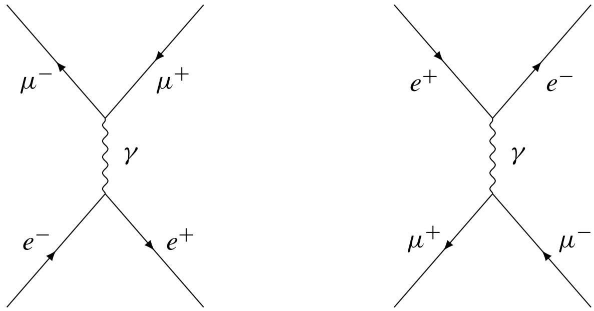

Figure 8.4 A Feynman diagram for an electron–positron decay (left), when vertically reflected, produces a description that both reverses time and exchanges matter and antimatter states (right).

the 'true' nature of time reversal? The fact that reversing a Feynman diagram is similar to an application of CT might be viewed as an artefect of the diagrams which does not adequately capture the symmetries of time itself. In my view, that is exactly what it is. However, let me first review an important argument to the contrary.

8.2.1 The Greaves Quantisation Argument

Arntzenius and Greaves have argued that the Feynman view is needed to explain two puzzles about the CPT theorem.[8] All known relativistic quantum field theories are invariant under CPT, and the 'CPT theorem' is a collection of results that explains this: it shows that a large class of relativistic quantum theories are CPT invariant.[9] However, there is still some room to be puzzled about these results. Greaves (2008, 2010) has put it in the following way.

> [O]ne can identify two positive sources of puzzlement:
>
> - How can it come about that one symmetry (e.g., Lorentz invariance) entails another (e.g., CPT) at all?
> - How can there be such an intimate relationship between spatiotemporal symmetries (Lorentz invariance, parity reversal, time reversal) on the one hand and charge conjugation, not obviously a spatiotemporal notion at all, on the other? (Greaves 2010, p.28)

[8] See Greaves (2008, 2010), Arntzenius and Greaves (2009), and Arntzenius (2011, 2012).
[9] See Bain (2016); Swanson (2018, 2019) for a summary and philosophical perspectives on these results.

Her solution to the second puzzle, echoed by Arntzenius (2011, 2012), is that we must adopt the Feynman view: the transformation normally called CT is what deserves the name 'time reversal'. Thus, both CT and CPT count as 'spatiotemporal', solving the second puzzle. As for the first puzzle, on how continuous relativistic symmetries can give rise to further discrete symmetries, Greaves argues that a "classical PT theorem" shows how this is forced by the geometry of relativistic spacetime.

This section will be concerned with Greaves' solution to the second puzzle.[10] Why is it more 'natural' to say that time reversal is really CT? Greaves (2010, §4) sketches an answer, which draws on her view of quantum field theory as the "quantisation" of a classical field theory. She begins with a classical field theory, making the standard choice for the classical time reversal operator. Standard textbook practice then converts the classical theory into a quantum theory, and Greaves draws on Bell (1955) to determine how classical time reversal is transformed by this procedure. According to Greaves, and perhaps also to Bell, quantisation automatically converts classical time reversal into what is normally called the CT transformation.[11] In this sense, she proposes, CT in quantum field theory is naturally interpreted as time reversal:

> [S]tart from a classical field theory, with assumptions about which *classical* transformations deserve the names 'time reversal' and 'parity reversal' already in place (never mind whence!); obtain a QFT by quantization; work out which transformations on QFT states and operators are induced by the already-named transformations on classical fields, and name the former accordingly. . . . when one carries out this *latter* project, with standard names for the classical transformations, *the transformation that is usually called 'TC' receives the name 'T'.* (Greaves 2010, p.39)

If this is right, then the quantisation of classical time reversal is CT, and so the quantisation of classical PT is CPT. That would be quite surprising: time reversal in classical and quantum theory are generally quite similar, in that both can be viewed as representations of time translation reversal.[12] In contrast, Greaves proposes that they are quite different: time reversal in classical theory becomes CT in quantum theory. In the remainder of this

[10] Her solution to the first draws on an argument of Bell (1955), which was generalised by Greaves and Thomas (2014) but which has some difficulties. For example, as Swanson (2019, p.120) points out, her PT theorem is too restrictive for the Standard Model, since it holds only for polynomial interactions of tensor fields and thus does not apply to non-polynomial interactions or to those involving spinorial tensors.

[11] Bell summarises the result of quantisation: "Thus the kind of reversal we have been considering implies an automatic change of sign of charge. From the field point of view, such a sign change is no more surprising than the sign change of a velocity, or an angular momentum, with time reversal in particle mechanics" (Bell 1955, p.483).

[12] See Chapter 3.

section, I will argue that it does not: on a rigorous quantisation procedure, classical time reversal does *not* induce a transformation that exchanges matter and antimatter.

8.2.2 Quantisation of T Is Not CT

There are many approaches to 'quantisation', or the practice of converting a classical description into a quantum one. This practice sometimes includes 'eye balling it', and when it does, it is called *canonical quantisation*. Unfortunately, canonical quantisation suffers from a large number of impossibility results.[13] So, it will be helpful to adopt a rigorous alternative. I will argue that, when quantisation is treated in a rigorous way, classical time reversal is not converted into an operator that exchanges matter and antimatter. So, the Greaves quantisation argument does not generally stand up to scrutiny. To illustrate, I will adopt one well-known approach: the *Segal quantisation* of a classical bosonic field.[14]

Begin with a general description of a complex classical field theory, such as a free Klein–Gordon field. Its space of solutions forms a real manifold S, which we will assume for simplicity is also a vector space: this holds, for example, in the linear approximation of solutions to a field equation. The manifold has a symplectic structure, which is to say a bilinear map $\omega : S \times S \to \mathbb{R}$ that is skew-symmetric and non-degenerate, arising from the structure of the dynamics.[15] It also has a Riemannian structure, which is to say a bilinear positive symmetric map $g : S \times S \to \mathbb{R}$, arising from the structure of the state space of initial conditions. These will usually satisfy a technical condition of being *compatible*, in the sense that $\frac{1}{2}|\omega(\psi,\phi)| \leq g(\psi,\psi)^{1/2} g(\phi,\phi)^{1/2}$, which helps to guarantee that g provides an inner product. In summary, I will refer to the triple (S, ω, g) as a *classical field theory* whenever S is a linear manifold, ω is a symplectic structure, and g is a Riemannian structure that is compatible with it.

