CONCERNING NON-MEASURABLE SUBSETS OF A GIVEN MEASURABLE SET

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Let R, μ and M_μ denote the set of real numbers, Lebesgue outer measure and the class of Lebesgue measurable subsets of R respectively. It is easy to prove that the complement E^c of $E \in M_\mu$ is a set of Lebesgue measure zero if the inequality $\mu(E \cap I) \geqq \delta\mu(I)$ holds for some $\delta > 0$ and all intervals I of R. However, in [1], Hewitt raised a problem whether the result is still true if E is not a priori measurable set. In this paper, a negative answer to this question is given through a counter-example. Also, it is proved that for a given set $E \in M_\mu$ with $\mu(E) > 0$ there is a non-measurable subset A of E satisfying $\mu(A) = \mu(E)$.

LEMMA 1. *Let $E \in M_\mu$ with $\mu(E) < \infty$ and $A \subset E$. Then $A \in M_\mu$ if and only if $\mu(E) = \mu(A) + \mu(E - A)$.*

For the proof, the reader is referred to [2].

LEMMA 2. *If $\{E_i\}$ is a sequence of pairwise disjoint sets of M_μ each having positive measure and $\{A_i\}$ is a sequence of non-measurable sets such that $A_i \subset E_i$ for each i, then $\bigcup_{i=1}^\infty A_i$ is non-measurable and $\mu(\bigcup_{i=1}^\infty A_i) = \sum_{i=1}^\infty \mu(A_i)$.*

PROOF. The non-measurability for $\bigcup_{i=1}^\infty A_i$ is obvious. We need only prove

$$\mu(\bigcup_{i=1}^{\infty} A_i) \geqq \sum_{i=1}^{n} \mu(A_i)$$

for every n, from which $\mu(\bigcup_{i=1}^\infty A_i) \geqq \sum_{i=1}^\infty \mu(A_i)$ follows, and the conclusion is obtained in view of subadditivity of μ. By monotoneity of μ,

$$\mu(\bigcup_{i=1}^{\infty} A_i) \geqq \mu(\bigcup_{i=1}^{n} A_i)$$

for all n. We shall show that $\mu(\bigcup_{i=1}^n A_i) = \sum_{i=1}^n \mu(A_i)$ by induction. The equality is trivial for $n = 1$. Assume that it holds for $n = k$. Since $A_{k+1} \subset E_{k+1}$ and $\bigcup_{i=1}^k A_i \subset E_{k+1}^c$, measurability of E_{k+1} implies that

$$\mu(\bigcup_{i=1}^{k+1} A_i) = \mu(\bigcup_{i=1}^{k} A_i) + \mu(A_{k+1}) = \sum_{i=1}^{k+1} \mu(A_i).$$

The last equality follows by inductive hypothesis. The proof is now completed.

LEMMA 3. *If* $E \in M_\mu$ *with* $\mu(E) > 0$, *then there is a non-measurable subset* A *of* E *such that* $\mu(A) \geqq \frac{1}{2}\mu(E)$.

PROOF. The existence of a non-measurable subset Q of E is well-known.

If $0 < \mu(E) < \infty$, then by lemma 1, $\mu(E) < \mu(Q)+\mu(E-Q)$. Thus we have $\mu(Q) > \frac{1}{2}\mu(E)$ or $\mu(E-Q) > \frac{1}{2}\mu(E)$. Since $Q \notin M_\mu$, $E-Q \notin M_\mu$. The conclusion follows.

If $\mu(E) = \infty$, then by σ-finiteness of μ, there is a sequence of pairwise disjoint sets $\{E_i\}$ of M_μ such that $E = \bigcup_{i=1}^\infty E_i$ and $\mu(E_i) < \infty$ for each i (we may assume $0 < \mu(E_i) < \infty$). By what we have just shown, there is a non-measurable subset A_i of E_i for each i such that $\mu(A_i) > \frac{1}{2}\mu(E_i)$. Let $A = \bigcup_{i=1}^\infty A_i$. By lemma 2, $A \notin M_\mu$ and

$$\mu(A) = \sum_{i=1}^\infty \mu(A_i) \geqq \frac{1}{2}\sum_{i=1}^\infty \mu(E_i) = \frac{1}{2}\mu(E).$$

THEOREM. *If* $E \in M_\mu$ *with* $\mu(E) > 0$, *then there is a non-measurable subset* A *of* E *such that* $\mu(A) = \mu(E)$.

PROOF. *Case* 1. $0 < \mu(E) < \infty$. We define $r_0 = \mu(E)$ and $B_0 = \emptyset$.

By lemma 3, there is $A_1 \subset E$ such that $A_1 \notin M_\mu$ and $\mu(A_1) \geqq r_0/2$. Also, there is $B_1 \in M_\mu$ such that $E-B_0 \supset B_1 \supset A_1$ and $\mu(B_1) = \mu(A_1)$. Let $r_1 = \mu(E-B_1)$. Clearly $0 \leqq r_1 \leqq r_0/2$.

If $r_1 = 0$, then we are through. Assume $r_1 > 0$. By the same reason, there are $A_2 \notin M_\mu$ and $B_2 \in M_\mu$ such that $A_2 \subset B_2 \subset E-\bigcup_{k=0}^1 B_k$ and $\mu(B_2) = \mu(A_2) \geqq r_1/2$. Let $r_2 = \mu(E-\bigcup_{k=0}^2 B_k)$, then $0 \leqq r_2 \leqq r_1/2 \leqq r_0/2^2$.