The central result of Segal quantisation is that it is always possible to convert a classical field theory of this kind into a quantum theory, called a 'one particle structure', in a way that preserves the essential structure of the

[13] The canonical quantisation procedure of Dirac (1947), whereby classical observables are 'hatted' to produce quantum observables, suffers from the impossibility results of Groenewold (1946) and of the PhD thesis of Van Hove (1951). A class of similar results is referred to as the *Groenewold–Van Hove Theorem*; for an introduction, see Gotay (2000).

[14] For an introduction, see Kay (1979), Kay and Wald (1991, Appendix A), Segal and Mackey (1963), or Wald (1994). For some alternative approaches see Landsman (1998) or Woodhouse (1991).

[15] For example, if the field theory is formulated with a symplectic manifold as its state space (see Section 3.3), then the solutions' Hamiltonian evolution preserves the symplectic form on state space, which allows one to define a symplectic structure on solutions as well.

original field theory. This structure then forms the basis for a Fock space representation of a quantum field theory. The result is summarised by Kay and Wald (1991, Proposition 3.1):

Proposition 8.2 (Segal Quantisation) *For every classical field theory* (S, ω, g) *there is a Hilbert space* $\mathcal{H}$ *and a 'Segal quantisation' map* $K : S \to \mathcal{H}$ *such that:*

1. *(adequacy) The range of* K *is dense in* $\mathcal{H}$.
2. *(Riemannian preservation)* $\mathrm{Re}\langle K\psi, K\phi\rangle = g(\psi, \phi)$ *for all* $\psi, \phi \in S$.
3. *(symplectic preservation)* $2\,\mathrm{Im}\langle K\psi, K\phi\rangle = \omega(\psi, \phi)$ *for all* $\psi, \phi \in S$.

The pair $(K, \mathcal{H})$ *is unique up to unitary equivalence, in the sense that if* $(K', \mathcal{H}')$ *satisfies (1)–(3), then there is some unitary* $U : \mathcal{H} \to \mathcal{H}'$ *such that* $UK = K'$.

The Hilbert space $\mathcal{H}$ built by Segal quantisation can be used to define a symmetric Fock space,

$$\mathcal{F} = \mathbb{C} \oplus \mathcal{H} \oplus \mathcal{S}(\mathcal{H} \otimes \mathcal{H}) \oplus \cdots , \tag{8.3}$$

where $\mathcal{S}$ is the projection onto the symmetric subspace. One can then follow a standard procedure to define creation, annihilation, and particle number operators of a bosonic quantum field system.[16] Proposition 8.2 ensures that each one-particle structure $\mathcal{H}$ is the unique quantum system capturing the essential properties of the classical bosonic field (S, g, ω): *adequacy* ensures that every state in the Hilbert space is 'reasonably close' to representing a state in the original classical field theory. And, *Riemannian preservation* and *symplectic preservation* ensure that the metrical and dynamical information are preserved, respectively. In this sense, the resulting Fock space representation is an adequate representation of a bosonic quantum field theory.

The Segal construction moreover guarantees that each one-particle structure can be written as $\mathcal{H} = \mathcal{H}' \oplus \mathcal{H}'$ in a canonical way, with the summands interpreted as 'positive frequency' and 'negative frequency' subspaces. The exchange of these subspaces $(\psi^+ \oplus \psi^-) \mapsto (\psi^- \oplus \psi^+)$ is called *matter–antimatter exchange*. So, to see whether the quantisation of classical time reversal exchanges matter and antimatter, as Greaves proposes it does, we can check whether the Segal quantisation of classical time reversal has this property. As I have suggested, the resulting quantum time reversal operator does *not* exchange matter and antimatter. However, confirming this requires introducing a few details of the Segal construction, which I will now sketch; the remaining details can be found in Kay and Wald (1991, Appendix A).

16 This Fock space construction can be shown to be essentially unique (see Baez, Segal, and Zhou 1992, Theorem 1.10). For an introduction, see Araki (1999, §3.5).

The results of Chapter 3 help to identify the key properties of a time reversal map $T_S : S \to S$ for a classical field theory (S, ω, g). In summary, time reversal must reverse the symplectic structure, $\omega(T_S\phi, T_S\psi) = -\omega(\phi, \psi)$, and it must preserve the Riemannian structure, $g(T_S\phi, T_S\psi)$ for all $\phi, \psi \in S$. The former follows from the fact that classical time reversal must be antisymplectic (Proposition 3.1). The latter can be motivated from the assumption that time reversal does not alter classical metrical structure. It also follows directly from the fact that any quantum time reversal operator that we end up with must be antiunitary (Proposition 3.4).[17]

How does Segal quantisation convert classical time reversal into a quantum time reversal operator? Given any $T_S : S \to S$, the Segal quantisation map K induces a quantum operator $T_{\mathcal{H}}$ defined by,

$$T_{\mathcal{H}}(K\phi) := K(T_S\phi), \tag{8.4}$$

for all $K\phi \in \mathcal{H}$. This definition of $T_{\mathcal{H}}$ uniquely extends to all of $\mathcal{H}$, because the range of K is dense. Since classical time reversal T_S reverses ω and preserves g, one can show that this $T_{\mathcal{H}}$ must be antiunitary operator, as we will see shortly.

To determine whether quantum time reversal exchanges matter and antimatter, one can check whether $T_{\mathcal{H}}$ exchanges the canonical positive and negative frequency subspaces in $\mathcal{H} = \mathcal{H}' \oplus \mathcal{H}'$. In the Segal construction, the quantisation map K is defined so as to transform each classical solution $\phi \in S$ to a quantum state $\psi^+ \oplus \psi^- \in \mathcal{H}$ of the form

$$K(\phi) = \psi^+ \oplus \psi^- := E\phi \oplus FC\phi, \tag{8.5}$$

where E and F are positive operators, and C is an antilinear map defined on $\mathcal{H}'$, which is a 'complexified copy' of S. Omitting some details, here is the main fact about these operators that we will need: whenever T_S preserves g and reverses ω, it commutes with all three of these operators, E, F, and C. This immediately implies our main conclusion, that for any quantum state $\psi^+ \oplus \psi^- = K\phi$,

$$\begin{aligned} T_{\mathcal{H}}(\psi^+ \oplus \psi^-) &:= K(T_S\phi) = E(T_S\phi) \oplus FC(T_S\phi) \\ &= T_S(E\phi) \oplus T_S(FC\phi) = (T_{\mathcal{H}}\psi^+) \oplus (T_{\mathcal{H}}\psi^-). \end{aligned} \tag{8.6}$$

Since the positive and negative frequency states are not exchanged, the quantised time reversal operator is not associated with matter–antimatter exchange.

[17] Antiunitarity implies that $\langle T_{\mathcal{H}}\psi, T_{\mathcal{H}}\psi'\rangle = \langle \psi, \psi'\rangle^*$. On the Segal construction of the inner product in Eq. (8.7), this is only possible if g is preserved.

Here are a few more details, which might help to avoid one possible confusion. Segal quantisation begins by constructing a 'complex structure' $J : S \to S$ on the classical solution space, which is a linear map $J : S \to S$ satisfying $J^2 = -I$ and which turns g into a 'Kähler form': $\frac{1}{2}\omega(\psi,\phi) = g(\psi, J\phi)$ and $\omega(J\psi, J\phi) = \omega(\psi,\phi)$ for all $\phi,\psi \in S$. One then converts S into the 'positive frequency' Hilbert space $\mathcal{H}'$, by defining $i\psi := -J\psi$ and adopting the inner product,

$$\langle\psi,\phi\rangle := g(\psi,\phi) + \tfrac{i}{2}\omega(\psi,\phi). \tag{8.7}$$

Classical time reversal preserves g and reverses ω, which guarantees it is antiunitary with respect to this inner product,

$$\begin{aligned}\langle T_S\psi, T_S\phi\rangle &= g(T_S\psi, T_S\phi) + \tfrac{i}{2}\omega(T_S\psi, T_S\phi)\\ &= g(\psi,\phi) - \tfrac{i}{2}\omega(\psi,\phi) = \langle\psi,\phi\rangle^*.\end{aligned} \tag{8.8}$$

Using these facts, a short calculation shows[18] that any such T_S must also reverse the complex structure, in that $T_S J = -J T_S$. Since this J intuitively captures 'multiplication by i' in the classical field theory, the fact that $T_S J T_S^{-1} = -J$ does capture a sense in which T_S 'conjugates' classical fields.

But, this kind of conjugation is not what captures matter–antimatter exchange in quantum theory and should not be conflated with the conjugation operator C appearing in the Segal quantisation map of Eq. (8.5). The latter is relevant to the exchange of positive and negative frequency subspaces of the quantised field theory, whereas the former is not. On the contrary, the fact that T_S reverses J helps to establish that T_S commutes with E, F, and C and thus that it does not exchange these subspaces.[19]

Classical time reversal is not converted into a CT transformation by quantisation. However, one might still wonder about Greaves' two puzzles. How can assuming continuous symmetries give rise to more symmetries? And, wouldn't CPT symmetry be a lot easier to explain if charge conjugation were somehow associated with spacetime?

The first puzzle is in fact not so surprising considering the account of discrete symmetries that I have given in Chapter 2. By viewing these symmetries as automorphisms of the continuous symmetries, they arise naturally as further 'higher-order' symmetries, through the semidirect product

[18] Irreducibility considerations generally guarantee that $TJT^{-1} = \pm J$ (cf. Wallace 2009, p.218). Moreover, our compatibility assumption for the case that $\phi = \psi$ implies that $g(\phi,\phi) \geq 0$ and hence that $-g(\phi,\phi) = \frac{1}{2}\omega(\phi, J\phi) \leq 0$ for all $\phi \in S$. But this makes $TJT^{-1} = J$ impossible, for then we would have $0 \geq \omega(\phi, J\phi) = -\omega(T\phi, TJ\phi) = -\omega(T\phi, JT\phi) \geq 0$ and hence that $\omega(\phi, J\phi) = g(\phi,\phi) = 0$ for all $\phi, \in S$.

[19] Similar remarks have been made by Baker and Halvorson (2010); Wallace (2009).

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construction that I have given in Section 2.6. I will discuss this in more detail in Section 8.4. What is more puzzling is how the structure of a given state space can possibly prevent them from being symmetries. My perspective on how this can happen is given in the discussion of T violation in Chapter 7.

On the other hand, I find the second question to be tantalising. The ubiquitous symmetry of relativistic quantum field theory under CPT certainly would be a lot less puzzling if C could be viewed as a spacetime symmetry. Quantisation theory is unlikely to make the connection, as we have seen above. But, might there be another way? As Swanson (2019) has pointed out, such a link does exist in algebraic formulations of charge conjugation in quantum field theory. In the next two sections, I will rather consider a somewhat non-standard proposal about how this link can be made. In particular, there is a structure called the 'covering group' of the restricted Poincaré group, which makes a central appearance in the foundations of both spacetime symmetries and matter–antimatter exchange. Section 8.3 proposes a sense in which the spacetime symmetries might be 'enlarged' in a way that goes beyond the Poincaré group, while still remaining empirically adequate. Section 8.4 then considers a possible account of matter–antimatter exchange that makes use of this structure. Even based on this unusual account, I find that time reversal is still just T.

8.3 Local Symmetries beyond the Poincaré Group

A 'local' spacetime symmetry is one defined on scales for which gravitation and Planck-scale phenomena can be ignored. These are the scales on which relativistic quantum field theory is formulated. What are the local spacetime symmetries? The success of special relativity provides strong evidence that they consist of at least three things:[20]

1. *Lorentz boosts* describing a change of inertial reference frame;
2. *spatial rotations* describing rigid rotations of the spatial surfaces orthogonal to a timelike line; and
3. *the spacetime translations*, describing rigid translations along a timelike line.

The first two categories can be collected to form a Lie group called the *restricted Lorentz group* $L_+^\uparrow$, written with '$\uparrow$' and '$+$' to indicate that they do not reverse temporal or total orientation, respectively. The inclusion of

[20] For an introduction, see Landsman (1998, esp. §2.2 and Part IV) or Varadarajan (2007, §IX.2).

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translations through a semidirect product[21] produces the *restricted Poincaré group* $\mathcal{P}_+^\uparrow = \mathbb{R}^4 \rtimes L^\uparrow$. They are all isometries of Minkowski spacetime, sometimes referred to as the 'continuous symmetries', since as elements of a Lie group they are all continuously connected to the identity. The 'discrete' symmetries are isometries too, consisting of a continuous symmetry composed with either time reversal, parity, or both. Including these produces what is called the *complete Poincaré group* $\mathcal{P}$.

Of course, the discovery of P, T, and PT violation in electroweak theory indicates that these are not actually spacetime symmetries, as I have argued in Chapter 7. So, the structure of spacetime is not *quite* Minkowski. Instead, common wisdom has it that the local spacetime symmetries are described by the restricted Poincaré group $\mathcal{P}_+^\uparrow$.

Let me take a little inspiration from the proposal of Greaves (2010) discussed above. Is it possible that the spacetime symmetries might include something like matter–antimatter exchange as well? Perhaps common wisdom is not the whole story – the local spacetime symmetries might not be correctly described by the restricted Poincaré group! – if there is an alternative structure that could do the same job and offer a little more.