Suppose $\{A_j\}_{j=1}^n$, $\{B_j\}_{j=0}^n$ and $\{r_j\}_{j=0}^n$ have been defined such that $A_j \notin M_\mu$, $B_j \in M_\mu$, $A_j \subset B_j \subset E-\bigcup_{k=0}^{j-1} B_k$,

$$\mu(B_j) = \mu(A_j) \geqq r_{j-1}/2, r_j = \mu(E-\bigcup_{k=0}^{j} B_k) \leqq r_0/2^j \text{ for } j = 1, 2, \cdots n.$$

Clearly $\{B_j\}_{j=1}^n$ is pairwise disjoint. By lemma 2, $\bigcup_{j=1}^n A_j \notin M_\mu$ and

$$\mu(\bigcup_{j=1}^n A_j) = \sum_{j=1}^n \mu(A_j) = \sum_{j=1}^n \mu(B_j) = \mu(\bigcup_{j=1}^n B_j) = \mu(E)-r_n.$$

If $r_n = 0$, we may take $A = \bigcup_{j=1}^n A_j$. Otherwise, $\mu(E-\bigcup_{j=0}^n B_j) = r_n > 0$ and there are $A_{n+1} \notin M_\mu$, $B_{n+1} \in M_\mu$ such that $A_{n+1} \subset B_{n+1} \subset E-\bigcup_{k=0}^n B_k$,

$$\mu(B_{n+1}) = \mu(A_{n+1}) \geqq r_n/2, r_{n+1} = \mu(E-\bigcup_{k=0}^{n+1} B_k) \leqq r_0/2^{n+1}.$$

If this process does not terminate, we obtain infinite sequences $\{A_i\}$, $\{B_i\}$ and $\{r_i\}$. Let $A = \bigcup_{i=1}^\infty A_i$. By lemma 2 again, $A \notin M_\mu$ and

$$\mu(A) = \sum_{i=1}^\infty \mu(A_i) = \sum_{i=1}^\infty \mu(B_i) = \mu(\bigcup_{i=1}^\infty B_i).$$

Thus

$$\mu(E) \geqq \mu(A) \geqq \mu(\bigcup_{i=1}^{n} B_i) = \mu(E) - r_n \geqq \mu(E)(1-1/2^n)$$

for all n. It follows that $\mu(A) = \mu(E)$.

Case 2. $\mu(E) = \infty$. By σ-finiteness of μ, there is a sequence of pairwise disjoint sets $\{E_i\}$ of M_μ such that $E = \cup E_i$, $0 < \mu(E_i) < \infty$ for each i. The conclusion follows easily from case 1 and lemma 2.

Finally, we proceed to the construction of a *counter-example*. Let $E = [0, 1]$. By the above theorem, there is a $Q \subset E$ such that $Q \notin M_\mu$ and $\mu(Q) = \mu(E) = 1$. Let

$$A = (-\infty, 0) \cup Q \cup (1, \infty).$$

Obviously $A \notin M_\mu$, and therefore $\mu(A^c) \neq 0$. We assert that $\mu(A \cap I) = \mu(I)$ for every interval I of R.

Case 1. $I \subset [0, 1]$.

1.1. $0 \in I$ or $1 \in I$: There is a subinterval J of $[0, 1]$ such that $I \cap J = \emptyset$, $I \cup J = [0, 1]$, where J may be empty or a singleton. Thus

$$1 = \mu(Q) = \mu(Q \cap I) + \mu(Q \cap I^c) = \mu(Q \cap I) + \mu(Q \cap J).$$

If $\mu(Q \cap I) < \mu(I)$, then we would have

$$1 = \mu(Q) = \mu(Q \cap I) + \mu(Q \cap J) < \mu(I) + \mu(J) = 1.$$

This leads to a contradiction. Thus $\mu(Q \cap I) = \mu(I)$ and hence $\mu(A \cap I) = \mu(Q \cap I) = \mu(I)$.

1.2. $0 \notin I$ and $1 \notin I$: There are two subintervals J_1, J_2 of $[0, 1]$ such that J_1, I, J_2 are pairwise disjoint and $J_1 \cup I \cup J_2 = [0, 1]$. Since $J_1 \cup I \in M_\mu$, $J_1 \in M_\mu$, we have

$$\begin{aligned} 1 = \mu(Q) &= \mu(Q \cap (J_1 \cup I)) + \mu(Q \cap J_2) \\ &= \mu(Q \cap J_1) + \mu(Q \cap I) + \mu(Q \cap J_2). \end{aligned}$$

If $\mu(Q \cap I) < \mu(I)$, then we would have

$$1 = \mu(Q) < \mu(J_1) + \mu(I) + \mu(J_2) = 1.$$

This leads to a contradiction too. Thus $\mu(A \cap I) = \mu(Q \cap I) = \mu(I)$.

Case 2. $I \not\subset [0, 1]$. Let $I_1 = I \cap (-\infty, 0)$, $I_2 = I \cap [0, 1]$ and $I_3 = I \cap (1, \infty)$ (some of them may be empty). Since $I_1, I_2 \in M_\mu$,

$$\begin{aligned} \mu(A \cap I) &= \mu(A \cap I_1) + \mu(A \cap (I_2 \cup I_3)) \\ &= \mu(A \cap I_1) + \mu(A \cap I_2) + \mu(A \cap I_3) \\ &= \mu(I_1) + \mu(A \cap I_2) + \mu(I_3). \end{aligned}$$

By case 1, we have

$$\mu(A \cap I) = \mu(I_1)+\mu(A \cap I_2)+\mu(I_3) = \mu(I_1)+\mu(I_2)+\mu(I_3) = \mu(I).$$

References

[1] E. Hewitt and K. Stromberg, *Real and Abstract Analysis* (Springer-Verlag, Berlin, 1965, p. 295).

[2] M. E. Munroe, *Introduction to Measure and Integration* (Addison-Wesley, Reading, Massachusetts, 1959, pp. 96—97).

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