In this section, I will consider a proposal of this kind, which identifies the spacetime symmetries with the 'universal covering' of the restricted Poincaré group. The elements of the universal covering can all be interpreted as falling into one of the three categories above: boosts, rotations, and translations. But, this group is only locally (and not globally) isomorphic to the restricted Poincaré group. To some extent, this is an example of what philosophers call, 'underdetermination of theory by evidence': two different spacetime symmetry groups are compatible with existing evidence. However, in Section 8.4, I will illustrate a sense in which the universal covering might offer some advantages, through an account of matter–antimatter exchange as a spacetime symmetry.

I will begin in the next section with an example to motivate how the spacetime symmetries might be 'extended' in this way. I will then review the universal covering of the restricted Lorentz group in Section 8.3.2, before suggesting how one might construct a notion of matter–antimatter exchange in Section 8.4. The result might be viewed as a 'spacetime account' of matter–antimatter exchange, or at least the beginning of one. However, even based on this account, I can find no sense in which time reversal requires matter–antimatter exchange.

[21] Semidirect products were introduced in Section 2.6. The group multiplication rule for two elements $(x, \Lambda), (x', \Lambda') \in \mathbb{R}^4 \rtimes L^\uparrow$ is given by $(x, \Lambda)(x', \Lambda') = (x + \Lambda x', \Lambda\Lambda')$.

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8.3.1 What Is the Group of Spatial Rotations?

Let me begin with an example that is easy to visualise. The continuous symmetries of Euclidean space at a point can be collected together to form a Lie group $SO(3)$ of rigid Euclidean rotations. Those symmetries were given up following the discovery of general relativity. But, in this discussion, let me focus on local regimes in which gravity can be neglected and where one still might wish to postulate that the symmetry group of space at a point is $SO(3)$.

There is still another option. The elements of a different group, $SU(2)$, can also be interpreted as symmetries of space. We have seen this group already in our discussion of spin in Section 3.4.4: its elements are the 'rotations' given by

$$R_x(\theta) = e^{(i/2)\theta\sigma_x}, \qquad R_y(\theta) = e^{(i/2)\theta\sigma_y}, \qquad R_z(\theta) = e^{(i/2)\theta\sigma_z}, \tag{8.9}$$

where each σ_j for $j = x, y, z$ is a Pauli spin observable. As we have seen, this group has the interesting property that a 'rotation' through 2π does not return a system to where it started. Instead, $R(2\pi) = -I$, and a second rotation through 2π is needed to restore the identity, $R(4\pi) = I$. Nevertheless, each element of $SU(2)$ can be viewed as a spatial rotation, in the sense that there is a neighbourhood of each element that is isomorphic[22] to a neighbourhood of the ordinary rotation group $SO(3)$. The groups $SU(2)$ and $SO(3)$ have the same local structure but differ in their global description of how the rotations fit together.

The structure of $SU(2)$ obviously does not capture our experience of spatial symmetry on the scale of everyday objects: by rotating a ball through 2π, it appears to go back to where it started.[23] However, the correct spatial symmetry group might still be $SU(2)$, if its strange behaviour is invisible on the scales that are currently accessible to us. Recall the analogy of an arrow on a Möbius strip: when an arrow is transported orthogonally along a loop, a rotation through 4π is needed to restore it to its original state. However, if we imagine that the arrow is scaled to be very small, as in Figure 8.5, or if it is only detectable in the presence of certain kinds of matter, then it might appear that a rotation through 2π is enough.

[22] Alternatively: these Lie groups have isomorphic Lie algebras at every point; this property will be defined more generally in Section 8.3.2 in the definition of a covering group.

[23] You *can* still experience $SU(2)$ with relative ease, not as a symmetry of space, but through the following procedure: hold your palm upright in front of you and notice that by raising your elbow, it is possible to rotate your hand through 360 degrees, all while keeping your palm upright. Your arm will now be quite twisted; but, by continuing to rotate in the same direction through another 360 degrees, always palm up, you can then untwist your arm into its original state!

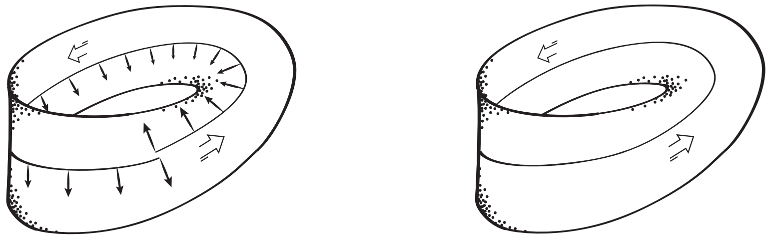

Figure 8.5 The global structure of $SU(2)$ might be hidden from us for scales on which the 'arrows' are small enough to appear invisible.

This is an ordinary case of underdetermination of theory by evidence: either the spatial symmetry group is $SO(3)$, or else it is secretly $SU(2)$ but in a way that gives the illusion of being $SO(3)$ on familiar scales. Of course, we do not need to take $SU(2)$ too seriously in the absence of evidence for it. In contrast, if we had evidence that $SU(2)$ were a more effective model of some phenomenon, in a way that explains the illusion of rotations on ordinary scales, then we might reject $SO(3)$ in favour of $SU(2)$. I do not know of any compelling evidence for this in the case of rotations, at least when considered in isolation. However, I will point out a sense in which this kind of expansion of the Poincaré group does have modelling advantages, in that it might allow one to characterise the matter–antimatter relationship as a spacetime symmetry.

Before I turn to this idea, let me head off a possible source of confusion: the structure of the rotation group for a spin-1/2 system does not provide any evidence for $SU(2)$ over $SO(3)$, as far as I am aware. It is true that the Hilbert space description of a spin-1/2 particle admits an irreducible unitary representation of $SU(2)$, and not of $SO(3)$. However, this does not imply that descriptions related by a rotation of $R(2\pi) = -I$ describe factually different states of affairs.

On the contrary, the statistical predictions of quantum theory are the same for all states ψ on the same *ray*, or set of vectors related by a phase factor. The 'true' state space of quantum theory is thus a ray space; correspondingly, its symmetries are only defined up to a phase factor as well, as discussed in Section 3.4.2. From this perspective, the transformations $R(2\pi) = -I$ and $R(0) = I$ refer to the very same rotation, as Hegerfeldt and Kraus (1968) have pointed out.[24] Indeed, Bargmann (1954) famously showed that the ray space of a spin-1/2 system does admit a representation of $SO(3)$.

[24] Their article, written in response to Aharonov and Susskind (1967), is in fact the once-rejected paper that led to their collaboration with Wigner, as reported in Footnote 18.

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My point here is rather that we still face a certain amount of underdetermination: whenever we have a representation of $SO(3)$, we always formally have a degenerate representation of $SU(2)$. And, when we turn to the case of the Poincaré group, I will argue that a very similar structure may provide a more effective model of reality, by allowing us to characterise the matter–antimatter relationship.

8.3.2 The Covering Group of Spacetime Symmetries

Let me describe an analogous construction to the one above for the case of the Poincaré group. It will be helpful to first identify the nature of this construction in a more general way.[25]

> **Definition 8.1** A *covering group* or *cover* over a Lie group G is a Lie group $\hat{G}$, together with a continuous homomorphism or 'covering map' from $\hat{G}$ onto G such that the induced map on Lie algebras is an isomorphism. The *universal covering group* is the unique, simply connected covering group over G.

A cover over a Lie group has the same local structure but may not be isomorphic. The 'universal' covering gets its name from the fact that it is a cover for all other covers. For example, $SU(2)$ is a cover for $SO(3)$, and indeed it is simply connected, and so it is the universal covering group over $SO(3)$.

The universal covering group for the restricted Lorentz group $L_+^\uparrow$ is called $SL(2, \mathbb{C})$. It was famously applied by Bargmann (1954) as part of a more general technique for finding unitary representations of Lie groups using their universal coverings. Like the case of $SU(2)$, the group $SL(2, \mathbb{C})$ is a doubly-degenerate cover over $L_+^\uparrow$, and the choice between $L_+^\uparrow$ and $SL(2, \mathbb{C})$ is underdetermined in the same sense that we have discussed in Section 8.3.1. Locally, it behaves just like the restricted Lorentz group. But, it might be that its global structure is somehow hidden from us on ordinary scales. Indeed, $SU(2)$ is a subgroup of $SL(2, \mathbb{C})$, and so this is in fact a more general instance of the problem discussed in the previous section. To understand the symmetries of $SL(2, \mathbb{C})$ and its relationship to Minkowski spacetime, let me introduce a few more details of a classic construction.[26]

The name $SL(2, \mathbb{C})$ stands for the 'special linear group' of 2×2 complex-valued matrices with unit determinant. To compare it to Minkowski

[25] See Hochschild (1965, Chapters IV and XII) for an introduction to covering groups.
[26] Classic presentations include Gel'fand, Minlos, and Shapiro (1958, Part II Chapter 1 §1.8) and Naimark (1964, §3.9).

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spacetime, we first give a clever expression of the latter using the set M of 2×2 matrices that are self-adjoint, $A = A^*$. Note that I write A^* to refer to the conjugate-transpose or *adjoint* of A; I will later write $\bar{A}$ to refer to conjugation of each matrix element and $A^\top$ to refer to the matrix transpose.

The set M of self-adjoint matrices is closed under matrix addition and multiplication by real scalars. It thus forms a real vector space of four dimensions, with an explicit basis set given by the Pauli matrices:

$$\sigma_0 = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \quad \sigma_1 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} & -i \\ i & \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}. \tag{8.10}$$

We can identify a metric on this space by defining a symmetric bilinear form, $\eta(\sigma_i, \sigma_j) := g_{ij}$, where $g_{ij} = \mathrm{diag}(1, -1, -1, -1)$ is the Minkwoski metric. Then the pair (M, η) is isometric to Minkowski spacetime! In particular, since it is a linear space, we can interpret it as the tangent space of Minkowski spacetime at a point. Thus, each element $v \in M$ can be interpreted as a vector at a point in the underlying spacetime manifold, corresponding to an 'instantaneous' translation along spacelike, timelike, or null curves.

It is illuminating to write these elements explicitly in terms of our basis elements and some real numbers $u = (t, x_1, x_2, x_3)$, which can be viewed as representing translations in spacetime. Then the general form of a vector in Minkowski spacetime at a point is

$$v = u \cdot \sigma = \begin{pmatrix} t + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & t - x_3 \end{pmatrix}. \tag{8.11}$$

Writing the Minkowski norm as $|v|^2 := \eta_{\mu\nu} v^\mu v^\nu$ in the Einstein summation convention can now express it in a particularly simple way:

$$\begin{aligned} |v|^2 &:= \eta_{\mu\nu} v^\mu v^\nu = t^2 - x_1^2 - x_2^2 - x_3^2 \\ &= (t + x_3)(x_0 - x_3) - (x_1 - ix_2)(x_1 + ix_2) = \det(v), \end{aligned} \tag{8.12}$$

where the determinant is associated with the matrix of Eq. (8.11). The complete Lorentz group L consists of the maps $\Lambda : M \to M$ that preserve the Minkowski norm, $\eta(\Lambda v, \Lambda v) = \eta(v, v)$. By Eq. (8.12), this is equivalent to $\det(\Lambda v) = \det(v)$. The *restricted Lorentz group* $L_+^\uparrow$ then consists of those elements that are continuously connected to the identity.

I have said that $SL(2, \mathbb{C})$ is a cover for $L_+^\uparrow$. The covering map $\Lambda : S \mapsto \Lambda_S$ can be explicitly defined by

$$\Lambda_S v := S v S^*, \tag{8.13}$$

for all $S \in SL(2,\mathbb{C})$ and all $v \in M$. This map is straightforwardly shown[27] to be a continuous, surjective homomorphism $\Lambda : SL(2,\mathbb{C}) \to L_+^\uparrow$ that is a doubly-degenerate covering, meaning that $\ker \Lambda = \{I, -I\}$. The latter implies the map is a local isomorphism and thus induces an isomorphism of Lie algebras, making this a covering map. Since $SL(2,\mathbb{C})$ is simply connected, it follows that this is in fact the universal covering group over $L_+^\uparrow$. Each of its elements can be written in polar decomposition as $S = Ue^A$, where U is unitary and A is self-adjoint. They are generated by operators of the form $R_j(\theta) = e^{i(\theta/2)\sigma_j}$ and $B_j(s) = e^{(s/2)\sigma_j}$ with $j = x, y, z$, where the former give rise to a subgroup of 'rotations' isomorphic to $SU(2)$, and the latter can be interpreted as Lorentz boosts.

This completes our brief review of $SL(2,\mathbb{C})$.[28] It can be extended to include spacetime translations in a straightforward way, just as they are included in the restricted Poincaré group $\mathcal{P}_+^\uparrow$, by adopting the semidirect product group $\mathbb{R}^4 \rtimes SL(2,\mathbb{C})$ with multiplication given by $(v,S)(v',S') = (v + Sv'S^*, SS')$. I will denote this group $\overline{\mathcal{P}}_+^\uparrow$, with a 'bar' covering the letter to remind one that it is the universal covering.

This development of $SL(2,\mathbb{C})$ and its relationship to Minkowski spacetime provides a well-known technique for constructing projective representations of the restricted Lorentz group $L_+^\uparrow$ on a Hilbert space. However, the interpretation I am considering here goes beyond this usage, in order to give serious consideration to the possibility that C is a spacetime symmetry.

Namely, if physical spacetime were locally described by Minkowski spacetime, then its symmetry group would be the complete Poincaré group. But, spacetime is *not* locally Minkowski, because P, T, and PT symmetry are all violated. So, my plan is now to consider the possibility that it is different in other ways too and in particular that its global structure is given by the universal covering group $\overline{\mathcal{P}}_+^\uparrow$. This is an unusual way to look at this group. But, in the next section, I will indicate how it suggests an account of matter–antimatter exchange that is closely related to a spacetime symmetry and which allows it to be effectively compared to time reversal.

[27] See Varadarajan (2007, p.334–5). To fill in some of the details: Λ_S is in the Lorentz group because it preserves the Minkowski norm, $\eta(\Lambda_S v, \Lambda_S v) = \det(\Lambda_S v) = \det(SvS^*) = \det(v) = \eta(v,v)$. It is obviously continuous, and one can check that it is a homomorphism, $\Lambda_{RS}(v) = (RS)v(RS)^* = R(SvS^*)R^* = \Lambda_R\Lambda_S(v)$. To see that it is a twofold covering, let $S \in \ker\Lambda$, so that $v = \Lambda_S v = SvS^*$ for all v. It follows by Schur's lemma that $S = cI$ for some $c \in \mathbb{C}$. We thus have that $1 = \det(S) = c^2$ and hence that $S = \pm I$, which is to say that $\ker\Lambda = \{\pm I\}$.

[28] For a more detailed introduction, see Varadarajan (2007, p.334–7), or Naimark (1964, §3.9).

8.4 C as a Spacetime Symmetry

I proposed a general strategy for determining the discrete symmetries associated with a collection of continuous spacetime symmetries in Section 2.6. In particular, I showed how time reversal arises as an automorphism of the time translations. In this section, I will argue that matter–antimatter exchange can be understood in a way that is very similar to this: not as an automorphism of spacetime symmetries, but of the space of representations of those symmetries. This brings C, P, T, and their combinations closer to being on par, so long as the symmetries of spacetime are associated with the covering group $\overline{\mathcal{P}}_{+}^{\uparrow}$.

Two caveats about this account: first, the concept of 'charge conjugation' is clearly not exhausted by any purely spacetime description. Charges in relativistic field theory are conserved quantities associated with a global gauge group, which in the Standard Model is postulated to be $SU(3) \times SU(2) \times U(1)$. I do not claim that this group is a spacetime structure. What I would like to indicate is how *one aspect* of charge conjugation, which might be called 'matter–antimatter exchange', can be viewed as intimately connected to the spacetime symmetry. I find this to be a natural way to explore the proposal of Greaves (2010). But, as I will argue, its relationship to time reversal does not provide a clean vindication of the Feynman view, and it does not erase the arrow of time associated with T violation.

8.4.1 Extending the Discrete Symmetries

Let me begin by revisiting my account of discrete symmetries. When we restrict attention to the subgroup $(\mathbb{R}, +)$ of time translations in a reference frame, we find that time reversal is the unique non-trivial automorphism, $\tau : t \mapsto -t$ (Proposition 2.1). It is not an element of the original symmetry group but rather a 'symmetry of symmetries', which I referred to as 'higher order' in Section 4.3. However, time reversal can always be 'added' into the group through the construction of the semidirect product $(\mathbb{R}, +) \rtimes \{\iota, \tau\}$. In Section 2.6, I showed how this produces a group element that reverses time translations, $\tau t \tau^{-1} = -t$.

Extending this thinking to the restricted Poincaré group $\mathcal{P}_{+}^{\uparrow}$, we find three non-trivial automorphisms: time reversal τ, together with the parity p and their combination $p\tau$. These automorphisms can be defined for each spacetime translation $u = (t,, x_1, x_2, x_3) \in \mathbb{R}^4$ in a foliation, as: $u^p := (t, -x_1, -x_2, -x_3)$ and $u^\tau := (-t, x_1, x_2, x_3)$, and hence $u^{p\tau} = -u$. They can also be defined for each Lorentz boost, as: $p(\Lambda) = \tau(\Lambda) = \pi\Lambda\pi$, where

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$\pi : (t, x_1, x_2, x_3) = (t, -x_1, -x_2, -x_3)$ is a spatial reversal. The latter arises because both parity and time reversal 'turn around' the spatial direction in which each boost occurs so that the three-velocity is reversed. Thus, the ordinary p, τ, and $p\tau$ transformations can be viewed as automorphisms of the Poincaré group. In summary:

$$\begin{aligned} p &: (u, \Lambda) \mapsto (u^p, \pi\Lambda\pi) \\ \tau &: (u, \Lambda) \mapsto (u^\tau, \pi\Lambda\pi) \\ p\tau &: (u, \Lambda) \mapsto (-u, \Lambda). \end{aligned} \tag{8.14}$$

These turn out to exhaust the non-trivial automorphisms α of $\mathcal{P}_+^\uparrow$ that are also involutions.[29] So, constructing the 'complete' Poincaré group from these automorphisms produces a group that really is complete.

Similar automorphisms can be found for the covering group $\overline{\mathcal{P}}_+^\uparrow$ as well. Let $v \in M$ be a vector in Minkowski spacetime represented as a 2×2 self-adjoint matrix, as introduced in Eq. (8.11). The covering map $\Lambda_S v = SvS^*$ defines the restricted Lorentz transformation Λ_S associated with each $S \in SL(2, \mathbb{C})$. With a little effort, one can check[30] that $S \mapsto (S^{-1})^*$ is an automorphism of $SL(2, \mathbb{C})$ and that it induces the transformation $\Lambda_S \mapsto \Lambda_{(S^{-1})^*} = \pi\Lambda_S\pi$ on the restricted Lorentz group via the covering map. As a result, $S \mapsto (S^{-1})^*$ induces the same parity and time reversal transformations, as described for the ordinary Poincaré group in Eq. (8.14).[31] We can summarise these as:

$$\begin{aligned} p &: (u, S) \longmapsto (u^p, (S^{-1})^*) \\ \tau &: (u, S) \longmapsto (u^\tau, (S^{-1})^*) \\ p\tau &: (u, S) \longmapsto (u^{pt}, S). \end{aligned} \tag{8.15}$$

Can we now introduce matter–antimatter exchange as an automorphism in a similar way? Not quite. There are exactly three non-trivial 'outer' automorphisms of $SL(2, \mathbb{C})$: the inverse-transpose $S \mapsto (S^{-1})^\top$, the complex conjugate $S \mapsto (\bar{S})$, and their combination $S \mapsto (S^{-1})^*$. There are also

[29] The spacetime translations admit further automorphisms of the form $a \mapsto \lambda a$ for $\lambda \in \mathbb{R}$; however, the argument of Proposition 2.1 ensures that if they are required to be involutions, then $\lambda = \pm q$.

[30] Conjugation $S \mapsto \bar{S}$ and the inverse-transpose $S \mapsto (S^{-1})^\top$ are both automorphisms of $SL(2, \mathbb{C})$, and thus so is their composition $S \mapsto (S^{-1})^*$. Viewing $S = UA$ in its polar decomposition, one can show that $\Lambda_{S^*} = \Lambda_S^\top$ where '$\top$' is the 2×2 matrix transpose (Varadarajan 2007, pp.335, Eq. (55)). Now, defining $\pi(t, x) = (t, -x)$, one can see by inspection of the form of v in Eq. (8.11) that $v^{-1} = \frac{1}{\det(v)}\pi v$, and that $\det(\pi v) = \det(v)$. This implies that $\Lambda_{S^{-1}} v = S^{-1} v (S^{-1})^* = (S^* v^{-1} S)^{-1} = (S^* \frac{1}{\det(v)} \pi v S)^{-1} = \det(v)(\Lambda_{S^*}\pi v)^{-1} = (\pi\Lambda_S^\top\pi)v$. Combining this with our equation $\Lambda_{S^*} = \Lambda_S^\top$ now establishes that $\Lambda_{(S^{-1})^*} = \pi\Lambda_S\pi$.

[31] This was observed already by Gel'fand, Minlos, and Shapiro (1958, p.160).

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conjugate:	$\Lambda_S \mapsto \Lambda_{\bar{S}} = \zeta \Lambda_S \zeta$
transpose:	$\Lambda_S \mapsto \Lambda_{S^\top} = \zeta \Lambda_S^\top \zeta$
inverse:	$\Lambda_S \mapsto \Lambda_{S^{-1}} = \pi \Lambda_S^\top \pi$

Figure 8.6 Automorphisms and antiautomorphisms of $SL(2, \mathbb{C})$, and their induced effect on the Lorentz group.

four antiautomorphisms: the inverse $S \mapsto S^{-1}$ and the transpose $S \mapsto S^\top$, together with the inverse-conjugate and the transpose-conjugate. One can calculate the way that each transforms elements Λ_S of the restricted Lorentz group: three of them are summarised in the table of Figure 8.6, from which the effects of the remaining one can be determined. Here, π is a total spatial reversal, while ζ is defined by $\zeta(t, x_1, x_2, x_3) = (t, x_1, -x_2, x_3)$, reversing the spatial translations on just one axis. The problem is that *all* the automorphisms of $SL(2, \mathbb{C})$ induce a non-trivial automorphism of the Lorentz transformations Λ_S. This makes none of them appropriate for matter–antimatter exchange, which would require a transformation that does not effect any element of the Lorentz group.

As a result, a little more structure is needed to express matter–antimatter exchange. This makes sense, because we clearly do not have enough structure at this abstract level to define something like 'positive frequency' and 'negative frequency' subspaces. However, that kind of structure is afforded at the level of *representations* of $SL(2, \mathbb{C})$. We will now see that, on this space of representations, matter–antimatter exchange takes the form of a symmetry that is very similar to parity and time reversal.

8.4.2 Conjugation and Matter–Antimatter Exchange

There are no finite-dimensional unitary representations of $SL(2, \mathbb{C})$, owing to the fact that it is not a compact Lie group. However, the *non*-unitary linear representations of $SL(2, \mathbb{C})$ are the foundation for the theory of spinors, which play an important role in the description of many quantum fields.[32] Weyl (1946, Theorem 8.11.B) showed that these representations of $SL(2, \mathbb{C})$

[32] Spinors appear in the classification of semi-simple Lie groups by Cartan (1913), who in the introduction to his later 1937 *Theory of Spinors* delighted in having 'discovered' spinors before Dirac. Brief modern overviews can be found in R. Geroch (1973, "Special topics in particle physics". In: Unpublished Manunscript of Fall 1973, Version of 25 May 2006, §18) and Wald (1984, Chapter 13); for a detailed introduction, see Carmeli and Malin (2000) or Penrose and Rindler (1984).

have a particular canonical form. And, it is this form that can in some contexts be interpreted as distinguishing between matter and antimatter. Let me briefly set out what that is.

Viewing $SL(2,\mathbb{C})$ as a group of 2×2 matrices, it is by definition a representation amongst the linear transformations of the vector space $V^A := \mathbb{C}^2$, whose elements are called *spinors* (or sometimes *Weyl spinors*). Note that this concept is not the same as that of a (four-component) Dirac spinor. The tensors built from the vector space V^A are called *spinorial tensors*. I will be concerned with the symmetric spinorial tensors defined on the n-fold tensor product $V^A \otimes V^A \otimes \cdots \otimes V^A$ for some n. Writing $n = 2j$ for $j = 0, \frac{1}{2}, 1, \frac{3}{2} \cdots$, let D^j denote the subspace of symmetric spinorial tensors defined on this tensor product. Following a typical presentation (cf. Varadarajan 2007, p.336), let $\varphi^j : SL(2,\mathbb{C}) \to D^j$ be the representation

$$\varphi^j : S \mapsto S \otimes S \otimes \cdots \otimes S. \tag{8.16}$$

Since spinors are defined on a complex vector space, there exists another inequivalent representation of $SL(2,\mathbb{C})$ defined by complex conjugation with respect to the first. One writes $V^{A'} = \mathbb{C}^2$ to denote the complex-conjugate vector space[33] and $D^{j'}$ to denote the subspace of symmetric tensors built from this vector space. Then the 'conjugate representation' $\varphi^{j'}$ is given by

$$S \mapsto \bar{S} \otimes \bar{S} \otimes \cdots \bar{S}. \tag{8.17}$$

Weyl's result is that every finite-dimensional irreducible representation of $SL(2,\mathbb{C})$ is equivalent (up to an intertwining) to one of the form

$$\varphi^{(j,j')} := \varphi^j \otimes \varphi^{j'}, \tag{8.18}$$

for some $j, j' = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$. In the description of a quantum field system, one may wish to interpret the first factor as a positive-frequency description, corresponding to an ordinary matter field, while the other is interpreted as a negative-frequency description, corresponding to an antimatter field. Given such an interpretation, the exchange of these factors can be understood as matter–antimatter exchange. Indeed, in the first rigorous statement of a CPT theorem due to Jost (1965), the 'charge conjugation' aspect of CPT is characterised by conjugation of field operators that exchanges these 'undotted' and 'dotted' spinor subspaces.[34]

[33] Writing the scalar product in V^A as av, where $a \in \mathbb{C}$ and $v \in V^A$, we define $V^{A'}$ to consist of the same set of vectors as V^A but with a scalar product $a \cdot v$ defined by $a \cdot v := a^* v$.

[34] See Haag (1996, §II.5.1) for a modern (and English-language) overview of this theorem.

Of course, to make this connection concretely, a further unitary representation of this structure is needed on a Hilbert space of infinite dimension. This can be carried out by appeal to the representation theory of spinors, which is found in many places, and so I will not develop it here.[35]

What I would like to point out is that these representations of matter and antimatter themselves admit a symmetry, viewed separately from the symmetries of $SL(2, \mathbb{C})$, which exchanges a given representation with its conjugate. Namely, we define the map c on the space of irreducible representations by the definition

$$c\left(\varphi^{(j,j')}\right) := \varphi^{(j',j)}. \tag{8.19}$$

In terms of the canonical direct tensor product $\varphi^j \otimes \varphi^{j'}$, this transformation exchanges the two components. Thus, interpreting one as a 'positive frequency' subspace and the other as a 'negative frequency' subspace, we arrive at a definition of matter–antimatter exchange.

This transformation C is fundamentally different from P and T, which are automorphisms of the spacetime symmetry group $\overline{P}_+^\uparrow$. In contrast, C can only be defined as a symmetry of its representations. However, the definitions of φ^j and $\varphi^{j'}$ allow us to define time reversal and parity as transformations of these representations as well, allowing one to compare all three. Both time reversal and parity take the form $\tau(S) = p(S) = (S^{-1})^*$, since these transformations are only distinct in their transformation of spacetime translations. This induces a transformation of φ^j through Eq. (8.16), and on φ^j through Eq. (8.17), and therefore on each irreducible representation $\varphi^{(j,j')} = \varphi^j \otimes \varphi^{j'}$.

Moreover, there is a sense in which both P and T are closely connected to matter–antimatter exchange. Both of them involve the conjugation automorphism $S \mapsto \bar{S}$, albeit together with the inverse-transpose as well. Viewed as transformations from one representation $\varphi^{(j,j')}$ of $SL(2, \mathbb{C})$ to another, both are also 'equivalent' to matter–antimatter exchange, in the following sense: consider the element $i\sigma_2 \in SL(2, \mathbb{C})$, where σ_2 is a Pauli matrix. It can be shown through simple matrix multiplication to transform time reversal $\tau(S) = (S^{-1})^*$ to the conjugate automorphism,

$$(i\sigma_2)\tau(S)(i\sigma_2)^{-1} = \bar{S}. \tag{8.20}$$

[35] Comments on this representation theory can be found in Carmeli and Malin (2000, §4.3), Penrose and Rindler (1984), Varadarajan (2007, Chapter IX), or Wald (1984, 357–9).

But, the automorphism $S \mapsto (i\sigma_2)\tau(S)(i\sigma_2)^{-1} = \bar{S}$ transforms a representation in the same way that matter–antimatter exchange does, $c\left(\varphi^{(j,j')}\right) := \varphi^{(j',j)}$. As a result, parity, time reversal, and matter–antimatter exchange are all related by an intertwining map $i\sigma_2$. More precisely, the transformations p, τ, and c are related by the property that

$$(i\sigma_2) \circ p = (i\sigma_2) \circ \tau = c \circ (i\sigma_p). \tag{8.21}$$

In the language of representation theory, this means that they are 'equivalent' as symmetries of the space of representations. Although it is more common to propose that matter–antimatter exchange is relevant to time reversal, this suggests a sense in which it is relevant to parity as well. Indeed, Wigner (1957, p.258) himself considered the possibility that "the mirror image of matter is antimatter" in response to Wu's discovery of parity violation.

This provides a neat connection between C, P, and T. Let me now compare it to the remarks about time reversal considered at the beginning of this chapter. In the first place, matter–antimatter exchange is not *quite* a spacetime symmetry – not even adopting the idiosyncratic view of spacetime symmetries as given by $SL(2, \mathbb{C})$ – but rather a symmetry of its representations. So, there is really no sense in which time reversal is 'really' CT or CPT, since neither can be viewed as an automorphism of the continuous spacetime symmetries. This provides a clear sense in which CT and CPT do not behave in an appropriate way to be deserving of the name 'time reversal', following the discussion of Section 8.1. These are not spacetime symmetries but rather symmetries of a representation. As a result, a symmetry involving matter–antimatter exchange cannot 'erase' the time asymmetry established by T violation. Only the latter is relevant for describing the symmetries of 'time alone'.

However, there is still an interesting sense in which the Feynman proposal is perhaps vindicated: time reversal, parity, and matter–antimatter exchange can all be viewed as transforming 'spinor' representations to their conjugates. One might interpret this as the statement that both parity and time reversal 'automatically' include matter–antimatter exchange in a spinor representation. However, exploring this possibility in detail would require considerably more development.

8.5 Summary

T violation means that the standard time reversal operator T does not provide a representation of time translation reversal, $t \mapsto -t$. The experimental evidence for this provides strong evidence that time itself has an arrow, as I have argued in Chapter 7. In this chapter, I considered a possible concern about this conclusion: that there is always a large number of representations that 'restore' temporal symmetry, which might include transformations like PT or CPT. One might worry that these would 'erase' the arrow established by T violation and provide an appropriate representation of time reversal symmetry. But, this would be to ignore the broader context of what these transformations mean. The transformation PT reverses spatial translations and so does not deserve the name 'time reversal'. And, when we are considering an irreducible representation, the choice of time reversal operator is generally unique.

There are other arguments that might seek to establish that time reversal is 'really' CT or CPT. Quantisation theory is one; but, on careful mathematical treatments, this transforms classical time reversal to the ordinary quantum time reversal operator T. A more ambitious reinterpretation of C brings us a little closer: by moving up from the ordinary spacetime symmetries to the more exotic universal covering group, I exhibited a close relationship between time reversal and matter–antimatter exchange. However, these transformations are still defined on different spaces. In particular, there is no plausible way to define matter–antimatter exchange as a true spacetime symmetry, even for the universal covering group. Transformations like CT and CPT are certainly important in the foundations of quantum field theory. However, they should not be mistaken for transformations that determine whether time has an arrow.